Local Search Techniques

Marijn J.H. Heule

Carnegie Mellon University

http://www.cs.cmu.edu/~mheule/15816-f22/

Automated Reasoning and Satisfiability
September 28, 2022

Random k-SAT

Stochastic Local Search

WalkSAT and ProbSAT

Weight Transfer

UnitWalk

Random k-SAT

Stochastic Local Search

WalkSAT and ProbSAT

Weight Transfer

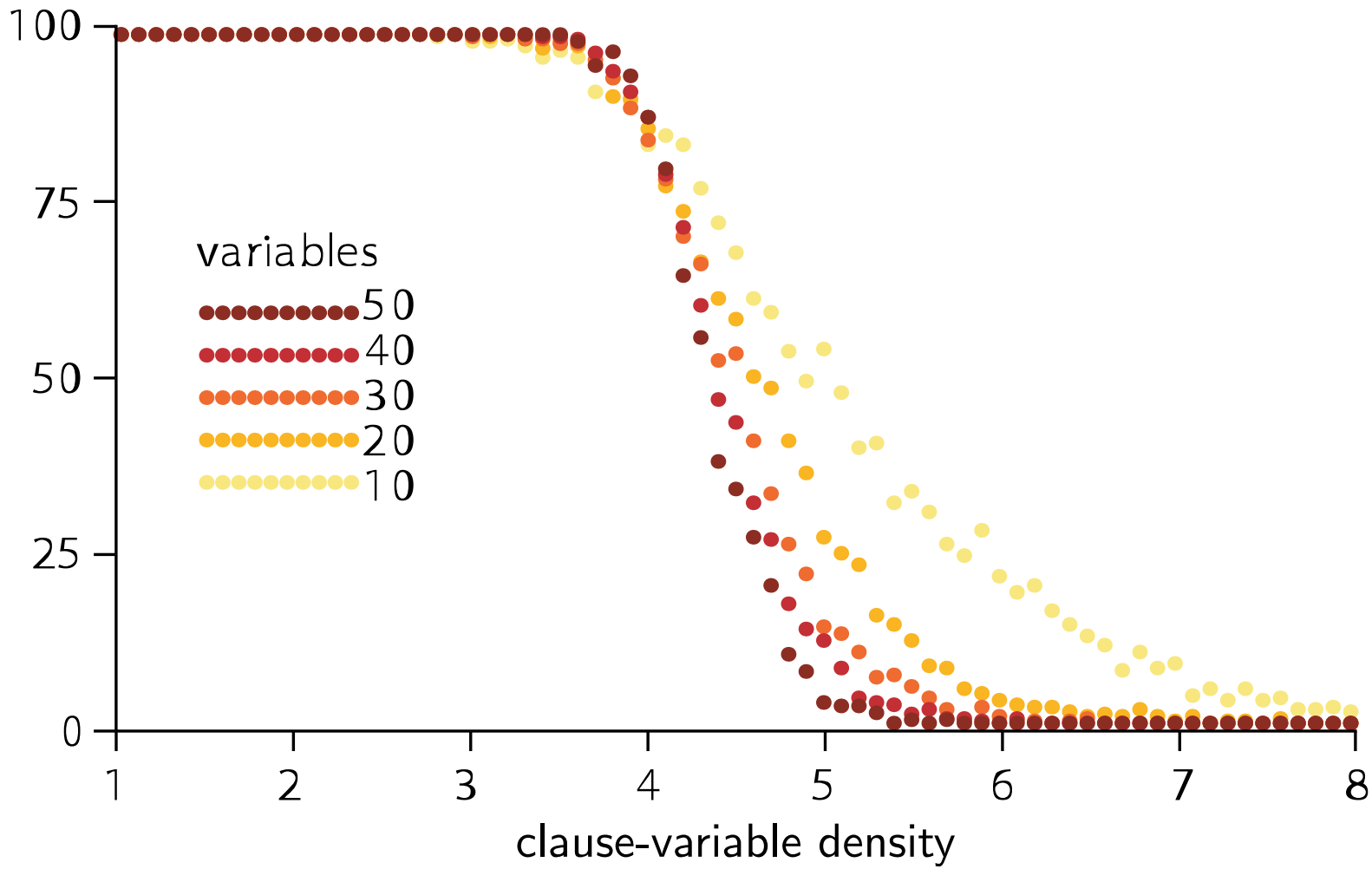
UnitWalk

Random k-SAT

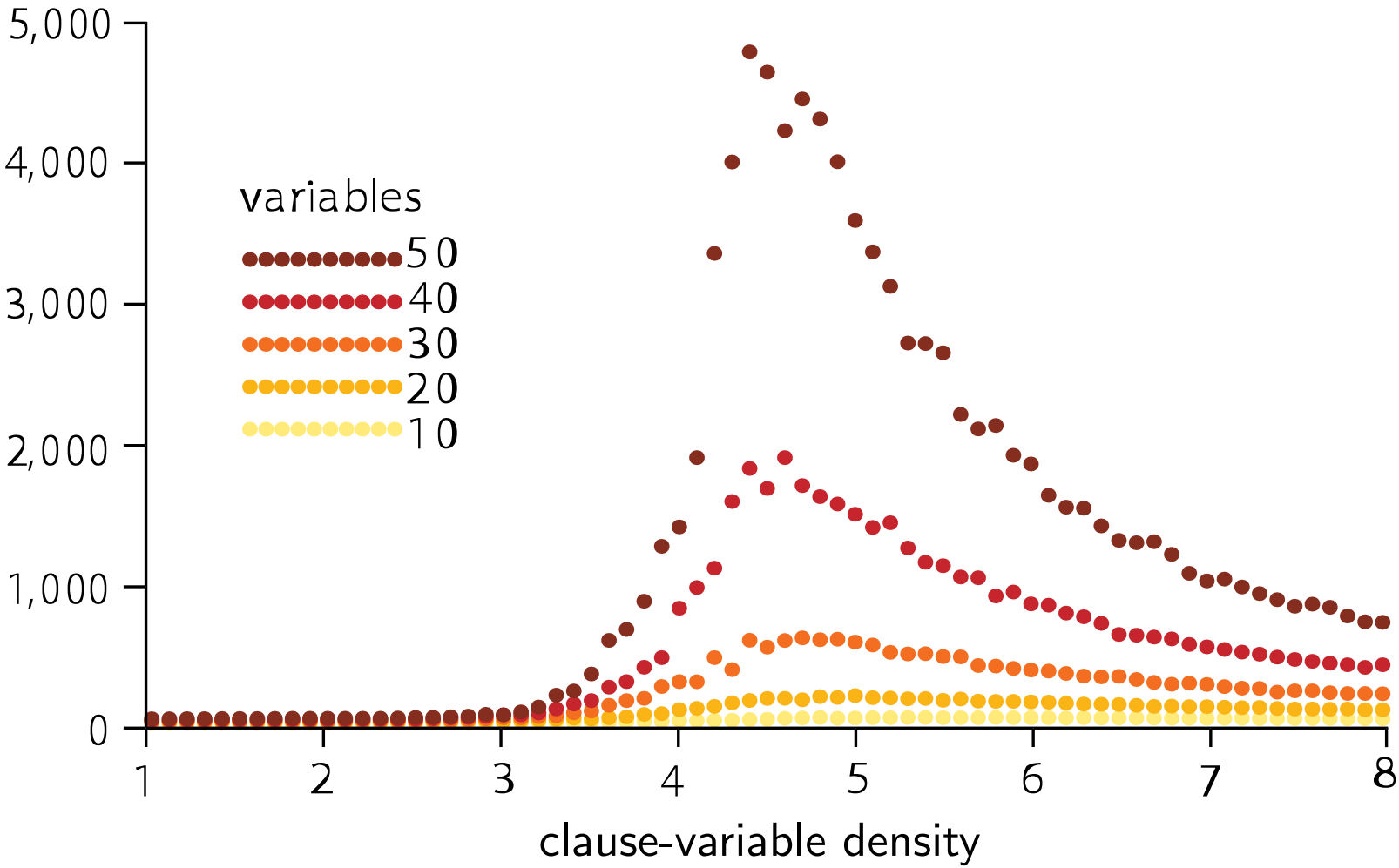
Local search solvers are particularly effective (and optimized) on hard uniform random (satisfiable) k-SAT problems

- All clauses have length k
- Variables have the same probability to occur
- Each literal is negated with probability of 50%
- Density is ratio Clauses to Variables

Random 3-SAT: % satisfiable, the phase transition



Random 3-SAT: exponential runtime, the threshold



Random k-SAT

Stochastic Local Search

WalkSAT and ProbSAT

Weight Transfer

UnitWalk

Local Search: Generic structure

Generic structure of local search SAT solvers

```
1:  for i in 1 to MAX_TRIES do
2:      α := random initial assignment
3:      for j in 1 to MAX_STEPS do
4:          if α satisfies F then
5:              return satisfiable
6:          end if
7:          α := Flip (α)
8:      end for
9:  end for
10: return unknown
```

Local Search: Global vs Local flips

Global flips

- Pro: Big improvements
- Neg: Probabilistic incomplete

Local flips

- Neg: Small improvements
- Pos: Probabilistic complete

Random k-SAT

Stochastic Local Search

WalkSAT and ProbSAT

Weight Transfer

UnitWalk

Local Search: Types of Flips

Select a random unsatisfied clause C

- Free flip
- Random flip
- Heuristic flip

Local Search: WalkSAT Code [Selman, Kautz, and Cohen '93]

FLIP_WALKSAT (α)

1: $C :=$ random falsified clause by $\alpha \circ \mathcal{F}$
2: **if** a variable in C can be flipped for free **then**
3: flip in α that variable
4: **else**
5: flip in α with p a random $x_i \in C$
6: flip in α with $1 - p$ the "optimal" $x_i \in C$
7: **end if**
8: **return** α

Local Search: ProbSAT [Balint and Schöning '12]

ProbSAT generalizes the WalkSAT code.

Let $\mathrm{break}(x, \alpha)$ denote the number of clauses that are only satisfied by x or $\overline{x}$ under the assignment α

- $C :=$ random falsified clause by $\alpha \circ \mathcal{F}$
- randomly pick a variable x in C using weights $c^{-\mathrm{break}(x,\alpha)}$
- an effective constant for random 3-SAT: $c = 2.5$
- update α by flipping x

Random k-SAT

Stochastic Local Search

WalkSAT and ProbSAT

Weight Transfer

UnitWalk

Local Search: Weight Transfer

- All clauses have a weight
- Only do global flips
- Pick the variable that reduces the falsified weight the most
- If there is no weight-reducing variable, modify the weights

Local Search: Weight Transfer Pseudo-code

Generic structure of local search SAT solvers

```
1:  for i in 1 to MAX_TRIES do
2:      α := random initial assignment
3:      for j in 1 to MAX_STEPS do
4:          if α satisfies F then
5:              return satisfiable
6:          else if there exists a weight-reducing variable then
7:              flip the most weight-reducing variable in α
8:          else
9:              increase the weight of clauses falsified by α
10:         end if
11:     end for
12: end for
13: return unknown
```

Random k-SAT

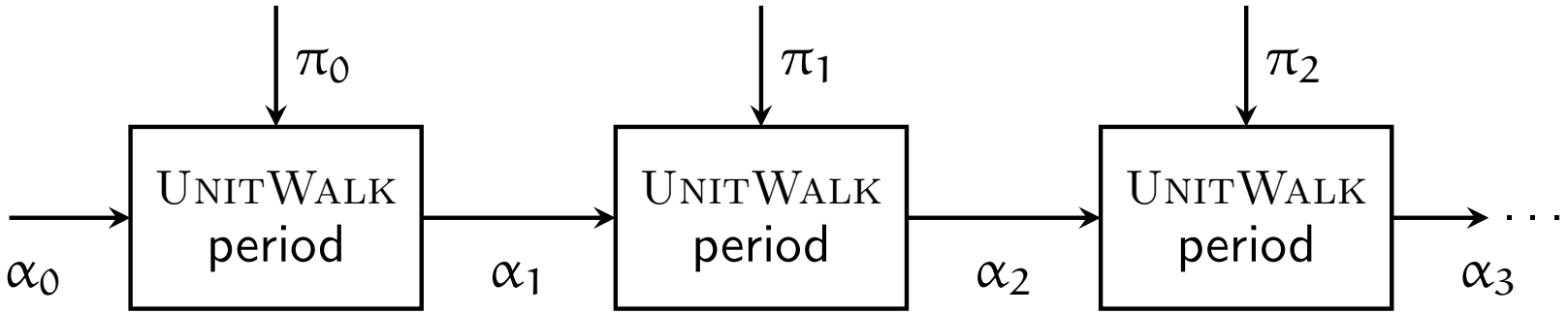
Stochastic Local Search

WalkSAT and ProbSAT

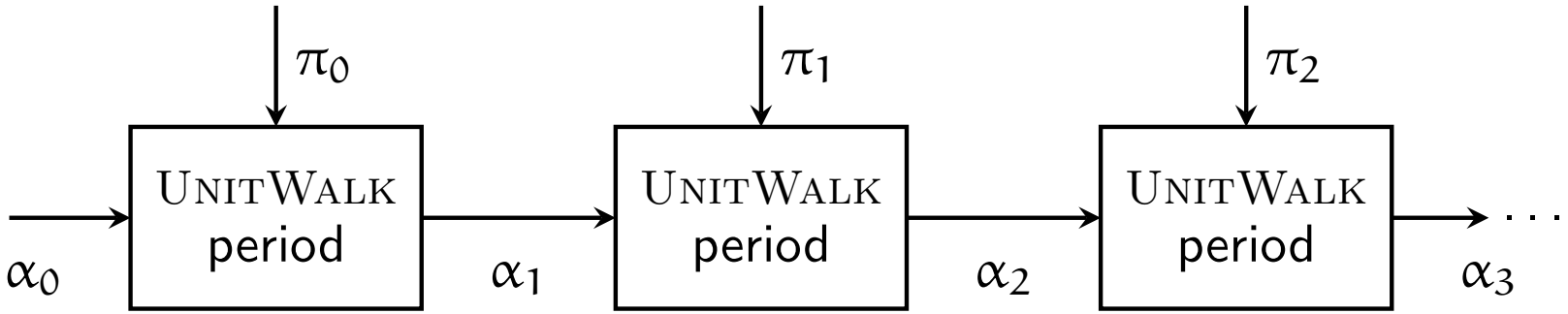
Weight Transfer

UnitWalk

The UnitWalk Algorithm



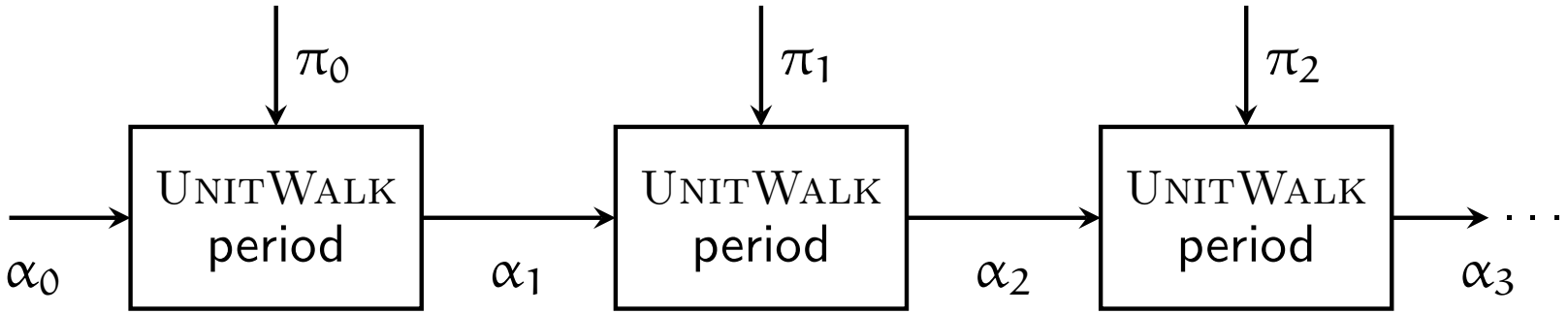
The UnitWalk Algorithm



The general idea of a UnitWalk period:

- Within each unsatisfied clause in $\alpha_i \circ \mathcal{F}$ the assignment to the least important variable (based on π_i) is flipped

The UnitWalk Algorithm



The general idea of a UnitWalk period:

- Within each unsatisfied clause in $\alpha_i \circ \mathcal{F}$ the assignment to the least important variable (based on π_i) is flipped

For example:

- $\mathcal{F} = (x \vee \overline{y}),\ \alpha_0 = \{x = 0, y = 1\},\ \pi_0 = \{y, x\}$

UnitWalk Period Example

$$
\begin{aligned}
\mathcal{F}_{\text{example}} &:= (x_1 \vee x_2) \wedge (\overline{x}_1 \vee x_2 \vee x_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge \\
&\quad (\overline{x}_2 \vee x_3 \vee \overline{x}_4) \wedge (\overline{x}_2 \vee x_3 \vee x_4) \wedge (\overline{x}_3 \vee \overline{x}_4) \\
\alpha_{\text{master}} &:= \{x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0\} \\
\alpha_{\text{active}} &:= \{x_1 = *, x_2 = *, x_3 = *, x_4 = *\} \\
\pi &:= (x_2, x_1, x_4, x_3)
\end{aligned}
$$

do

iterative propagate unit clauses in $\alpha_{\text{active}} \circ \mathcal{F}_{\text{example}}$

extend α_{active} with most important free variable according to π

while α_{active} contains $*$'s

UnitWalk Period Example

$$
\begin{aligned}
\mathcal{F}_{\text{example}} &:= (x_1 \vee x_2) \wedge (\overline{x}_1 \vee x_2 \vee x_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge \\
&\quad (\overline{x}_2 \vee x_3 \vee \overline{x}_4) \wedge (\overline{x}_2 \vee x_3 \vee x_4) \wedge (\overline{x}_3 \vee \overline{x}_4) \\
\alpha_{\text{master}} &:= \{x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0\} \\
\alpha_{\text{active}} &:= \{x_1 = *, x_2 = *, x_3 = *, x_4 = *\} \\
\pi &:= (x_2, x_1, x_4, x_3)
\end{aligned}
$$

do
$\rightarrow$ iterative propagate unit clauses in $\alpha_{\text{active}} \circ \mathcal{F}_{\text{example}}$
extend α_{active} with most important free variable according to π
while α_{active} contains $*$'s

Action:

- no unit clauses in $\alpha_{\text{active}} \circ \mathcal{F}_{\text{example}}$

UnitWalk Period Example

$$
\begin{aligned}
\mathcal{F}_{\text{example}} &:= (x_1 \vee x_2) \wedge (\overline{x}_1 \vee x_2 \vee x_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge \\
&\quad (\overline{x}_2 \vee x_3 \vee \overline{x}_4) \wedge (\overline{x}_2 \vee x_3 \vee x_4) \wedge (\overline{x}_3 \vee \overline{x}_4) \\
\alpha_{\text{master}} &:= \{x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0\} \\
\alpha_{\text{active}} &:= \{x_1 = *, x_2 = 1, x_3 = *, x_4 = *\} \\
\pi &:= (x_2, x_1, x_4, x_3)
\end{aligned}
$$

do
iterative propagate unit clauses in $\alpha_{\text{active}} \circ \mathcal{F}_{\text{example}}$
$\rightarrow$ extend α_{active} with most important free variable according to π
while α_{active} contains $*$'s

Action:

- extend α_{active} with $x_2 := 1$ (the truth value in α_{master})

UnitWalk Period Example

$$
\begin{aligned}
\mathcal{F}_{\text{example}} &:= (x_1 \vee x_2) \wedge (\overline{x}_1 \vee x_2 \vee x_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge \\
&\quad (\overline{x}_2 \vee x_3 \vee \overline{x}_4) \wedge (\overline{x}_2 \vee x_3 \vee x_4) \wedge (\overline{x}_3 \vee \overline{x}_4) \\
\alpha_{\text{master}} &:= \{x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0\} \\
\alpha_{\text{active}} &:= \{x_1 = *, x_2 = 1, x_3 = 0, x_4 = *\} \\
\pi &:= (x_2, x_1, x_4, x_3)
\end{aligned}
$$

do
$\rightarrow$ iterative propagate unit clauses in $\alpha_{\text{active}} \circ \mathcal{F}_{\text{example}}$
extend α_{active} with most important free variable according to π
while α_{active} contains $*$'s

Action:

- detected unit clause $\overline{x}_3 \rightarrow x_3 := 0$

UnitWalk Period Example

$$
\begin{aligned}
\mathcal{F}_{\text{example}} &:= (x_1 \vee x_2) \wedge (\overline{x}_1 \vee x_2 \vee x_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge \\
&\quad (\overline{x}_2 \vee x_3 \vee \overline{x}_4) \wedge (\overline{x}_2 \vee x_3 \vee x_4) \wedge (\overline{x}_3 \vee \overline{x}_4) \\
\alpha_{\text{master}} &:= \{x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0\} \\
\alpha_{\text{active}} &:= \{x_1 = *, x_2 = 1, x_3 = 0, x_4 = 0\} \\
\pi &:= (x_2, x_1, x_4, x_3)
\end{aligned}
$$

do
$\rightarrow$ iterative propagate unit clauses in $\alpha_{\text{active}} \circ \mathcal{F}_{\text{example}}$
extend α_{active} with most important free variable according to π
while α_{active} contains $*$'s

Action:

- detected unit clauses x_4 and $\overline{x}_4 \rightarrow$ conflict
- assign x_4 to truth value in $\alpha_{\text{master}} \rightarrow x_4 := 0$

UnitWalk Period Example

$$
\begin{aligned}
\mathcal{F}_{\text{example}} &:= (x_1 \vee x_2) \wedge (\overline{x}_1 \vee x_2 \vee x_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge \\
&\quad (\overline{x}_2 \vee x_3 \vee \overline{x}_4) \wedge (\overline{x}_2 \vee x_3 \vee x_4) \wedge (\overline{x}_3 \vee \overline{x}_4) \\
\alpha_{\text{master}} &:= \{x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0\} \\
\alpha_{\text{active}} &:= \{x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 0\} \\
\pi &:= (x_2, x_1, x_4, x_3)
\end{aligned}
$$

do
iterative propagate unit clauses in $\alpha_{\text{active}} \circ \mathcal{F}_{\text{example}}$
$\rightarrow$ extend α_{active} with most important free variable according to π
while α_{active} contains $*$'s

Action:

- extend α_{active} with $x_1 := 0$ (the truth value in α_{master})

UnitWalk Period Example

$$
\begin{aligned}
\mathcal{F}_{\text{example}} &:= (x_1 \vee x_2) \wedge (\overline{x}_1 \vee x_2 \vee x_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge \\
&\quad (\overline{x}_2 \vee x_3 \vee \overline{x}_4) \wedge (\overline{x}_2 \vee x_3 \vee x_4) \wedge (\overline{x}_3 \vee \overline{x}_4) \\
\alpha_{\text{master}} &:= \{x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0\} \\
\alpha_{\text{active}} &:= \{x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 0\} \\
\pi &:= (x_2, x_1, x_4, x_3)
\end{aligned}
$$

do

iterative propagate unit clauses in $\alpha_{\text{active}} \circ \mathcal{F}_{\text{example}}$

extend α_{active} with most important free variable according to π

$\longrightarrow$ **while** α_{active} contains $*$'s

Action:

- end of period because all variables are assigned in α_{active}