

Preprocessing Techniques

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Automated Reasoning and Satisfiability
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Motivation

Subsumption

Variable Elimination

Bounded Variable Addition

Blocked Clause Elimination

Hyper Binary Resolution

Unhiding Redundancy

Concluding Remarks

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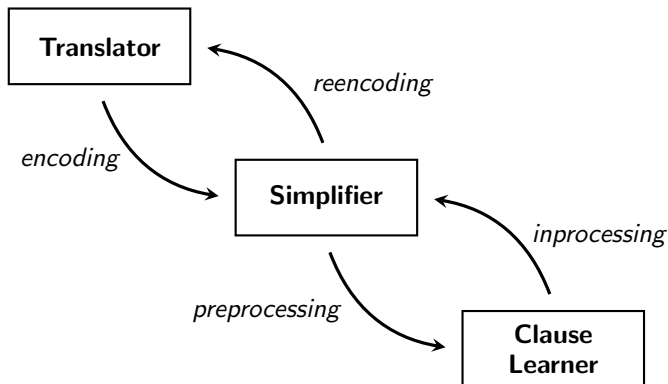
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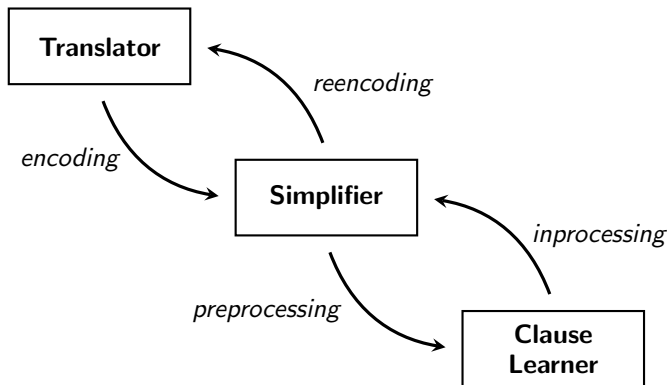
Unhiding Redundancy

Concluding Remarks

Interaction between different solving approaches



Interaction between different solving approaches



It all comes down to adding and removing redundant clauses

Redundant clauses

A clause is redundant with respect to a formula if adding it to the formula preserves satisfiability.

- For unsatisfiable formulas, all clauses can be added, including the empty clause \perp .

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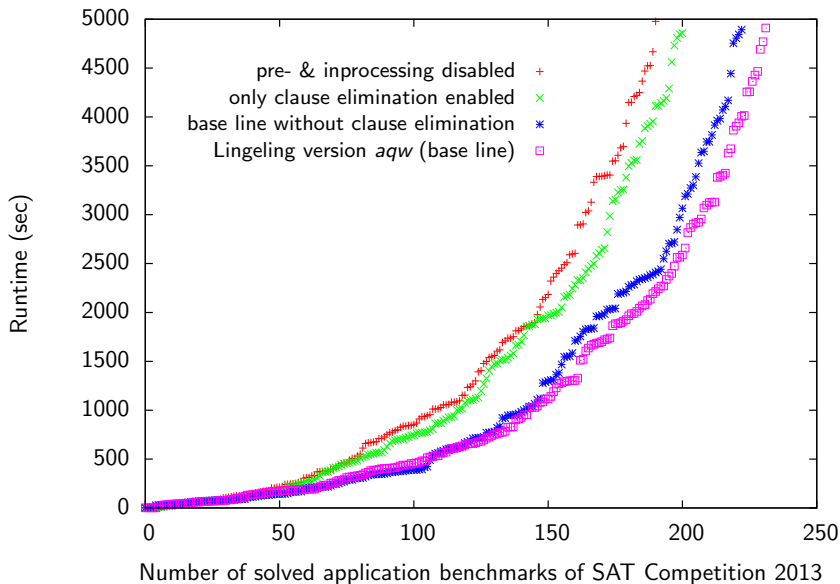
A clause is redundant with respect to a formula if removing it from the formula preserves unsatisfiability.

- For satisfiable formulas, all clauses can be removed.

Challenge regarding redundant clauses:

- How to check redundancy in polynomial time?
- Ideally find redundant clauses in linear time

Preprocessing and Inprocessing in Practice



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Tautologies and Subsumption

Definition (Tautology)

A clause C is a tautology if it contains two complementary literals x and \bar{x} .

Example

The clause $(a \vee b \vee \bar{b})$ is a tautology.

Definition (Subsumption)

Clause C subsumes clause D if and only if $C \subset D$.

Example

The clause $(a \vee b)$ subsumes clause $(a \vee b \vee \bar{c})$.

Self-Subsuming Resolution

Self-Subsuming Resolution

$$\frac{C \vee x \quad D \vee \bar{x}}{D} \quad C \subseteq D \quad \frac{(a \vee b \vee x) \quad (a \vee b \vee c \vee \bar{x})}{(a \vee b \vee c)}$$

resolvent D subsumes second antecedent $D \vee \bar{x}$

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Example

Assume a CNF contains both antecedents

$\dots (a \vee b \vee x)(a \vee b \vee c \vee \bar{x}) \dots$

If D is added, then $D \vee \bar{x}$ can be removed

which in essence removes \bar{x} from $D \vee \bar{x}$

$\dots (a \vee b \vee x)(a \vee b \vee c) \dots$

Initially in the SATeLite preprocessor, [EenBiere'07]
now common in most solvers (i.e., as pre- and inprocessing)

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Example: Remove literals using self-subsumption

$$\begin{aligned} &(a \vee b \vee c) \wedge (\bar{a} \vee b \vee c) \wedge \\ &(\bar{a} \vee b \vee \bar{c}) \wedge (a \vee \bar{b} \vee c) \wedge \\ &(\bar{a} \vee \bar{b} \vee d) \wedge (\bar{a} \vee \bar{b} \vee \bar{d}) \wedge \\ &(a \vee \bar{c} \vee d) \wedge (a \vee \bar{c} \vee \bar{d}) \end{aligned}$$

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Implementing Subsumption

Definition (Subsumption)

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The clause $(a \vee b)$ subsumes clause $(a \vee b \vee \bar{c})$.

Forward Subsumption

for each clause C in formula F **do**

if C is subsumed by a clause D in $F \setminus C$ **then**
 remove C from F

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Backward Subsumption

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for each clause  $C$  in formula  $F$  do  
  remove all clauses  $D$  in  $F$  that are subsumed by  $C$ 
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for each clause C in formula F **do**
 if C is subsumed by a clause D in $F \setminus C$ **then**
 remove C from F

Backward Subsumption

for each clause C in formula F **do**
 pick a literal x in C
 remove all clauses D in F_x that are subsumed by C

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Variable Elimination [DavisPutnam'60]

Definition (Resolution)

Given two clauses $C = (x \vee a_1 \vee \dots \vee a_i)$ and $D = (\bar{x} \vee b_1 \vee \dots \vee b_j)$, the *resolvent* of C and D on variable x (denoted by $C \bowtie_x D$) is $(a_1 \vee \dots \vee a_i \vee b_1 \vee \dots \vee b_j)$

Resolution on sets of clauses F_x and $F_{\bar{x}}$ (denoted by $F_x \bowtie_x F_{\bar{x}}$) generates all non-tautological resolvents of $C \in F_x$ and $D \in F_{\bar{x}}$.

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Given a CNF formula F , *variable elimination* (or DP resolution) removes a variable x by replacing F_x and $F_{\bar{x}}$ by $F_x \bowtie_x F_{\bar{x}}$

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Proof procedure [DavisPutnam60]

VE is a complete proof procedure. Applying VE until fixpoint results in either the empty formula (satisfiable) or empty clause (unsatisfiable)

Example VE by clause distribution [DavisPutnam'60]

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Example of clause distribution

		F_x		
		$(x \vee c)$	$(x \vee \bar{d})$	$(x \vee \bar{a} \vee \bar{b})$
$F_{\bar{x}}$	$\left\{ \begin{array}{l} (\bar{x} \vee a) \\ (\bar{x} \vee b) \\ (\bar{x} \vee \bar{e} \vee f) \end{array} \right.$	$(a \vee c)$	$(a \vee \bar{d})$	$(a \vee \bar{a} \vee \bar{b})$
		$(b \vee c)$	$(b \vee \bar{d})$	$(b \vee \bar{a} \vee \bar{b})$
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			$(a \vee \bar{a} \vee \bar{b})$
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In the example: $|F_x \bowtie_x F_{\bar{x}}| > |F_x| + |F_{\bar{x}}|$

Exponential growth of clauses in general

VE by substitution [EenBiere07]

General idea

Detect gates (or definitions) $x = \text{GATE}(a_1, \dots, a_n)$ in the formula and use them to reduce the number of added clauses

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Possible gates

gate	G_x	$G_{\bar{x}}$
AND(a_1, \dots, a_n)	$(x \vee \bar{a}_1 \vee \dots \vee \bar{a}_n)$	$(\bar{x} \vee a_1), \dots, (\bar{x} \vee a_n)$
OR(a_1, \dots, a_n)	$(x \vee \bar{a}_1), \dots, (x \vee \bar{a}_n)$	$(\bar{x} \vee a_1 \vee \dots \vee a_n)$
ITE(c, t, f)	$(x \vee \bar{c} \vee \bar{t}), (x \vee c \vee \bar{f})$	$(\bar{x} \vee \bar{c} \vee t), (\bar{x} \vee c \vee f)$

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Variable elimination by substitution [EenBiere07]

Let $R_x = F_x \setminus G_x$; $R_{\bar{x}} = F_{\bar{x}} \setminus G_{\bar{x}}$.

Replace $F_x \wedge F_{\bar{x}}$ by $G_x \bowtie_x R_{\bar{x}} \wedge G_{\bar{x}} \bowtie_x R_x$.

Always less than $F_x \bowtie_x F_{\bar{x}}$!

VE by substitution [EenBiere'07]

Example of gate extraction: $x = \text{AND}(a, b)$

$$F_x = (x \vee c) \wedge (x \vee \bar{d}) \wedge (x \vee \bar{a} \vee \bar{b})$$

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Example of substitution

	R_x		G_x
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using substitution: $|F_x \bowtie F_{\bar{x}}| < |F_x| + |F_{\bar{x}}|$

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Main Idea

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Reverse of Variable Elimination

For example, replace the clauses

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by

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Challenge: how to find suitable patterns for replacement?

Factoring Out Subclauses

Example

Replace

$$(a \vee b \vee c \vee d) \quad (a \vee b \vee c \vee e) \quad (a \vee b \vee c \vee f)$$

by

$$(x \vee d) \quad (x \vee e) \quad (x \vee f) \quad (\bar{x} \vee a \vee b \vee c)$$

adds 1 variable and 1 clause *reduces number of literals by 2*

Not compatible with VE, which would eliminate x immediately!

... so this does not work ...

Bounded Variable Addition

Example

Smallest pattern that is compatible: Replace

$$\begin{array}{cc} (a \vee d) & (a \vee e) \\ (b \vee d) & (b \vee e) \\ (c \vee d) & (c \vee e) \end{array}$$

by

$$\begin{array}{ccc} (\bar{x} \vee a) & (\bar{x} \vee b) & (\bar{x} \vee c) \\ (x \vee d) & (x \vee e) & \end{array}$$

adds 1 variable

removes 1 clause

Bounded Variable Addition

Possible Patterns

$$\begin{array}{ccc} (X_1 \vee L_1) & \dots & (X_1 \vee L_k) \\ \vdots & & \vdots \\ (X_n \vee L_1) & \dots & (X_n \vee L_k) \end{array} \equiv \bigwedge_{i=1}^n \bigwedge_{j=1}^k (X_i \vee L_j)$$

replaced by $\bigwedge_{i=1}^n (y \vee X_i) \wedge \bigwedge_{j=1}^k (\bar{y} \vee L_j)$

- Every k clauses share sets of literals L_j
- There are n sets of literals X_i that appear in clauses with L_j

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- Every k clauses share sets of literals L_j
- There are n sets of literals X_i that appear in clauses with L_j
- Reduction: $nk - n - k$ clauses are removed by replacement

Bounded Variable Addition on AtMostOne (1)

Example encoding of AtMostOne (x_1, x_2, \dots, x_n)

$$\begin{aligned} &(\bar{x}_1 \vee \bar{x}_2) \wedge (\bar{x}_9 \vee \bar{x}_{10}) \wedge (\bar{x}_8 \vee \bar{x}_{10}) \wedge (\bar{x}_7 \vee \bar{x}_{10}) \wedge (\bar{x}_6 \vee \bar{x}_{10}) \wedge \\ &(\bar{x}_1 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_8 \vee \bar{x}_9) \wedge (\bar{x}_7 \vee \bar{x}_9) \wedge (\bar{x}_6 \vee \bar{x}_9) \wedge \\ &(\bar{x}_1 \vee \bar{x}_4) \wedge (\bar{x}_2 \vee \bar{x}_4) \wedge (\bar{x}_3 \vee \bar{x}_4) \wedge (\bar{x}_7 \vee \bar{x}_8) \wedge (\bar{x}_6 \vee \bar{x}_8) \wedge \\ &(\bar{x}_1 \vee \bar{x}_5) \wedge (\bar{x}_2 \vee \bar{x}_5) \wedge (\bar{x}_3 \vee \bar{x}_5) \wedge (\bar{x}_4 \vee \bar{x}_5) \wedge (\bar{x}_6 \vee \bar{x}_7) \wedge \\ &(\bar{x}_1 \vee \bar{x}_6) \wedge (\bar{x}_2 \vee \bar{x}_6) \wedge (\bar{x}_3 \vee \bar{x}_6) \wedge (\bar{x}_4 \vee \bar{x}_6) \wedge (\bar{x}_5 \vee \bar{x}_6) \wedge \\ &(\bar{x}_1 \vee \bar{x}_7) \wedge (\bar{x}_2 \vee \bar{x}_7) \wedge (\bar{x}_3 \vee \bar{x}_7) \wedge (\bar{x}_4 \vee \bar{x}_7) \wedge (\bar{x}_5 \vee \bar{x}_7) \wedge \\ &(\bar{x}_1 \vee \bar{x}_8) \wedge (\bar{x}_2 \vee \bar{x}_8) \wedge (\bar{x}_3 \vee \bar{x}_8) \wedge (\bar{x}_4 \vee \bar{x}_8) \wedge (\bar{x}_5 \vee \bar{x}_8) \wedge \\ &(\bar{x}_1 \vee \bar{x}_9) \wedge (\bar{x}_2 \vee \bar{x}_9) \wedge (\bar{x}_3 \vee \bar{x}_9) \wedge (\bar{x}_4 \vee \bar{x}_9) \wedge (\bar{x}_5 \vee \bar{x}_9) \wedge \\ &(\bar{x}_1 \vee \bar{x}_{10}) \wedge (\bar{x}_2 \vee \bar{x}_{10}) \wedge (\bar{x}_3 \vee \bar{x}_{10}) \wedge (\bar{x}_4 \vee \bar{x}_{10}) \wedge (\bar{x}_5 \vee \bar{x}_{10}) \end{aligned}$$

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Replace $(\bar{x}_i \vee \bar{x}_j)$ with $i \in \{1..5\}, j \in \{6..10\}$ by $(\bar{x}_i \vee y), (\bar{x}_j \vee \bar{y})$

Bounded Variable Addition on AtMostOne (2)

Example encoding of AtMostOne (x_1, x_2, \dots, x_n)

$$\begin{aligned} &(\bar{x}_1 \vee \bar{x}_2) \wedge (\bar{x}_9 \vee \bar{x}_{10}) \wedge (\bar{x}_8 \vee \bar{x}_{10}) \wedge (\bar{x}_7 \vee \bar{x}_{10}) \wedge (\bar{x}_6 \vee \bar{x}_{10}) \wedge \\ &(\bar{x}_1 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_8 \vee \bar{x}_9) \wedge (\bar{x}_7 \vee \bar{x}_9) \wedge (\bar{x}_6 \vee \bar{x}_9) \wedge \\ &(\bar{x}_1 \vee \bar{x}_4) \wedge (\bar{x}_2 \vee \bar{x}_4) \wedge (\bar{x}_3 \vee \bar{x}_4) \wedge (\bar{x}_7 \vee \bar{x}_8) \wedge (\bar{x}_6 \vee \bar{x}_8) \wedge \\ &(\bar{x}_1 \vee \bar{x}_5) \wedge (\bar{x}_2 \vee \bar{x}_5) \wedge (\bar{x}_3 \vee \bar{x}_5) \wedge (\bar{x}_4 \vee \bar{x}_5) \wedge (\bar{x}_6 \vee \bar{x}_7) \wedge \\ &(\bar{x}_1 \vee \mathbf{y}) \wedge (\bar{x}_2 \vee \mathbf{y}) \wedge (\bar{x}_3 \vee \mathbf{y}) \wedge (\bar{x}_4 \vee \mathbf{y}) \wedge (\bar{x}_5 \vee \mathbf{y}) \wedge \\ &(\bar{x}_6 \vee \bar{\mathbf{y}}) \wedge (\bar{x}_7 \vee \bar{\mathbf{y}}) \wedge (\bar{x}_8 \vee \bar{\mathbf{y}}) \wedge (\bar{x}_9 \vee \bar{\mathbf{y}}) \wedge (\bar{x}_{10} \vee \bar{\mathbf{y}}) \end{aligned}$$

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Replace matched pattern

$$\begin{aligned} &(\bar{x}_1 \vee \mathbf{z}) \wedge (\bar{x}_2 \vee \mathbf{z}) \wedge (\bar{x}_3 \vee \mathbf{z}) \wedge \\ &(\bar{x}_4 \vee \bar{\mathbf{z}}) \wedge (\bar{x}_5 \vee \bar{\mathbf{z}}) \wedge (\mathbf{y} \vee \bar{\mathbf{z}}) \end{aligned}$$

Bounded Variable Addition on AtMostOne (3)

Example encoding of AtMostOne (x_1, x_2, \dots, x_n)

$$\begin{aligned} &(\bar{x}_1 \vee \bar{x}_2) \wedge (\bar{x}_9 \vee \bar{x}_{10}) \wedge (\bar{x}_8 \vee \bar{x}_{10}) \wedge (\bar{x}_7 \vee \bar{x}_{10}) \wedge (\bar{x}_6 \vee \bar{x}_{10}) \wedge \\ &(\bar{x}_1 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_8 \vee \bar{x}_9) \wedge (\bar{x}_7 \vee \bar{x}_9) \wedge (\bar{x}_6 \vee \bar{x}_9) \wedge \\ &(\bar{x}_1 \vee z) \wedge (\bar{x}_2 \vee z) \wedge (\bar{x}_3 \vee z) \wedge (\bar{x}_7 \vee \bar{x}_8) \wedge (\bar{x}_6 \vee \bar{x}_8) \wedge \\ &(\bar{x}_4 \vee \bar{z}) \wedge (\bar{x}_5 \vee \bar{z}) \wedge (y \vee \bar{z}) \wedge (\bar{x}_4 \vee \bar{x}_5) \wedge (\bar{x}_6 \vee \bar{x}_7) \wedge \\ &(\bar{x}_4 \vee y) \wedge (\bar{x}_5 \vee y) \wedge (\bar{x}_6 \vee \bar{y}) \wedge (\bar{x}_7 \vee \bar{y}) \wedge (\bar{x}_8 \vee \bar{y}) \\ &(\bar{x}_9 \vee \bar{y}) \wedge (\bar{x}_{10} \vee \bar{y}) \end{aligned}$$

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Replace matched pattern

$$\begin{aligned} &(\bar{x}_6 \vee w) \wedge (\bar{x}_7 \vee w) \wedge (\bar{x}_8 \vee w) \wedge \\ &(\bar{x}_9 \vee \bar{w}) \wedge (\bar{x}_{10} \vee \bar{w}) \wedge (\bar{y} \vee \bar{w}) \end{aligned}$$

Motivation

Subsumption

Variable Elimination

Bounded Variable Addition

Blocked Clause Elimination

Hyper Binary Resolution

Unhiding Redundancy

Concluding Remarks

Blocked Clauses [Kullmann 1999]

Definition (Block Clause)

A clause $(C \vee x)$ is a **blocked** on x w.r.t. a CNF formula F if for every clause $(D \vee \bar{x}) \in F$, resolvent $C \vee D$ is a **tautology**.

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Example

Consider the formula $(a \vee b) \wedge (a \vee \bar{b} \vee \bar{c}) \wedge (\bar{a} \vee c)$.

First clause is not blocked.

Second clause is blocked by both a and \bar{c} .

Third clause is blocked by c

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Theorem

Adding or removing a blocked clause preserves (un)satisfiability.

Blocked Clause Elimination (BCE)

Definition (BCE)

While there is a blocked clause C in a CNF F , remove C from F .

Example

Consider $(a \vee b) \wedge (a \vee \bar{b} \vee \bar{c}) \wedge (\bar{a} \vee c)$.

*After removing either $(a \vee \bar{b} \vee \bar{c})$ or $(\bar{a} \vee c)$, the clause $(a \vee b)$ becomes blocked (*no clause* with either \bar{b} or \bar{a}).*

An extreme case in which BCE removes all clauses!

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Proposition

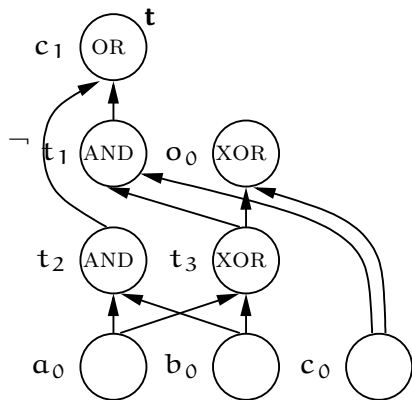
BCE is confluent, i.e., has a unique fixpoint

- Blocked clauses stay blocked w.r.t. removal

BCE very effective on circuits [JärvisaloBiereHeule'10]

BCE converts the Tseitin encoding to Plaisted Greenbaum
BCE simulates Pure literal elimination, Cone of influence, etc.

Example of circuit simplification by BCE on Tseitin encoding

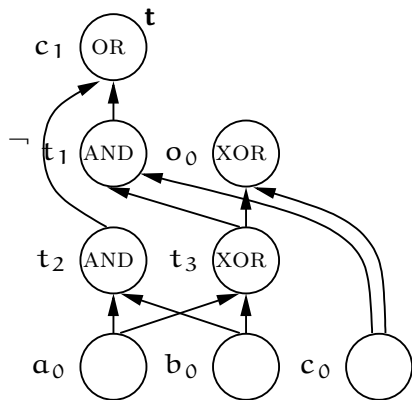


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	$(\bar{t}_2 \vee b_0)$
$(\bar{o}_0 \vee t_3 \vee c_0)$	
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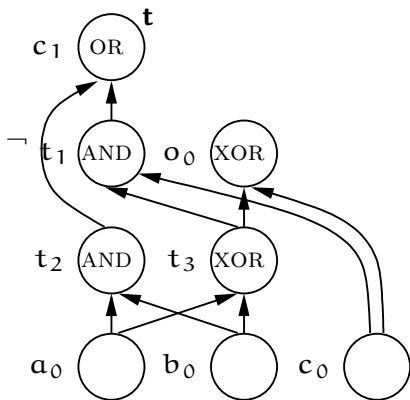


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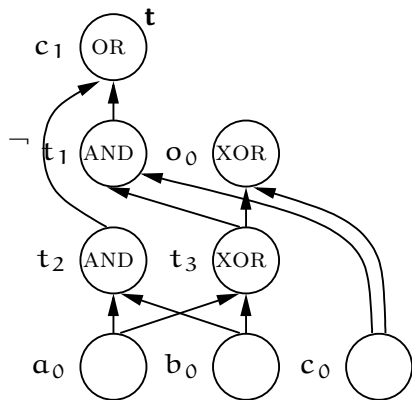


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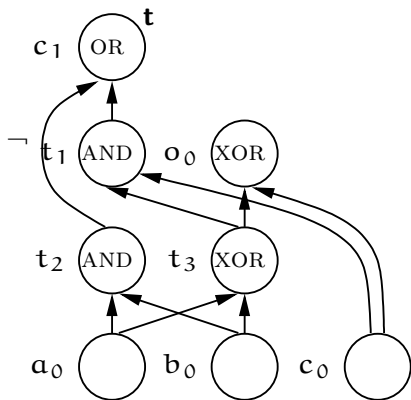


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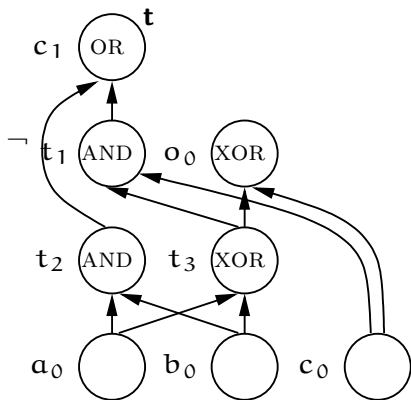


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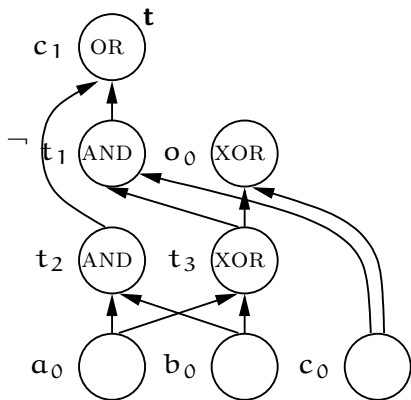


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$(c_1 \vee \bar{t}_2)$	$(\bar{t}_2 \vee a_0)$
	$(\bar{t}_2 \vee b_0)$
$(\bar{c}_0 \vee t_3 \vee c_0)$	
$(\bar{c}_0 \vee \bar{t}_3 \vee \bar{c}_0)$	$(\bar{t}_3 \vee a_0 \vee b_0)$
$(c_0 \vee t_3 \vee \bar{c}_0)$	$(\bar{t}_3 \vee \bar{a}_0 \vee \bar{b}_0)$
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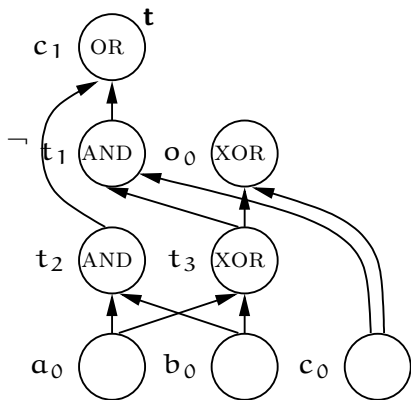


BCE very effective on circuits [JärvisaloBiereHeule'10]

BCE converts the Tseitin encoding to Plaisted Greenbaum
BCE simulates Pure literal elimination, Cone of influence, etc.

Example of circuit simplification by BCE on Tseitin encoding

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$(\bar{c}_0 \vee t_3 \vee c_0)$	
$(\bar{c}_0 \vee \bar{t}_3 \vee \bar{c}_0)$	$(\bar{t}_3 \vee a_0 \vee b_0)$
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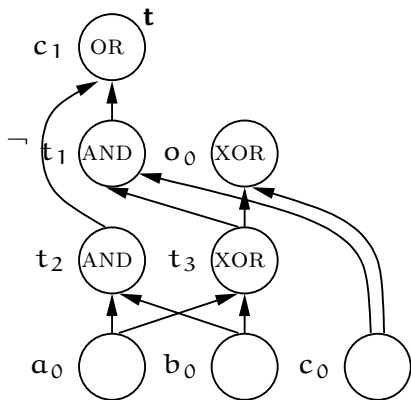


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$(c_1 \vee t_2)$	
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$(\bar{o}_0 \vee \bar{t}_3 \vee \bar{c}_0)$	$(\bar{t}_2 \vee a_0)$
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$(o_0 \vee \bar{t}_3 \vee c_0)$	$(\bar{t}_3 \vee a_0 \vee b_0)$
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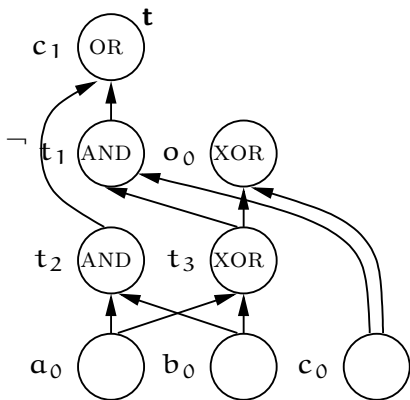


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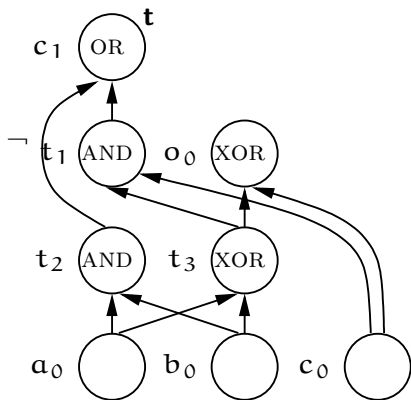


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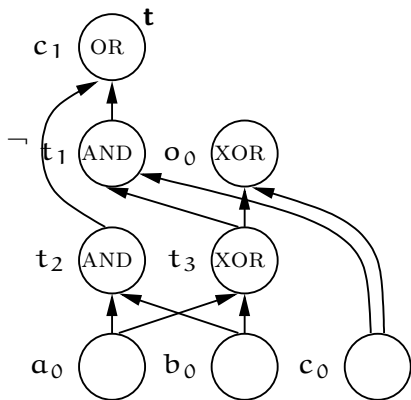


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	$(t_3 \vee a_0 \vee \bar{b}_0)$
	$(t_3 \vee \bar{a}_0 \vee b_0)$



BCE: Solution Reconstruction

Input:

- stack S of eliminated blocked clauses
- formula F (without the blocked clauses)
- assignment α that satisfies F

Output: an assignment that satisfies $F \wedge S$

```
1: while  $S.size$  () do  
2:    $\langle C, l \rangle := S.pop$  ()  
3:   if  $\alpha$  falsifies  $C$  then  $\alpha := \alpha_l$   
4: end while  
5: return  $\alpha$ 
```


Motivation

Subsumption

Variable Elimination

Bounded Variable Addition

Blocked Clause Elimination

Hyper Binary Resolution

Unhiding Redundancy

Concluding Remarks

Hyper Binary Resolution [Bacchus-AAAI02]

Definition (Hyper Binary Resolution Rule)

$$\frac{(x \vee x_1 \vee x_2 \vee \dots \vee x_n) \quad (\bar{x}_1 \vee x') \quad (\bar{x}_2 \vee x') \quad \dots \quad (\bar{x}_n \vee x')}{(x \vee x')}$$

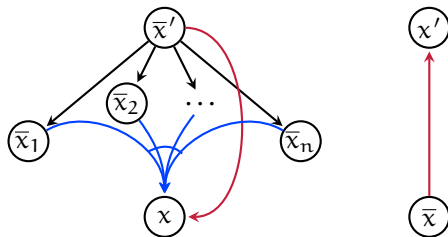
binary edge



hyper edge



hyper binary edge



Hyper Binary Resolution Rule:

- combines multiple resolution steps into one
- uses one n-ary clauses and multiple binary clauses
- special case *hyper unary resolution* where $x = x'$

Hyper Binary Resolution (HBR)

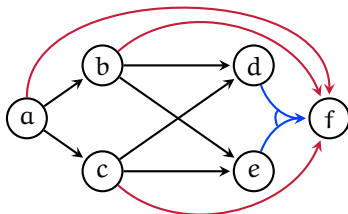
Definition (Hyper Binary Resolution)

Apply the hyper binary resolution rule until fixpoint

Example

Consider

$$(\bar{a} \vee b) \wedge (\bar{a} \vee c) \wedge (\bar{b} \vee d) \wedge (\bar{b} \vee e) \wedge (\bar{c} \vee d) \wedge (\bar{c} \vee e) \wedge (\bar{d} \vee \bar{e} \vee f).$$



hyper binary resolvents:

$$(\bar{a} \vee f), (\bar{b} \vee f), (\bar{c} \vee f)$$

HBR is confluent, i.e., has a unique fixpoint

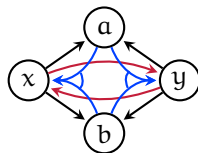
Structural Hashing of AND-gates via HBR

gate g	$g \Rightarrow f(g_1, \dots, g_n)$ "positive"	$g \Leftarrow f(g_1, \dots, g_n)$ "negative"
$g := \text{OR}(g_1, \dots, g_n)$	$(\bar{g} \vee g_1 \vee \dots \vee g_n)$	$(g \vee \bar{g}_1), \dots, (g \vee \bar{g}_n)$
$g := \text{AND}(g_1, \dots, g_n)$	$(\bar{g} \vee g_1), \dots, (\bar{g} \vee g_n)$	$(g \vee \bar{g}_1 \vee \dots \vee \bar{g}_n)$
$g := \text{XOR}(g_1, g_2)$	$(\bar{g} \vee \bar{g}_1 \vee \bar{g}_2), (\bar{g} \vee g_1 \vee g_2)$	$(g \vee \bar{g}_1 \vee \bar{g}_2), (g \vee g_1 \vee \bar{g}_2)$
$g := \text{ITE}(g_1, g_2, g_3)$	$(\bar{g} \vee \bar{g}_1 \vee g_2), (\bar{g} \vee g_1 \vee g_3)$	$(g \vee \bar{g}_1 \vee \bar{g}_2), (g \vee g_1 \vee \bar{g}_3)$

Definition (Structural Hashing of AND-gates)

Given a Boolean circuit with two equivalent gates, merge the gates.

Example



$$x = \text{AND}(a,b) : (\bar{x} \vee a) \wedge (\bar{x} \vee b) \wedge (x \vee \bar{a} \vee \bar{b})$$

$$y = \text{AND}(a,b) : (\bar{y} \vee a) \wedge (\bar{y} \vee b) \wedge (y \vee \bar{a} \vee \bar{b})$$

the two HBRs $(\bar{x} \vee y)$ and $(x \vee \bar{y})$ express that $x = y$

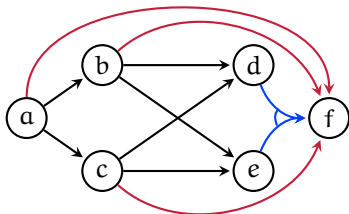
Non-transitive Hyper Binary Resolution (NHBR)

A problem with classic HBR is that it adds many **transitive** binary clauses

Example

Consider

$$(\bar{a} \vee b) \wedge (\bar{a} \vee c) \wedge (\bar{b} \vee d) \wedge (\bar{b} \vee e) \wedge (\bar{c} \vee d) \wedge (\bar{c} \vee e) \wedge (\bar{d} \vee \bar{e} \vee f).$$



adding $(\bar{b} \vee f)$ or $(\bar{c} \vee f)$
makes $(\bar{a} \vee f)$ transitive

Solution [HeuleJärvisaloBiere 2013]

Add only non-transitive hyper binary resolvents

Can be implemented using an alternative unit propagation style

Space Complexity of NHBR: Quadratic

Question regarding complexity [Biere 2009]

- Are there formulas where the transitively reduced hyper binary resolution closure is quadratic in size w.r.t. to the size of the original?
- where size = #clauses or size = #literals or size = #variables

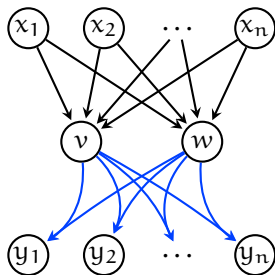
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Yes!

Consider the formula $F_n = \bigwedge_{1 \leq i \leq n} ((\bar{x}_i \vee v) \wedge (\bar{x}_i \vee w) \wedge (\bar{v} \vee \bar{w} \vee y_i))$



#variables: $2n + 2$

#clauses: $3n$

#literals: $7n$

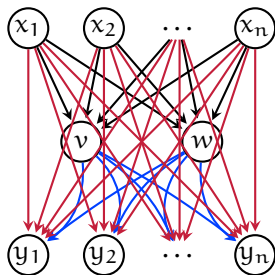
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#clauses: $3n$

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n^2 hyper binary resolvents:

$(\bar{x}_i \vee y_j)$ for $1 \leq i, j \leq n$

Motivation

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Concluding Remarks

Redundancy

Redundant clauses:

- Removal of $C \in F$ preserves unsatisfiability of F
- Assign all $x \in C$ to false and check for a conflict in $F \setminus \{C\}$

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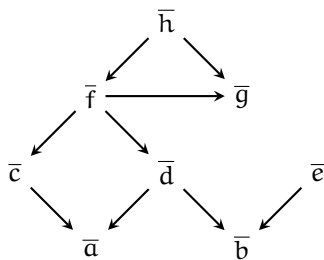
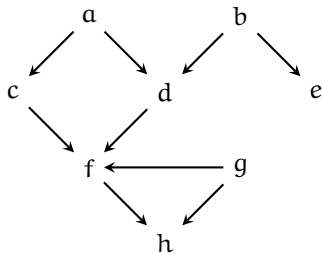
- Removal of $x \in C$ preserves satisfiability of F
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Redundancy elimination during pre- and in-processing

- Distillation [JinSomenzi2005]
- ReVivAI [PietteHamadiSaïs2008]
- Unhiding [HeuleJärvisaloBiere2011]

Unhide: Binary implication graph (BIG)

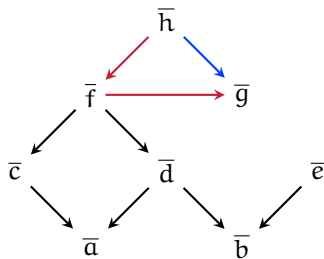
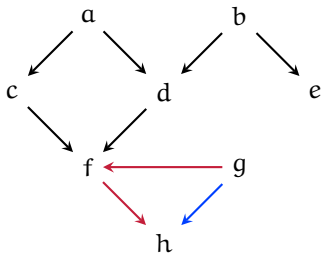
unhide: use the binary clauses to detect redundant clauses and literals



$$\begin{aligned} &(\bar{a} \vee c) \wedge (\bar{a} \vee d) \wedge (\bar{b} \vee d) \wedge (\bar{b} \vee e) \wedge \\ &(\bar{c} \vee f) \wedge (\bar{d} \vee f) \wedge (\bar{g} \vee f) \wedge (\bar{f} \vee h) \wedge \\ &(\bar{g} \vee h) \wedge \underbrace{(\bar{a} \vee \bar{e} \vee h) \wedge (\bar{b} \vee \bar{c} \vee h) \wedge (a \vee b \vee c \vee d \vee e \vee f \vee g \vee h)}_{\text{non binary clauses}} \end{aligned}$$

Unhide: Transitive reduction (TRD)

transitive reduction: remove shortcuts in the binary implication graph



$$(\bar{a} \vee c) \wedge (\bar{a} \vee d) \wedge (\bar{b} \vee d) \wedge (\bar{b} \vee e) \wedge$$

$$(\bar{c} \vee f) \wedge (\bar{d} \vee f) \wedge (\bar{g} \vee f) \wedge (\bar{f} \vee h) \wedge$$

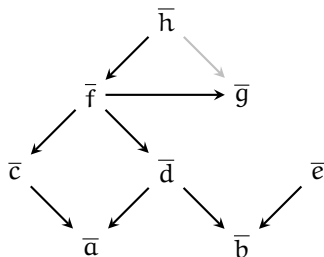
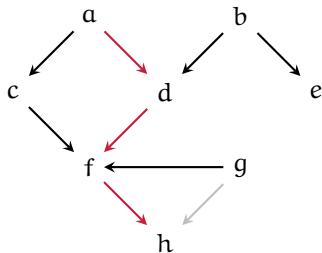
$$(\bar{g} \vee h) \wedge (\bar{a} \vee \bar{e} \vee h) \wedge (\bar{b} \vee \bar{c} \vee h) \wedge (a \vee b \vee c \vee d \vee e \vee f \vee g \vee h)$$

TRD

$$g \rightarrow f \rightarrow h$$

Unhide: Hidden tautology elimination (HTE) (1)

HTE removes clauses that are subsumed by an implication in BIG



$$(\bar{a} \vee c) \wedge (\bar{a} \vee d) \wedge (\bar{b} \vee d) \wedge (\bar{b} \vee e) \wedge$$

$$(\bar{c} \vee f) \wedge (\bar{d} \vee f) \wedge (\bar{g} \vee f) \wedge (\bar{f} \vee h) \wedge$$

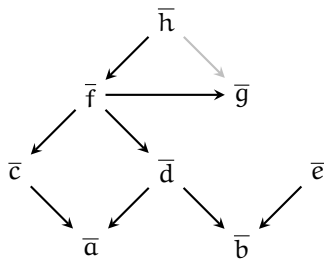
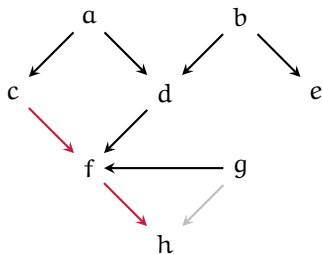
$$(\bar{a} \vee \bar{e} \vee h) \wedge (\bar{b} \vee \bar{c} \vee h) \wedge (a \vee b \vee c \vee d \vee e \vee f \vee g \vee h)$$

HTE

$$a \rightarrow d \rightarrow f \rightarrow h$$

Unhide: Hidden tautology elimination (HTE) (2)

HTE removes clauses that are subsumed by an implication in BIG



$$(\bar{a} \vee c) \wedge (\bar{a} \vee d) \wedge (\bar{b} \vee d) \wedge (\bar{b} \vee e) \wedge$$

$$(\bar{c} \vee f) \wedge (\bar{d} \vee f) \wedge (\bar{g} \vee f) \wedge (\bar{f} \vee h) \wedge$$

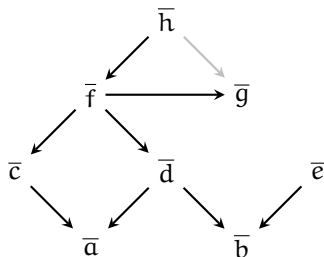
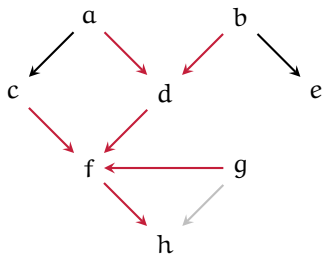
$$\cancel{(\bar{b} \vee \bar{c} \vee h)} \wedge (a \vee b \vee c \vee d \vee e \vee f \vee g \vee h)$$

HTE

$$c \rightarrow f \rightarrow h$$

Unhide: Hidden literal elimination (HLE)

HLE removes literal using the implication in BIG



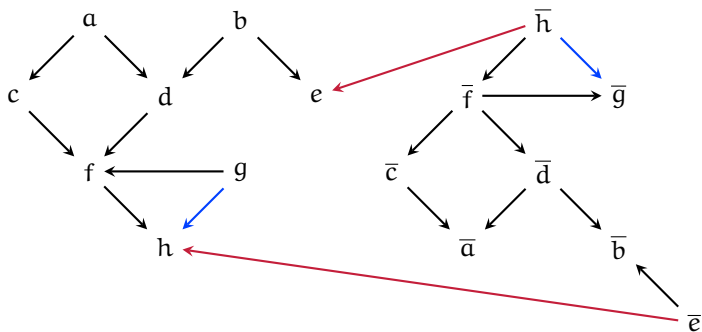
$$(\bar{a} \vee c) \wedge (\bar{a} \vee d) \wedge (\bar{b} \vee d) \wedge (\bar{b} \vee e) \wedge$$
$$(\bar{c} \vee f) \wedge (\bar{d} \vee f) \wedge (\bar{g} \vee f) \wedge (\bar{f} \vee h) \wedge$$

$$(\bar{a} \vee \bar{b} \vee \bar{c} \vee \bar{d} \vee e \vee \bar{f} \vee \bar{g} \vee h)$$

HLE
all but e imply h
also b implies e

Unhide: TRD + HTE + HLE

unhide: redundancy elimination removes and adds arcs from BIG(F)



$$(\bar{a} \vee c) \wedge (\bar{a} \vee d) \wedge (\bar{b} \vee d) \wedge (\bar{b} \vee e) \wedge$$
$$(\bar{c} \vee f) \wedge (\bar{d} \vee f) \wedge (\bar{g} \vee f) \wedge (\bar{f} \vee h) \wedge (e \vee h)$$

Motivation

Subsumption

Variable Elimination

Bounded Variable Addition

Blocked Clause Elimination

Hyper Binary Resolution

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Concluding Remarks

Many Techniques

Many pre- or in-processing techniques in SAT solvers:

- (Self-)Subsumption
- Variable Elimination
- Blocked Clause Elimination
- Hyper Binary Resolution
- Bounded Variable Addition
- Equivalent Literal Substitution
- Failed Literal Elimination
- Autarky Reasoning
- ...

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- ...

... and the list is growing:

- Propagation Redundant Clauses [CADE'17]