

# Logic and Mechanized Reasoning

## Normal Forms

Marijn J.H. Heule

Carnegie  
Mellon  
University

# Motivation

Why a normal form?

Advantages:

- ▶ Easier to reason about
- ▶ Some techniques only work on normal forms
- ▶ Uniform input format for tools
- ▶ Canonical representations

Disadvantages:

- ▶ Some structure may be lost
- ▶ Harder to construct

## Complete Sets of Connectives

Negation Normal Form

Disjunctive Normal Form

Conjunctive Normal Form

# Complete Sets of Connectives

Negation Normal Form

Disjunctive Normal Form

Conjunctive Normal Form

## Complete Sets: OR and NOT

The chosen set of connectives has redundancies. The connectives can be replaced by other connectives:

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

## Complete Sets: OR and NOT

The chosen set of connectives has redundancies. The connectives can be replaced by other connectives:

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \rightarrow B \equiv \neg A \vee B$$

## Complete Sets: OR and NOT

The chosen set of connectives has redundancies. The connectives can be replaced by other connectives:

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \rightarrow B \equiv \neg A \vee B$$

$$A \wedge B \equiv \neg(\neg A \vee \neg B)$$

## Complete Sets: OR and NOT

The chosen set of connectives has redundancies. The connectives can be replaced by other connectives:

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \rightarrow B \equiv \neg A \vee B$$

$$A \wedge B \equiv \neg(\neg A \vee \neg B)$$

$$\perp \equiv \neg \top$$

## Complete Sets: OR and NOT

The chosen set of connectives has redundancies. The connectives can be replaced by other connectives:

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \rightarrow B \equiv \neg A \vee B$$

$$A \wedge B \equiv \neg(\neg A \vee \neg B)$$

$$\perp \equiv \neg \top$$

$$\top \equiv p \vee \neg p$$

## Complete Sets: OR and NOT

The chosen set of connectives has redundancies. The connectives can be replaced by other connectives:

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \rightarrow B \equiv \neg A \vee B$$

$$A \wedge B \equiv \neg(\neg A \vee \neg B)$$

$$\perp \equiv \neg \top$$

$$\top \equiv p \vee \neg p$$

A set of connectives is **complete** if it can express all Boolean functions

## Complete Sets: AND and NOT

Now let's do the same for AND and NOT:

$$A \leftrightarrow B \equiv$$

## Complete Sets: AND and NOT

Now let's do the same for AND and NOT:

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \rightarrow B \equiv$$

## Complete Sets: AND and NOT

Now let's do the same for AND and NOT:

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \rightarrow B \equiv \neg A \vee B$$

$$A \vee B \equiv$$

## Complete Sets: AND and NOT

Now let's do the same for AND and NOT:

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \rightarrow B \equiv \neg A \vee B$$

$$A \vee B \equiv \neg(\neg A \wedge \neg B)$$

$$\top \equiv$$

## Complete Sets: AND and NOT

Now let's do the same for AND and NOT:

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \rightarrow B \equiv \neg A \vee B$$

$$A \vee B \equiv \neg(\neg A \wedge \neg B)$$

$$\top \equiv \neg \perp$$

$$\perp \equiv$$

## Complete Sets: AND and NOT

Now let's do the same for AND and NOT:

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \rightarrow B \equiv \neg A \vee B$$

$$A \vee B \equiv \neg(\neg A \wedge \neg B)$$

$$\top \equiv \neg \perp$$

$$\perp \equiv p \wedge \neg p$$

## Complete Sets: What about BIIMP and NOT?

What about the set  $\{\leftrightarrow, \neg\}$  ?

$\top \equiv$

## Complete Sets: What about BIIMP and NOT?

What about the set  $\{\leftrightarrow, \neg\}$  ?

$$\top \equiv p \leftrightarrow p$$

$$\perp \equiv$$

## Complete Sets: What about BIIMP and NOT?

What about the set  $\{\leftrightarrow, \neg\}$  ?

$$\top \equiv p \leftrightarrow p$$

$$\perp \equiv p \leftrightarrow \neg p$$

## Complete Sets: What about BIIMP and NOT?

What about the set  $\{\leftrightarrow, \neg\}$  ?

$$\top \equiv p \leftrightarrow p$$

$$\perp \equiv p \leftrightarrow \neg p$$

It is impossible to express  $\vee$ :

$A$	$B$	$\llbracket A \vee B \rrbracket_\tau$	$\llbracket A \leftrightarrow B \rrbracket_\tau$	$\llbracket \neg A \leftrightarrow B \rrbracket_\tau$	$\llbracket A \leftrightarrow \neg B \rrbracket_\tau$	$\llbracket \neg A \leftrightarrow \neg B \rrbracket_\tau$
$\top$	$\top$	$\top$	$\top$	$\perp$	$\perp$	$\top$
$\top$	$\perp$	$\top$	$\perp$	$\top$	$\top$	$\perp$
$\perp$	$\top$	$\top$	$\perp$	$\top$	$\top$	$\perp$
$\perp$	$\perp$	$\perp$	$\top$	$\perp$	$\perp$	$\top$

Complete Sets of Connectives

Negation Normal Form

Disjunctive Normal Form

Conjunctive Normal Form

## Negation Normal Form: Introduction

The set of propositional formulas in **negation normal form** (NNF) is generated inductively as follows:

- ▶ Each variable  $p_i$  is in negation normal form.
- ▶ The negation  $\neg p_i$  of a propositional variable is in negation normal form.
- ▶  $\top$  and  $\perp$  are in negation normal form.
- ▶ If  $A$  and  $B$  are in negation normal form, then so are  $A \wedge B$  and  $A \vee B$ .

## Negation Normal Form: Introduction

The set of propositional formulas in **negation normal form** (NNF) is generated inductively as follows:

- ▶ Each variable  $p_i$  is in negation normal form.
- ▶ The negation  $\neg p_i$  of a propositional variable is in negation normal form.
- ▶  $\top$  and  $\perp$  are in negation normal form.
- ▶ If  $A$  and  $B$  are in negation normal form, then so are  $A \wedge B$  and  $A \vee B$ .

### Example (Which formulas are in NNF?)

- ▶  $p \vee (q \wedge \neg p)$
- ▶  $p \rightarrow q$
- ▶  $\neg A \wedge (B \vee A)$

## Negation Normal Form: Recall Harder Example

Recall: For any propositional variables  $p$ ,  $q$ , and  $r$ , we have

$$\neg((p \vee q) \wedge (q \rightarrow r)) \equiv (\neg p \vee q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg r).$$

**Proof.**

$$\begin{aligned}\neg((p \vee q) \wedge (q \rightarrow r)) &\equiv \neg((p \vee q) \wedge (\neg q \vee r)) \\ &\equiv \neg(p \vee q) \vee \neg(\neg q \vee r) \\ &\equiv (\neg p \wedge \neg q) \vee (q \wedge \neg r) \\ &\equiv (\neg p \vee (q \wedge \neg r)) \wedge (\neg q \vee (q \wedge \neg r)) \\ &\equiv (\neg p \vee (q \wedge \neg r)) \wedge (\neg q \vee q) \wedge (\neg q \vee \neg r) \\ &\equiv (\neg p \vee (q \wedge \neg r)) \wedge \top \wedge (\neg q \vee \neg r) \\ &\equiv (\neg p \vee (q \wedge \neg r)) \wedge (\neg q \vee \neg r) \\ &\equiv (\neg p \vee q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg r).\end{aligned}$$

**Which formulas are in NNF?**



## Negation Normal Form: Lemma

### Lemma

*Every propositional formula is equivalent to one in negation normal form.*

### Proof.

First use the identities  $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$  and  $A \rightarrow B \equiv \neg A \vee B$  to get rid of  $\leftrightarrow$  and  $\rightarrow$ . Then use De Morgan's laws together with  $\neg\neg A \equiv A$ ,  $\neg T \equiv \perp$ , and  $\neg\neg T \equiv \perp$  to push negations down to the atomic formulas. □

## Negation Normal Form: Lemma

### Lemma

*Every propositional formula is equivalent to one in negation normal form.*

### Proof.

First use the identities  $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$  and  $A \rightarrow B \equiv \neg A \vee B$  to get rid of  $\leftrightarrow$  and  $\rightarrow$ . Then use De Morgan's laws together with  $\neg\neg A \equiv A$ ,  $\neg T \equiv \perp$ , and  $\neg\neg T \equiv \perp$  to push negations down to the atomic formulas. □

**What is the complexity?**

## Complete Sets of Connectives

Negation Normal Form

Disjunctive Normal Form

Conjunctive Normal Form

## Disjunctive Normal Form: Introduction

A **literal** is a propositional variable  $p$  or its negation  $\neg p$ .

A propositional formula is in Disjunctive Normal Form (DNF) if it is written as a disjunction of conjunctions of literals.

$$\bigvee_{i < n} \left( \bigwedge_{j < m_i} (\neg) p_{i,j} \right)$$

## Disjunctive Normal Form: Introduction

A **literal** is a propositional variable  $p$  or its negation  $\neg p$ .

A propositional formula is in Disjunctive Normal Form (DNF) if it is written as a disjunction of conjunctions of literals.

$$\bigvee_{i < n} \left( \bigwedge_{j < m_i} (\neg) p_{i,j} \right)$$

A conjunction of literals is called a **cube**.  $\top$  is the empty cube.

## Disjunctive Normal Form: Introduction

A **literal** is a propositional variable  $p$  or its negation  $\neg p$ .

A propositional formula is in Disjunctive Normal Form (DNF) if it is written as a disjunction of conjunctions of literals.

$$\bigvee_{i < n} \left( \bigwedge_{j < m_i} (\neg) p_{i,j} \right)$$

A conjunction of literals is called a **cube**.  $\top$  is the empty cube.

Example (Which formulas are in DNF?)

- ▶  $p \vee q$
- ▶  $p \wedge q$
- ▶  $(p \wedge q) \vee \neg(p \wedge q)$

## Disjunctive Normal Form: Lemma

### Lemma

*The conjunction of two DNF formulas is equivalent to a DNF formula.*

## Disjunctive Normal Form: Lemma

### Lemma

*The conjunction of two DNF formulas is equivalent to a DNF formula.*

### Proof.

True. Recall that  $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ .

## Disjunctive Normal Form: Lemma

### Lemma

*The conjunction of two DNF formulas is equivalent to a DNF formula.*

### Proof.

True. Recall that  $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ .

By induction on  $n$ , we have that for every sequence of formulas  $B_0, \dots, B_{n-1}$  we have  $A \wedge \bigvee_{i < n} B_i \equiv \bigvee_{i < n} (A \wedge B_i)$ .

## Disjunctive Normal Form: Lemma

### Lemma

*The conjunction of two DNF formulas is equivalent to a DNF formula.*

### Proof.

**True.** Recall that  $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ .

By induction on  $n$ , we have that for every sequence of formulas  $B_0, \dots, B_{n-1}$  we have  $A \wedge \bigvee_{i < n} B_i \equiv \bigvee_{i < n} (A \wedge B_i)$ .

Then by induction on  $n'$  we have

$$\bigvee_{i' < n'} A_{i'} \wedge \bigvee_{i < n} B_i \equiv \bigvee_{i' < n'} \bigvee_{i < n} (A_{i'} \wedge B_i).$$

Since each  $A_{i'}$  and each  $B_i$  is a conjunction of literals, this yields the result. □

# Disjunctive Normal Form: Proposition 1

## Proposition

*Every propositional formula is equivalent to one in disjunctive normal form.*

True or false?

# Disjunctive Normal Form: Proposition 1

## Proposition

*Every propositional formula is equivalent to one in disjunctive normal form.*

True or false?

Proof.

True. Since we already know that every formula is equivalent to one in negation normal form, we can use induction on that set of formulas. The claim is clearly true of  $\top$ ,  $\perp$ ,  $p_i$ , and  $\neg p_i$ . By the previous lemma, whenever it is true of  $A$  and  $B$ , it is also true of  $A \wedge B$ . □

## Disjunctive Normal Form: Proposition 2

### Proposition

*For every DNF formula  $A$  one can determine satisfiability and unsatisfiability in linear time.*

True or false?

## Disjunctive Normal Form: Proposition 2

### Proposition

*For every DNF formula  $A$  one can determine satisfiability and unsatisfiability in linear time.*

True or false?

Proof.

True. A cube with a pair of complementary literals  $p_i$  and  $\neg p_i$  is equal to  $\perp$ . Computing whether a cube is equal to  $\perp$  can be done in linear time. A formula is satisfiable if  $A$  contains at least one cube that is not equal to  $\perp$  and unsatisfiable otherwise.



## Disjunctive Normal Form: Diplomacy Problem

“You are chief of protocol for the embassy ball. The crown prince instructs you either to invite *Peru* or to exclude *Qatar*. The queen asks you to invite either *Qatar* or *Romania* or both. The king, in a spiteful mood, wants to snub either *Romania* or *Peru* or both. Is there a guest list that will satisfy the whims of the entire royal family?”

$$(p \vee \neg q) \wedge (q \vee r) \wedge (\neg r \vee \neg p)$$

**How to convert this into DNF?**

## Disjunctive Normal Form: Truth Table to DNF

$$\Gamma = (p \vee \neg q) \wedge (q \vee r) \wedge (\neg r \vee \neg p)$$

$\tau(p)$	$\tau(q)$	$\tau(r)$	falsifies	$\llbracket \Gamma \rrbracket_\tau$
$\perp$	$\perp$	$\perp$	$(q \vee r)$	$\perp$
$\perp$	$\perp$	$\top$	—	$\top$
$\perp$	$\top$	$\perp$	$(p \vee \neg q)$	$\perp$
$\perp$	$\top$	$\top$	$(p \vee \neg q)$	$\perp$
$\top$	$\perp$	$\perp$	$(q \vee r)$	$\perp$
$\top$	$\perp$	$\top$	$(\neg r \vee \neg p)$	$\perp$
$\top$	$\top$	$\perp$	—	$\top$
$\top$	$\top$	$\top$	$(\neg r \vee \neg p)$	$\perp$

## Disjunctive Normal Form: Truth Table to DNF

$$\Gamma = (p \vee \neg q) \wedge (q \vee r) \wedge (\neg r \vee \neg p)$$

$\tau(p)$	$\tau(q)$	$\tau(r)$	falsifies	$\llbracket \Gamma \rrbracket_\tau$
$\perp$	$\perp$	$\perp$	$(q \vee r)$	$\perp$
$\perp$	$\perp$	$\top$	—	$\top$
$\perp$	$\top$	$\perp$	$(p \vee \neg q)$	$\perp$
$\perp$	$\top$	$\top$	$(p \vee \neg q)$	$\perp$
$\top$	$\perp$	$\perp$	$(q \vee r)$	$\perp$
$\top$	$\perp$	$\top$	$(\neg r \vee \neg p)$	$\perp$
$\top$	$\top$	$\perp$	—	$\top$
$\top$	$\top$	$\top$	$(\neg r \vee \neg p)$	$\perp$

The DNF consists of all assignments that satisfy the formula:

$$(\neg p \wedge \neg q \wedge r) \vee (p \wedge q \wedge \neg r)$$

## Disjunctive Normal Form: Applying Distributive Laws

An alternative approach is applying the distributive laws

$$(p \vee \neg q) \wedge (q \vee r) \wedge (\neg r \vee \neg p) \equiv$$

## Disjunctive Normal Form: Applying Distributive Laws

An alternative approach is applying the distributive laws

$$\begin{aligned}(p \vee \neg q) \wedge (q \vee r) \wedge (\neg r \vee \neg p) &\equiv \\ ((p \wedge (q \vee r)) \vee (\neg q \wedge (q \vee r))) \wedge (\neg r \vee \neg p) &\equiv\end{aligned}$$

## Disjunctive Normal Form: Applying Distributive Laws

An alternative approach is applying the distributive laws

$$\begin{aligned}(p \vee \neg q) \wedge (q \vee r) \wedge (\neg r \vee \neg p) &\equiv \\ ((p \wedge (q \vee r)) \vee (\neg q \wedge (q \vee r))) \wedge (\neg r \vee \neg p) &\equiv \\ ((p \wedge q) \vee (p \wedge r) \vee (\neg q \wedge q) \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv\end{aligned}$$

## Disjunctive Normal Form: Applying Distributive Laws

An alternative approach is applying the distributive laws

$$\begin{aligned}(p \vee \neg q) \wedge (q \vee r) \wedge (\neg r \vee \neg p) &\equiv \\ ((p \wedge (q \vee r)) \vee (\neg q \wedge (q \vee r))) \wedge (\neg r \vee \neg p) &\equiv \\ ((p \wedge q) \vee (p \wedge r) \vee (\neg q \wedge q) \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\ ((p \wedge q) \vee (p \wedge r) \vee \perp \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv\end{aligned}$$

## Disjunctive Normal Form: Applying Distributive Laws

An alternative approach is applying the distributive laws

$$\begin{aligned}(p \vee \neg q) \wedge (q \vee r) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge (q \vee r)) \vee (\neg q \wedge (q \vee r))) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee (\neg q \wedge q) \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee \perp \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv\end{aligned}$$

## Disjunctive Normal Form: Applying Distributive Laws

An alternative approach is applying the distributive laws

$$\begin{aligned}(p \vee \neg q) \wedge (q \vee r) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge (q \vee r)) \vee (\neg q \wedge (q \vee r))) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee (\neg q \wedge q) \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee \perp \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\(\neg r \wedge p \wedge q) \vee (\neg r \wedge p \wedge r) \vee (\neg r \wedge \neg q \wedge r) \vee\end{aligned}$$

## Disjunctive Normal Form: Applying Distributive Laws

An alternative approach is applying the distributive laws

$$\begin{aligned}(p \vee \neg q) \wedge (q \vee r) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge (q \vee r)) \vee (\neg q \wedge (q \vee r))) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee (\neg q \wedge q) \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee \perp \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\(\neg r \wedge p \wedge q) \vee (\neg r \wedge p \wedge r) \vee (\neg r \wedge \neg q \wedge r) \vee \\(\neg p \wedge p \wedge q) \vee (\neg p \wedge p \wedge r) \vee (\neg p \wedge \neg q \wedge r) &\equiv\end{aligned}$$

## Disjunctive Normal Form: Applying Distributive Laws

An alternative approach is applying the distributive laws

$$\begin{aligned}(p \vee \neg q) \wedge (q \vee r) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge (q \vee r)) \vee (\neg q \wedge (q \vee r))) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee (\neg q \wedge q) \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee \perp \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\(\neg r \wedge p \wedge q) \vee (\neg r \wedge p \wedge r) \vee (\neg r \wedge \neg q \wedge r) \vee \\(\neg p \wedge p \wedge q) \vee (\neg p \wedge p \wedge r) \vee (\neg p \wedge \neg q \wedge r) &\equiv \\(\neg r \wedge p \wedge q) \vee \perp \vee \perp \vee \perp \vee \perp \vee (\neg p \wedge \neg q \wedge r) &\equiv\end{aligned}$$

## Disjunctive Normal Form: Applying Distributive Laws

An alternative approach is applying the distributive laws

$$\begin{aligned}(p \vee \neg q) \wedge (q \vee r) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge (q \vee r)) \vee (\neg q \wedge (q \vee r))) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee (\neg q \wedge q) \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee \perp \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\((p \wedge q) \vee (p \wedge r) \vee (\neg q \wedge r)) \wedge (\neg r \vee \neg p) &\equiv \\(\neg r \wedge p \wedge q) \vee (\neg r \wedge p \wedge r) \vee (\neg r \wedge \neg q \wedge r) \vee \\(\neg p \wedge p \wedge q) \vee (\neg p \wedge p \wedge r) \vee (\neg p \wedge \neg q \wedge r) &\equiv \\(\neg r \wedge p \wedge q) \vee \perp \vee \perp \vee \perp \vee \perp \vee (\neg p \wedge \neg q \wedge r) &\equiv \\(\neg r \wedge p \wedge q) \vee (\neg p \wedge \neg q \wedge r).\end{aligned}$$

## Disjunctive Normal Form: Complexity

What is the worst case cost of applying the distributive laws?

## Disjunctive Normal Form: Complexity

What is the worst case cost of applying the distributive laws?

In some cases, converting a formula to DNF can have an **exponential** explosion on the size of the formula.

If we convert  $(p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge \dots \wedge (p_n \vee q_n)$  using the distributive laws to DNF:

$$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \vee (q_1 \wedge p_2 \wedge \dots \wedge p_n) \vee \dots \vee (q_1 \wedge q_2 \wedge \dots \wedge q_n)$$

## Complete Sets of Connectives

Negation Normal Form

Disjunctive Normal Form

Conjunctive Normal Form

## Conjunctive Normal Form: Introduction

A **literal** is a propositional variable  $p$  or its negation  $\neg p$ .

A propositional formula is in Conjunctive Normal Form (CNF) if it is written as a conjunction of disjunctions of literals.

$$\bigwedge_{i < n} \left( \bigvee_{j < m_i} (\neg) p_{i,j} \right)$$

## Conjunctive Normal Form: Introduction

A **literal** is a propositional variable  $p$  or its negation  $\neg p$ .

A propositional formula is in Conjunctive Normal Form (CNF) if it is written as a conjunction of disjunctions of literals.

$$\bigwedge_{i < n} \left( \bigvee_{j < m_i} (\neg) p_{i,j} \right)$$

A **clause** is a disjunction of literals.  $\perp$  denotes the empty clause.

## Conjunctive Normal Form: Introduction

A **literal** is a propositional variable  $p$  or its negation  $\neg p$ .

A propositional formula is in Conjunctive Normal Form (CNF) if it is written as a conjunction of disjunctions of literals.

$$\bigwedge_{i < n} \left( \bigvee_{j < m_i} (\neg) p_{i,j} \right)$$

A **clause** is a disjunction of literals.  $\perp$  denotes the empty clause.

Example (Which formulas are in CNF?)

- ▶  $p \vee q$
- ▶  $p \wedge q$
- ▶  $(p \vee q) \wedge \neg(p \vee q)$

# Conjunctive Normal Form: Proposition

## Proposition

*For every CNF formula  $A$  one can determine whether it is valid in linear time.*

True or false?

# Conjunctive Normal Form: Proposition

## Proposition

*For every CNF formula A one can determine whether it is valid in linear time.*

True or false?

Proof.

True. A clause with a pair of complementary literals  $p_i$  and  $\neg p_i$  is equal to  $\top$ . Computing whether a clause is equal to  $\top$  can be done in linear time. A formula is valid if and only if all clauses are equal to  $\top$ .



## Conjunctive Normal Form: Input Form of Reasoning Tools

Two formulas  $\Gamma$  and  $\Delta$  are **equisatisfiable** if and only if they are both satisfiable or if they are both unsatisfiable.

### Example

The formulas  $\Gamma = p \wedge q$  and  $\Delta = \neg p \wedge \neg q$  are equisatisfiable because they are both satisfiable, even though there doesn't exist an assignment that satisfies both.

## Conjunctive Normal Form: Input Form of Reasoning Tools

Two formulas  $\Gamma$  and  $\Delta$  are **equisatisfiable** if and only if they are both satisfiable or if they are both unsatisfiable.

### Example

The formulas  $\Gamma = p \wedge q$  and  $\Delta = \neg p \wedge \neg q$  are equisatisfiable because they are both satisfiable, even though there doesn't exist an assignment that satisfies both.

Most reasoning tools for propositional logic require CNF input

- ▶ Transforming a formula to CNF can also be **exponential**...
- ▶ But, it can be avoided by focusing on **equisatisfiability**.
- ▶ The **performance** of solvers depends on the transformation.
- ▶ Typically the **smaller** the CNF, the easier to solve it.

## Conjunctive Normal Form: Input Form of Reasoning Tools

Two formulas  $\Gamma$  and  $\Delta$  are **equisatisfiable** if and only if they are both satisfiable or if they are both unsatisfiable.

### Example

The formulas  $\Gamma = p \wedge q$  and  $\Delta = \neg p \wedge \neg q$  are equisatisfiable because they are both satisfiable, even though there doesn't exist an assignment that satisfies both.

Most reasoning tools for propositional logic require CNF input

- ▶ Transforming a formula to CNF can also be **exponential**...
- ▶ But, it can be avoided by focusing on **equisatisfiability**.
- ▶ The **performance** of solvers depends on the transformation.
- ▶ Typically the **smaller** the CNF, the easier to solve it.

Let's look at transforming common constraints into CNF

## Conjunctive Normal Form: AtLeastOne

Given a set of propositions  $p_1, \dots, p_n$ , how to express

$$p_1 + \dots + p_n \geq 1$$

in CNF (using 0 for  $\perp$  and 1 for  $\top$ )?

**Hint:** This is easy...

## Conjunctive Normal Form: AtLeastOne

Given a set of propositions  $p_1, \dots, p_n$ , how to express

$$p_1 + \dots + p_n \geq 1$$

in CNF (using 0 for  $\perp$  and 1 for  $\top$ )?

**Hint:** This is easy...

$$(p_1 \vee p_2 \vee \dots \vee p_n)$$

## Conjunctive Normal Form: Parity Constraints

Given a set of Boolean variables  $p_1, \dots, p_n$ , how to express

$$p_1 \oplus \dots \oplus p_n = 1$$

in CNF (using 0 for  $\perp$  and 1 for  $\top$ )?

## Conjunctive Normal Form: Parity Constraints

Given a set of Boolean variables  $p_1, \dots, p_n$ , how to express

$$p_1 \oplus \dots \oplus p_n = 1$$

in CNF (using 0 for  $\perp$  and 1 for  $\top$ )?

$p_1 \oplus \dots \oplus p_n = 1$  is *true* if and only if an **odd number** of  $p_i$  is assigned to *true*. Consider the case with two literals:

$\tau(p_1)$	$\tau(p_2)$	$\llbracket p_1 \oplus p_2 = 1 \rrbracket_\tau$
$\perp$	$\perp$	$\perp$
$\perp$	$\top$	$\top$
$\top$	$\perp$	$\top$
$\top$	$\top$	$\perp$

## Conjunctive Normal Form: Parity Constraints

Given a set of Boolean variables  $p_1, \dots, p_n$ , how to express

$$p_1 \oplus \dots \oplus p_n = 1$$

in CNF (using 0 for  $\perp$  and 1 for  $\top$ )?

$p_1 \oplus \dots \oplus p_n = 1$  is *true* if and only if an **odd number** of  $p_i$  is assigned to *true*. Consider the case with two literals:

$\tau(p_1)$	$\tau(p_2)$	$\llbracket p_1 \oplus p_2 = 1 \rrbracket_\tau$
$\perp$	$\perp$	$\perp$
$\perp$	$\top$	$\top$
$\top$	$\perp$	$\top$
$\top$	$\top$	$\perp$

$$(p_1 \vee p_2) \wedge (\neg p_1 \vee \neg p_2)$$

## Conjunctive Normal Form: Exponential Transformation

Given a set of Boolean variables  $p_1, \dots, p_n$ , how to express

$$p_1 \oplus \dots \oplus p_n = 1$$

in CNF (using 0 for  $\perp$  and 1 for  $\top$ )?

The direct encoding requires  $2^{n-1}$  clauses of length  $n$ :

$$\bigwedge_{\text{even } \#\neg} ((\neg)p_1 \vee (\neg)p_2 \vee \dots \vee (\neg)p_n)$$

## Conjunctive Normal Form: Exponential Transformation

Given a set of Boolean variables  $p_1, \dots, p_n$ , how to express

$$p_1 \oplus \dots \oplus p_n = 1$$

in CNF (using 0 for  $\perp$  and 1 for  $\top$ )?

The direct encoding requires  $2^{n-1}$  clauses of length  $n$ :

$$\bigwedge_{\text{even } \# \neg} ((\neg)p_1 \vee (\neg)p_2 \vee \dots \vee (\neg)p_n)$$

$$p_1 \oplus p_2 \oplus p_3 = 1 \leftrightarrow (p_1 \vee p_2 \vee p_3) \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge \\ (\neg p_1 \vee p_2 \vee \neg p_3) \wedge (p_1 \vee \neg p_2 \vee \neg p_3)$$

## Conjunctive Normal Form: Exponential Transformation

Given a set of Boolean variables  $p_1, \dots, p_n$ , how to express

$$p_1 \oplus \dots \oplus p_n = 1$$

in CNF (using 0 for  $\perp$  and 1 for  $\top$ )?

The direct encoding requires  $2^{n-1}$  clauses of length  $n$ :

$$\bigwedge_{\text{even } \# \neg} ((\neg)p_1 \vee (\neg)p_2 \vee \dots \vee (\neg)p_n)$$

$$p_1 \oplus p_2 \oplus p_3 = 1 \leftrightarrow (p_1 \vee p_2 \vee p_3) \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge \\ (\neg p_1 \vee p_2 \vee \neg p_3) \wedge (p_1 \vee \neg p_2 \vee \neg p_3)$$

**Question:** How many assignments satisfy this formula?

## Conjunctive Normal Form: Exponential Transformation

Given a set of Boolean variables  $p_1, \dots, p_n$ , how to express

$$p_1 \oplus \dots \oplus p_n = 1$$

in CNF (using 0 for  $\perp$  and 1 for  $\top$ )?

The direct encoding requires  $2^{n-1}$  clauses of length  $n$ :

$$\bigwedge_{\text{even } \# \neg} ((\neg)p_1 \vee (\neg)p_2 \vee \dots \vee (\neg)p_n)$$

$$p_1 \oplus p_2 \oplus p_3 = 1 \leftrightarrow (p_1 \vee p_2 \vee p_3) \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge \\ (\neg p_1 \vee p_2 \vee \neg p_3) \wedge (p_1 \vee \neg p_2 \vee \neg p_3)$$

**Question:** How many assignments satisfy this formula? 4

## Conjunctive Normal Form: Exponential Transformation

Given a set of Boolean variables  $p_1, \dots, p_n$ , how to express

$$p_1 \oplus \dots \oplus p_n = 1$$

in CNF (using 0 for  $\perp$  and 1 for  $\top$ )?

The direct encoding requires  $2^{n-1}$  clauses of length  $n$ :

$$\bigwedge_{\text{even } \# \neg} ((\neg)p_1 \vee (\neg)p_2 \vee \dots \vee (\neg)p_n)$$

Can we encode large parity constraints with **less clauses**?

## Conjunctive Normal Form: Exponential Transformation

Given a set of Boolean variables  $p_1, \dots, p_n$ , how to express

$$p_1 \oplus \dots \oplus p_n = 1$$

in CNF (using 0 for  $\perp$  and 1 for  $\top$ )?

The direct encoding requires  $2^{n-1}$  clauses of length  $n$ :

$$\bigwedge_{\text{even } \# \neg} ((\neg)p_1 \vee (\neg)p_2 \vee \dots \vee (\neg)p_n)$$

Can we encode large parity constraints with **less clauses**?

Compact:  $(p_1 \oplus p_2 \oplus p_3 \oplus \neg q = 1) \wedge (q \oplus p_4 \oplus \dots \oplus p_n = 1)$

**Note:**  $(p_1 \oplus p_2 \oplus p_3 \oplus \neg q = 1) \equiv q \leftrightarrow (p_1 \oplus p_2 \oplus p_3 = 1)$

**Tradeoff:** increases the number of variables but decreases the number of clauses!

## Conjunctive Normal Form: AtMostOne Pairwise Encoding

Given a set of Boolean variables  $p_1, \dots, p_n$ , how to express

$$p_1 + \dots + p_n \leq 1$$

in CNF (using 0 for  $\perp$  and 1 for  $\top$ )?

## Conjunctive Normal Form: AtMostOne Pairwise Encoding

Given a set of Boolean variables  $p_1, \dots, p_n$ , how to express

$$p_1 + \dots + p_n \leq 1$$

in CNF (using 0 for  $\perp$  and 1 for  $\top$ )?

The direct encoding requires  $n(n - 1)/2$  binary clauses:

$$\bigwedge_{1 \leq i < j \leq n} (\neg p_i \vee \neg p_j)$$

## Conjunctive Normal Form: AtMostOne Pairwise Encoding

Given a set of Boolean variables  $p_1, \dots, p_n$ , how to express

$$p_1 + \dots + p_n \leq 1$$

in CNF (using 0 for  $\perp$  and 1 for  $\top$ )?

The direct encoding requires  $n(n - 1)/2$  binary clauses:

$$\bigwedge_{1 \leq i < j \leq n} (\neg p_i \vee \neg p_j)$$

Is it possible to use fewer clauses?

## Conjunctive Normal Form: AtMostOne Linear Encoding

Given a set of propositions  $p_1, \dots, p_n$ , how to express

$$p_1 + \dots + p_n \leq 1$$

in CNF using a linear number of binary clauses?

## Conjunctive Normal Form: AtMostOne Linear Encoding

Given a set of propositions  $p_1, \dots, p_n$ , how to express

$$p_1 + \dots + p_n \leq 1$$

in CNF using a linear number of binary clauses?

Split the constraint using **additional variables**: Apply the direct encoding if  $n \leq 4$  otherwise replace  $p_1 + \dots + p_n \leq 1$  by

$$(p_1 + p_2 + p_3 + \neg q \leq 1) \wedge (q + p_4 + \dots + p_n \leq 1)$$

resulting in  **$3n - 6$  clauses** and  **$(n - 3)/2$  new variables**.

**Note:**  $p_1 + p_2 + p_3 + \neg q \leq 1 \equiv$

$$p_1 + p_2 + p_3 \leq 1 \wedge (\neg p_1 \vee q) \wedge (\neg p_2 \vee q) \wedge (\neg p_3 \vee q) \equiv$$

$$p_1 + p_2 + p_3 \leq 1 \wedge (p_1 \vee p_2 \vee p_3) \rightarrow q$$

## Conjunctive Normal Form: Order Matters

Split the constraint using **additional variables**: Apply the direct encoding if  $n \leq k$  otherwise replace  $p_1 + \cdots + p_n \leq 1$  by

$$\text{Linear : } (p_1 + \cdots + p_k + \neg q \leq 1) \wedge (q + p_{k+1} + \cdots + p_n \leq 1)$$

$$\text{Pooled : } (p_1 + \cdots + p_k + \neg q \leq 1) \wedge (p_{k+1} + \cdots + p_n + q \leq 1)$$

Is there a difference?

## Conjunctive Normal Form: Order Matters

Split the constraint using **additional variables**: Apply the direct encoding if  $n \leq k$  otherwise replace  $p_1 + \cdots + p_n \leq 1$  by

*Linear* :  $(p_1 + \cdots + p_k + \neg q \leq 1) \wedge (q + p_{k+1} + \cdots + p_n \leq 1)$

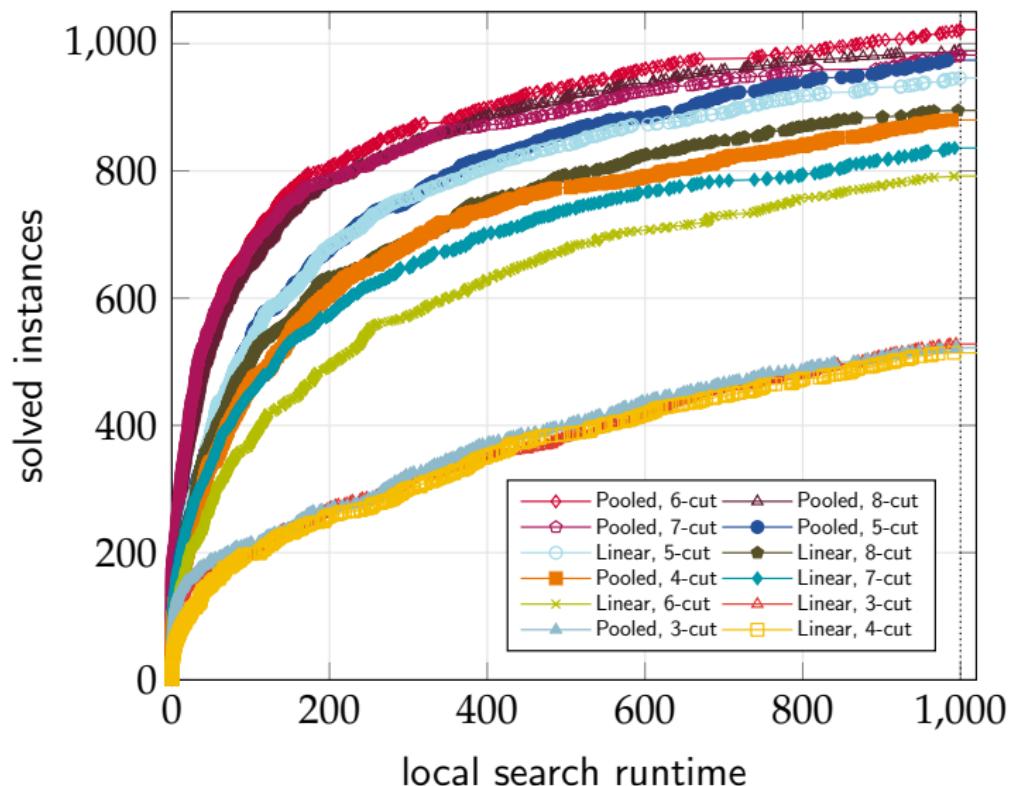
*Pooled* :  $(p_1 + \cdots + p_k + \neg q \leq 1) \wedge (p_{k+1} + \cdots + p_n + q \leq 1)$

Is there a difference?

*Linear* :  $(p_1 + p_2 + \neg q_1 \leq 1) \wedge (q_1 + p_3 + \neg q_2 \leq 1) \wedge$   
 $(q_2 + p_4 + \neg q_3 \leq 1) \wedge (q_3 + p_5 + p_6 \leq 1)$

*Pooled* :  $(p_1 + p_2 + \neg q_1 \leq 1) \wedge (p_3 + p_4 + \neg q_2 \leq 1) \wedge$   
 $(p_5 + p_6 + \neg q_3 \leq 1) \wedge (q_1 + q_2 + q_3 \leq 1)$

# Conjunctive Normal Form: Impact on Matrix Multiplication



## Conjunctive Normal Form: AtMostOne Equivalence

Are these two formulas of  $p_1 + p_2 \leq 1$  equivalent?

$A$ (direct encoding)	$B$ (split encoding)
$\neg p_1 \vee \neg p_2$	$\neg p_1 \vee q$ $\neg q \vee \neg p_2$

## Conjunctive Normal Form: AtMostOne Equivalence

Are these two formulas of  $p_1 + p_2 \leq 1$  equivalent?

$$\begin{array}{c|c} \begin{array}{c} A \text{ (direct encoding)} \\ \hline \neg p_1 \vee \neg p_2 \end{array} & \begin{array}{c} B \text{ (split encoding)} \\ \hline \neg p_1 \vee q \\ \neg q \vee \neg p_2 \end{array} \end{array}$$

**Question:** Is  $A$  equivalent to  $B$ ?

**Note:**  $A \leftrightarrow B$  if  $\neg A \wedge B$  and  $A \wedge \neg B$  are **unsatisfiable**.

## Conjunctive Normal Form: AtMostOne Equivalence

Are these two formulas of  $p_1 + p_2 \leq 1$  equivalent?

$A$ (direct encoding)	$B$ (split encoding)
$\neg p_1 \vee \neg p_2$	$\neg p_1 \vee q$ $\neg q \vee \neg p_2$

Is  $\neg A \wedge B$  unsatisfiable?

**Note:**  $\neg A \equiv p_1 \wedge p_2$

## Conjunctive Normal Form: AtMostOne Equivalence

Are these two formulas of  $p_1 + p_2 \leq 1$  equivalent?

$$\begin{array}{c|c} \begin{array}{c} A \text{ (direct encoding)} \\ \hline \neg p_1 \vee \neg p_2 \end{array} & \begin{array}{c} B \text{ (split encoding)} \\ \hline \neg p_1 \vee q \\ \neg q \vee \neg p_2 \end{array} \end{array}$$

Is  $\neg A \wedge B$  unsatisfiable? yes!

Note:  $\neg A \equiv p_1 \wedge p_2$

## Conjunctive Normal Form: AtMostOne Equivalence

Are these two formulas of  $p_1 + p_2 \leq 1$  equivalent?

$A$ (direct encoding)	$B$ (split encoding)
$\neg p_1 \vee \neg p_2$	$\neg p_1 \vee q$ $\neg q \vee \neg p_2$

Is  $A \wedge \neg B$  unsatisfiable?

**Note:**  $\neg B \equiv \neg((\neg p_1 \vee q) \wedge (\neg q \vee \neg p_2))$

$$\begin{aligned} &\equiv (\neg(\neg p_1 \vee q)) \vee (\neg(\neg q \vee \neg p_2)) \\ &\equiv (p_1 \wedge \neg q) \vee (q \wedge p_2) \\ &\equiv (p_1 \vee q) \wedge (p_1 \vee p_2) \wedge (\neg q \vee p_2) \end{aligned}$$

## Conjunctive Normal Form: AtMostOne Equivalence

Are these two formulas of  $p_1 + p_2 \leq 1$  equivalent?

$A$ (direct encoding)	$B$ (split encoding)
$\neg p_1 \vee \neg p_2$	$\neg p_1 \vee q$ $\neg q \vee \neg p_2$

Is  $A \wedge \neg B$  unsatisfiable? **no!**

**Note:**  $\neg B \equiv \neg((\neg p_1 \vee q) \wedge (\neg q \vee \neg p_2))$

$$\begin{aligned} &\equiv (\neg(\neg p_1 \vee q)) \vee (\neg(\neg q \vee \neg p_2)) \\ &\equiv (p_1 \wedge \neg q) \vee (q \wedge p_2) \\ &\equiv (p_1 \vee q) \wedge (p_1 \vee p_2) \wedge (\neg q \vee p_2) \end{aligned}$$

## Conjunctive Normal Form: AtMostOne Equivalence

Are these two formulas of  $p_1 + p_2 \leq 1$  equivalent?

$A$ (direct encoding)	$B$ (split encoding)
$\neg p_1 \vee \neg p_2$	$\neg p_1 \vee q$ $\neg q \vee \neg p_2$

$A$  and  $B$  are **equisatisfiable**:

- ▶  $A$  is satisfiable iff  $B$  is satisfiable.

**Note:** Equisatisfiability is weaker than equivalence but useful if all we want to do is determine satisfiability.