Name: _____

Logic and Mechanized Reasoning

Second Practice Exam

Spring 2026

Write your answers in the space provided, using the back of the page if necessary. You may use additional scratch paper. Justify your answers, and provide clear, readable explanations.

Problem	**Points**	**Score**
1	26	
2	40	
3	40	
4	60	
5	52	
6	40	
7	42	
Total	**300**	

Good luck!

Problem 1.

Remember that if Γ is a set of clauses, $[\![\Gamma]\!]_\tau$ is the set of clauses obtained by deleting any clause that has a literal that is assigned $\top$ and removing from each clause any literal that is assigned $\bot$.

Remember also that the DPLL search tries to find a satisfying assignment for a set of clauses Γ by doing a backtracking search on partial assignments τ. At a node τ in the search, DPLL tries to find a satisfying assignment to $[\![\Gamma]\!]_\tau$.

Part a) (6 points) What does it mean to say that a literal ℓ is a *pure literal* in $[\![\Gamma]\!]_\tau$?

Part b) (20 points) Suppose ℓ is pure in $[\![\Gamma]\!]_\tau$ and let τ' be the assignment $\tau[\ell \mapsto \top]$. Show that $[\![\Gamma]\!]_{\tau'}$ is satisfiable if and only if $[\![\Gamma]\!]_\tau$ is satisfiable.

Problem 2. Remember that in Lean we define the type `Clause` to be *List Lit*.

Part a) (20 points) Define a function `findComplement? : Clause → Clause → Option Lit` that finds a literal ℓ such that ℓ occurs in the first clause and $\neg\ell$ occurs in the second. If there isn't one, the function should return `none`. You can assume that you have a function `List.contains : α → List α → Bool` that determines whether an element is in a list.

Part b) (20 points) Define a function `resolve : Clause → Clause → Option Clause` that applies the resolution rule to two clauses, assuming there is a complementary pair, and returns `none` otherwise. You can assume that you have a function `List.erase : α → List α → List α` that deletes an element from a list, if it is present.

Problem 3. (40 points)

Remember that a resolution proof of a clause D from hypotheses in Γ is a sequence of clauses $C_0, C_1, \ldots, C_n$ containing D such that for each $i \leq n$, either C_i is in Γ or C_i is obtained from previous clauses C_j and C_k.

Let τ be any truth assignment and let τ' be $\tau[P \to \top]$. Suppose there is a resolution proof of a clause D from hypotheses in $[\![\Gamma]\!]_{\tau'}$. Show that there is a resolution proof of either D or $D \vee \neg P$ from hypotheses in $[\![\Gamma]\!]_{\tau}$.

Problem 4. Given the declaration `variable (P Q R : Prop)`, as best you can, try to write Lean proofs of the two theorems shown. For the tactic proofs, tell us what the goal looks like after each line in your proof, i.e. the hypotheses and conclusion. Don't worry too much about the syntax or writing the goal exactly like Lean does; we are more interested in seeing that you know what steps are allowed.

Part a) (18 points)

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example : P ∧ Q → Q ∨ R := by
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Part b) (12 points) Write down a proof term for the theorem in part a).

Part c) (18 points)

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example : (P → Q) → (Q → R) → (P → R) := by
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Part d) (12 points) Write down a proof term for the theorem in part c).

Problem 5

Suppose we color the natural numbers red, green, and blue, and design a language with predicates $\text{Red}(x)$, $\text{Green}(x)$, and $\text{Blue}(x)$ and a binary relation $<$ to talk about the coloring.

Part a) (16 points) Write down a first-order sentence in the language that says that every natural number has exactly one color.

Part b) (16 points) Write down a sentence that says between any two numbers that are colored red there is a number that is colored green or blue.

Part c) (20 points) Write down a sentence that says that there are three numbers in a row that are colored blue. (Hint: it takes some thought to say that x, y, and z are consecutive numbers using only $<$ and $=$.)

Problem 6

Part a) (20 points) Prove the following or provide a counterexample: for any two formulas $A(x)$ and $B(x)$ in first-order logic, if $\exists x.\, A(x) \wedge B(x)$ is satisfiable, then so are $\exists x.\, A(x)$ and $\exists x.\, B(x)$.

Part b) (20 points) Prove the following or provide a counterexample: for any two formulas $A(x)$ and $B(x)$ in first-order logic, if $\exists x.\, A(x)$ and $\exists x.\, B(x)$ are satisfiable, then so is $\exists x.\, A(x) \wedge B(x)$.

Problem 7

In each of the following cases, find a most general unifier of the pair of expressions or explain why no such unifier exists. In all these problems, $x, y, z, \ldots$ are variables and $a, b, c, \ldots$ are constants.

Part a) (14 points) $P(a, f(a, x), g(y))$ and $P(a, f(a, f(g(b), a)), x)$.

Part b) (14 points) $P(f(a), g(y))$ and $P(x, g(x))$.

Part c) (14 points) $P(f(x), g(x))$ and $P(f(f(a)), g(f(y)))$.