Automated Reasoning and Satisfiability
Assignment 1

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The homework is due at 6pm on Monday, September 27, 2021. Please email your answers to `marijn@cmu.edu` with subject "15-816 Homework Assignment 1". The questions below are mostly encoding questions. No external tools are not allowed to help answering question 1.

We prefer answers that consist of a generator that produces the requested DIMCAS file in a common programming language, such as Python or C(++). Encoding tools, such as PySAT, are allowed for questions 2 and 3. Alternatively, you can submit the encoding answers as a LaTeXdocument. However, questions 1(d), 2 (b), 2(c), 3(b), and 3(c) can only be solved using a generated DIMACS file.

The maximum number of regular points for this assignment is 50: the 30 points of question 1 + either 20 points of question 2 or 20 points of question 3 (a) and (b). Additionally, 10 bonus points can be earned in question 3 (c).

Question 1 (no encoding tools allowed)

(a) [10 points] Given the Boolean variables $x_1, \ldots, x_5$, construct two different encodings in conjunctive normal form (CNF) that express that at most two of them can be true: $x_1 + \ldots + x_5 \leq 2$. The first encoding can only use the variables $x_1, \ldots, x_5$, while the second encoding must also use auxiliary variables.

(b) [10 points] Let us refer to the above encodings as AtMostTwoA (w/o auxiliary variables) and AtMostTwoB (with auxiliary variables). Encode into CNF $y_1 \leftrightarrow \text{AtMostTwoA}(x_1, \ldots, x_5)$ and $y_2 \leftrightarrow \text{AtMostTwoB}(x_1, \ldots, x_5)$ using the Tseitin transformation.

(c) [5 points] Encode whether there exists an assignment to $x_1, \ldots, x_5$ that falsifies y_1 and satisfies y_2 by combining $y_1 \leftrightarrow \text{AtMostTwoA}(x_1, \ldots, x_5)$ and $y_2 \leftrightarrow \text{AtMostTwoB}(x_1, \ldots, x_5)$.

(d) [5 points] Solve the resulting formula using a SAT solver and show the output of the solver. (Hint: the formula should be unsatisfiable, so no local search solver can be used.)

Question 2 (answer this question or question 3)

(a) [10 points] Consider a $n \times m$ grid of squares and all possible rectangles within the grid whose length and width are at least 2. Encode whether there exists a coloring of the grid using three colors so that no such rectangle has the same color for its four corners. (Hint: The encoding requires two types of constraints. First, each square needs to have at least one color. Second, if four squares form the corners of a rectangle, then they cannot have the same color.)

```
0 0 1 1 2 2 0 1 2
2 0 0 1 1 2 2 0 1
1 2 0 0 1 1 2 2 0
0 1 2 0 0 1 1 2 2
2 0 1 2 0 0 1 1 2
2 2 0 1 2 0 0 1 1
1 2 2 0 1 2 0 0 1
1 1 2 2 0 1 2 0 0
0 1 1 2 2 0 1 2 0
```

(b) [5 points] Solve the encoding for a 10×10 grid using a SAT solver and decode the solution into a valid coloring. Show the output of the SAT solver and a valid 3-coloring similar to the one above of the 9×9 grid.

(c) [5 points] Solve the encoding for a 9×12 grid using a SAT solver and decode the solution into a valid coloring. Show the output of the SAT solver and a valid 3-coloring similar to the one above of the 9×9 grid.

Question 3 (answer this question or question 2)

An *almost square* is a $n \times (n+1)$ rectangle. One can cover the almost square 4×5 using the smallest three almost squares: 1×2, 2×3, and 3×4. A solution is shown below.

```
1 1 3 3 3
2 2 3 3 3
2 2 3 3 3
2 2 3 3 3
```

(a) [10 points] Encode whether the smallest k almost squares can cover an almost square. A satisfying assignment of the encoding should represent a covering. In case the smallest k almost squares don't add up to an almost square, the encoding should simply print a formula with only the empty clause.

(b) [10 points] Solve the encoding for the smallest 8 almost squares, which can cover the almost square 15×16, and decode the solution into a valid cover. Show the output of the SAT solver and valid cover similar to the one above of the 4×5 grid.

(c) [Bonus: 10 points] Construct a compact encoding for the smallest 20 almost squares, which can cover the almost square 55×56. Auxiliary variables are useful to reduce the size of the encoded formula. Bonus points are awarded for reasonably small encodings: 2 points for less than 3 million clauses; 4 points for less than 2 million clauses; 6 points for less than a million clauses; and 8 points for less than half a million clauses. All 10 points are awarded for any encoding for which you can show that a SAT solver can find a satisfying assignment. Warning: this problem is challenging.