Logic and Mechanized Reasoning
Using SAT Solvers

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Solving 2-SAT

SAT Solving First Steps

DPLL

Graph Coloring

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Solving 2-SAT: Complexity

A k-SAT formula is a CNF formula such that each clause has a length of at most k.

Solving a k-SAT formula is NP-complete for $k \geq 3$

However, 2-SAT can be solved in polynomial time using

- Unit propagation; and
- Autarky reasoning.

Solving 2-SAT: Unit Propagation

Let Γ be a 2-SAT formula, p a propositional variable occurring in Γ, and τ the assignment with $\tau(p) = \top$.

Unit propagation on Γ using τ has two possible outcomes:

- Unit propagation results in a conflict: All satisfying assignments of Γ assign p to false.

Solving 2-SAT: Unit Propagation

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Unit propagation on Γ using τ has two possible outcomes:

- Unit propagation results in a conflict: All satisfying assignments of Γ assign p to false.
- Unit propagation terminates without a conflict: Let τ' be the final assignment with unit propagation terminated. Now τ' is an autarky for Γ. Why?

Solving 2-SAT: Autarky

Given a 2-SAT formula Γ and a non-empty truth assignment. If unit propagation terminates without a conflict, then the extended assignment is an autarky for Γ.

- For a clause C and a non-conflicting assignment τ it holds that i) τ does not touch C, ii) τ satisfies C, or iii) τ reduces C to a unit clause (by falsifying the other literal);
- Unit clauses extend the assignment and maintain the above invariant;
- At the non-conflicting fixpoint, no touched clause is reduced in length; so
- All touched clauses are satisfied.

Solving 2-SAT: Decision Procedure

Given a 2-SAT formula Γ, the following procedure solves it in polynomial time:

- Pick an arbitrary variable p and let τ be $\tau(p) = \top$
- Let τ' be the extended assignment after applying unit propagation on Γ starting with τ
- If $[\![\Gamma]\!]_{\tau'}$ does not contain $\bot$, continue with $[\![\Gamma]\!]_{\tau'}$ (autarky)
- Otherwise continue with $[\![\Gamma]\!]_{\tau''}$ with $\tau''(p) = \bot$
- Stop if either $[\![\Gamma]\!]_{\tau'} = \top$ or $[\![\Gamma]\!]_{\tau'} = \bot$

Solving 2-SAT: Decision Procedure

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- Otherwise continue with $[\![\Gamma]\!]_{\tau''}$ with $\tau''(p) = \bot$
- Stop if either $[\![\Gamma]\!]_{\tau'} = \top$ or $[\![\Gamma]\!]_{\tau'} = \bot$

Tarjan's algorithm can be used to reduce it to linear runtime.

SAT Game

by Olivier Roussel

`https://www.cs.utexas.edu/~marijn/game/2SAT/`

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SAT Solving: Introduction

Dozens of (open source) SAT solvers have been developed.

International competition have been organized since 2002

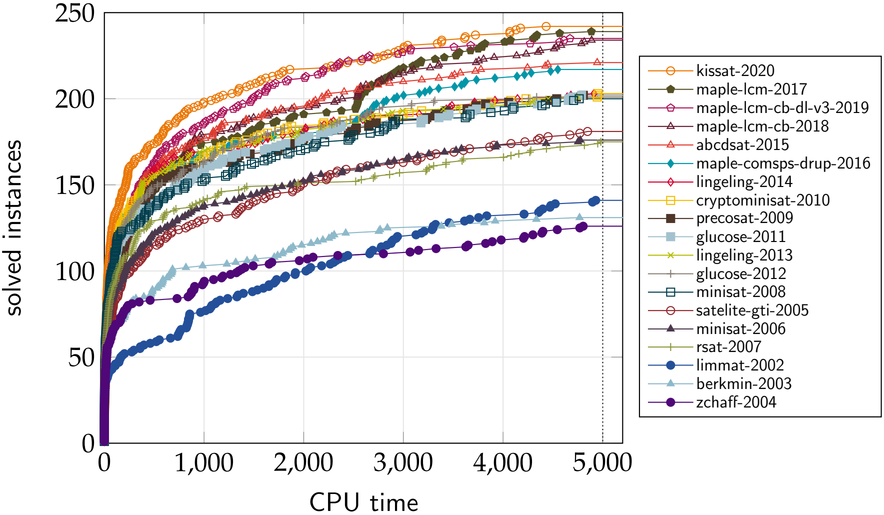
- Solvers are evaluated on a representative benchmark suite
- Practically every year clear progress is observed
- Arguably one of the drivers that advances the technology

CaDiCaL by Armin Biere is one of the strongest solvers

- Compiles easily on most operating systems
- Readable and understanable code and thus easy to modify
- Works normally from the command line, but also in Lean

Progress in SAT Solving (I)

SAT Competition Winners on the SC2011 Benchmark Suite

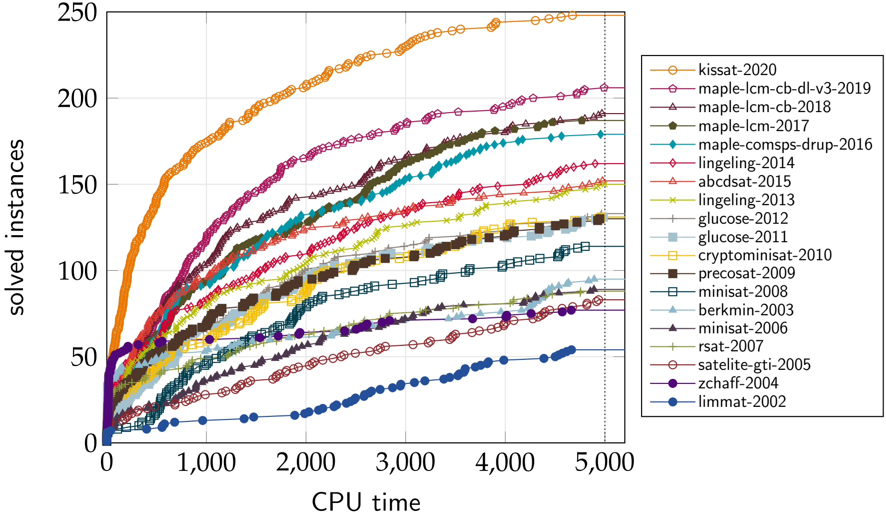


Benchmark suite used to show progress and get funding

Fisher: "instrumental ... for launching DARPA's HACMS program"

Progress in SAT Solving (II)

SAT Competition Winners on the SC2020 Benchmark Suite



Progress even larger due to harder instances

SAT Solving: Demo in Lean

```lean
/-
Examples of use of Cadical.
-/

-- textbook: SAT example
def cadicalExample : IO Unit := do
  let (s, result) ← callCadical exCnf0
  IO.println "Output from CaDiCaL :\n"
  --IO.println s
  --IO.println "\n\n"
  IO.println (formatResult result)
  pure ()

#eval cadicalExample
-- end textbook: SAT example
```

SAT Solving: DIMACS Input Format

The DIMACS format for SAT solvers has three types of lines:

- header: `p cnf n m` in which `n` denotes the highest variable index and `m` the number of clauses
- clauses: a sequence of integers ending with "0"
- comments: any line starting with "`c `"

$$
\begin{array}{l}
(p \vee q \vee \neg r) \wedge \\
(\neg p \vee \neg q \vee r) \wedge \\
(q \vee r \vee \neg s) \wedge \\
(\neg q \vee \neg r \vee s) \wedge \\
(p \vee r \vee s) \wedge \\
(\neg p \vee \neg r \vee \neg s) \wedge \\
(\neg p \vee q \vee s)
\end{array}
$$

```
c example
p cnf 4 7
 1  2 -3 0
-1 -2  3 0
 2  3 -4 0
-2 -3  4 0
 1  3  4 0
-1 -3 -4 0
-1  2  4 0
```

SAT Solving: DIMACS Output Format

The solution line of a SAT solver starts with "`s `":

- `s SATISFIABLE`: The formula is satisfiable
- `s UNSATISFIABLE`: The formula is unsatisfiable
- `s UNKNOWN`: The solver cannot determine satisfiability

In case the formula is satisfiable, the solver emits a certificate:

- lines starting with "`v `"
- a list of integers ending with `0`
- e.g. `v -1 2 4 0`

In case the formula is unsatisfiable, then most solvers support emitting a proof of unsatisfiability to a separate file

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DPLL: Introduction

Davis Putnam Logemann Loveland [DP60,DLL62]

Recursive procedure that in each recursive call:

- Simplifies the formula (using unit propagation)
- Splits the formula into two subformulas
  - Variable selection heuristics (which variable to split on)
  - Direction heuristics (which subformula to explore first)

DPLL: Example

$$\begin{aligned}\Gamma_{\text{DPLL}} := {} & (p_1 \vee p_2 \vee \neg p_3) \wedge (\neg p_1 \vee p_2 \vee p_3) \wedge \\ & (\neg p_1 \vee \neg p_2 \vee p_3) \wedge (p_1 \vee p_3) \wedge (\neg p_1 \vee \neg p_3)\end{aligned}$$

DPLL: Example

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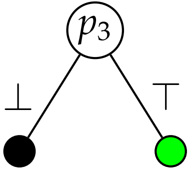

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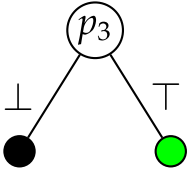

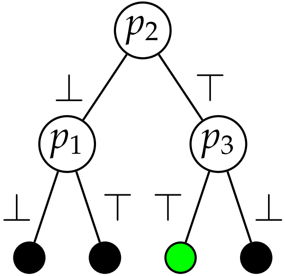

DPLL: Slightly Harder Example

Construct a DPLL tree for:

$$
\begin{aligned}
&(p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge \\
&(q \vee r \vee \neg s) \wedge (\neg q \vee \neg r \vee s) \wedge \\
&(p \vee r \vee s) \wedge (\neg p \vee \neg r \vee \neg s) \wedge \\
&(\neg p \vee q \vee s)
\end{aligned}
$$

What is a good heuristic?

DPLL: Slightly Harder Example

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\begin{aligned}
&(p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge \\
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&(p \vee r \vee s) \wedge (\neg p \vee \neg r \vee \neg s) \wedge \\
&(\neg p \vee q \vee s)
\end{aligned}
$$

What is a good heuristic?

A cheap and reasonably effective heuristic is MOMS:
Maximum Occurrence in clauses of Minimum Size

DPLL: Pseudocode

DPLL (τ, Γ)

$\tau' :=$ Simplify (τ, Γ)

if $[\![\Gamma]\!]_{\tau'} = \top$ **then return** `satisfiable`

if $[\![\Gamma]\!]_{\tau'} = \bot$ **then return** `unsatisfiable`

$l_{\text{decision}} :=$ Decide (τ', Γ)

if (DPLL($\tau' \cup l_{\text{decision}} := \top, \Gamma) =$ `satisfiable`) **then**

    return `satisfiable`

return DPLL $(\tau' \cup l_{\text{decision}} := \bot, \Gamma)$

DPLL: Demo in Lean

```lean
-- textbook: dpllSat
partial def dpllSatAux (τ : PropAssignment) (φ : CnfForm) : Option (PropAssignment × CnfForm) :=
  if φ.hasEmpty then none
  else match pickSplit? φ with
  -- No variables left to split on, we found a solution.
  | none => some (τ, φ)
  -- Split on `x`.
  -- `<|>` is the "or else" operator which tries one action, and if that failed tries the other.
  | some x => goWithNew x τ φ <|> goWithNew (-x) τ φ

where
  /-- Assigns `x` to true and continues out DPLL. -/
  goWithNew (x : Lit) (τ : PropAssignment) (φ : CnfForm) : Option (PropAssignment × CnfForm) :=
    let (τ', φ') := propagateWithNew x τ φ
    dpllSatAux τ' φ'

/-- Solve `φ` using DPLL. Return a satisfying assignment if found, otherwise `none`. -/
def dpllSat (φ : CnfForm) : Option PropAssignment :=
  let ⟨τ, φ⟩ := propagateUnits [] φ
  (dpllSatAux τ φ).map fun ⟨τ, _⟩ => τ
-- end textbook: dpllSat
```

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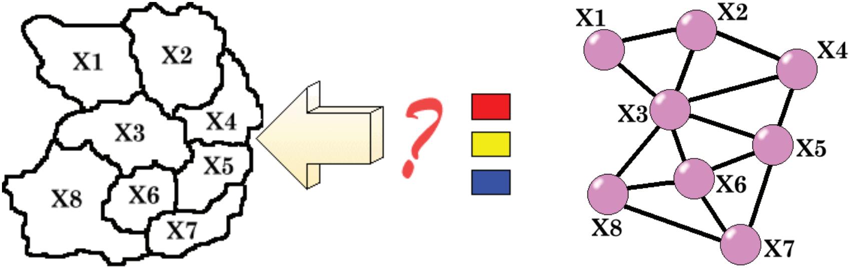
Graph Coloring

Graph Coloring: Introduction

Given a graph $G(V, E)$, can the vertices be colored with k colors such that for each edge $(v, w) \in E$, the vertices v and w are colored differently.



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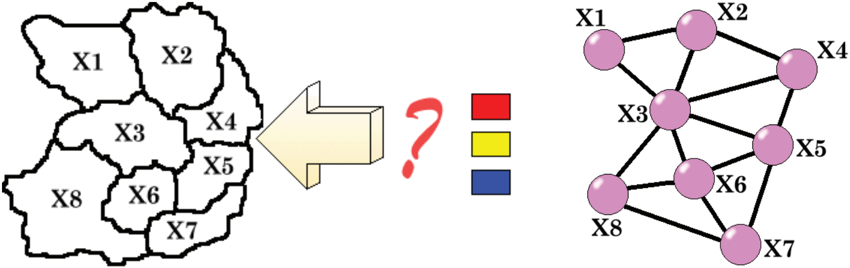
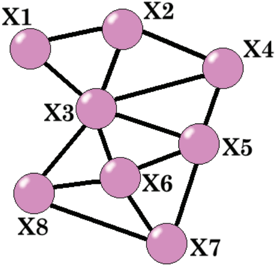


Possible problem: symmetries!

Graph Coloring: Format

- Header starts with `p edge`
- Followed by number of vertices and number of edges

```
p edge 8 13
e 1 2
e 1 3
e 2 3
e 2 4
e 3 5
e 3 6
e 3 8
e 4 5
e 5 6
e 5 7
e 6 7
e 6 8
e 7 8
```



Graph Coloring: Encoding

Variables	Range	Meaning
$p_{v,i}$	$i \in \{1, \ldots, c\}$ $v \in \{1, \ldots, \lvert V \rvert\}$	node v has color i

Clauses	Range	Meaning
$(p_{v,1} \vee p_{v,2} \vee \cdots \vee p_{v,c})$	$v \in \{1, \ldots, \lvert V \rvert\}$	v is colored
$(\neg p_{v,s} \vee \neg p_{v,t})$	$s \in \{1, \ldots, c-1\}$ $t \in \{s+1, \ldots, c\}$	v has at most one color
$(\neg p_{v,i} \vee \neg p_{w,i})$	$(v, w) \in E$	v and w have a different color
???	???	breaking symmetry

Graph Coloring: Lean Demo

```
def main (args : List String) : IO Unit := do
  let graphFname :: nColours :: _ ← args
    | do
      IO.println "Usage: <graph.edge> <colors>"
      return ()
  let some nColours ← nColours.toNat?
    | throwThe IO.Error s!"Invalid colour count: {nColours}"
  let g ← readEdgeFile graphFname
  match (← checkColourable g nColours) with
  | some vs =>
    IO.println s!"The graph is {nColours}-colourable! Satisfying assignment: {vs}"
  | none =>
    IO.println s!"The graph is not {nColours}-colourable."
```

Graph Coloring: Sudoku

Sudoku can be viewed as a graph coloring problem:

- Each cell is a vertex
- Vertices are connected if they occur in the same row / column / square
- There are 9 colors

The solution must be unique

- At least 17 givens

Who can solve this sudoku?

	4		3					
						7	9	
			6					
			1	4		5		
9							1	
2								6
				7	2			
	5					8		
				9				

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1	4	*7*	3	*8*	*9*	*2*	*6*	*5*
5	*8*	*6*	*2*	*1*	*4*	7	9	*3*
3	*9*	*2*	6	*5*	*7*	*1*	*8*	*4*
8	*7*	*3*	1	4	*6*	5	*2*	*9*
9	*6*	*4*	*7*	*2*	*5*	*3*	1	*8*
2	*1*	*5*	*9*	*3*	*8*	*4*	*7*	6
6	*3*	*8*	*5*	7	2	*9*	*4*	*1*
7	5	*9*	*4*	*6*	*1*	8	*3*	*2*
4	*2*	*1*	*8*	9	*3*	*6*	*5*	*7*

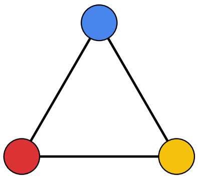
Graph Coloring: Sudoku in Lean

```
+-----------------------+
| 1 4 7 | 3 8 9 | 2 6 5 |
| 5 8 6 | 2 1 4 | 7 9 3 |
| 3 9 2 | 6 5 7 | 1 8 4 |
+-------+-------+-------+
| 8 7 3 | 1 4 6 | 5 2 9 |
| 9 6 4 | 7 2 5 | 3 1 8 |
| 2 1 5 | 9 3 8 | 4 7 6 |
+-------+-------+-------+
| 6 3 8 | 5 7 2 | 9 4 1 |
| 7 5 9 | 4 6 1 | 8 3 2 |
| 4 2 1 | 8 9 3 | 6 5 7 |
+-----------------------+
```

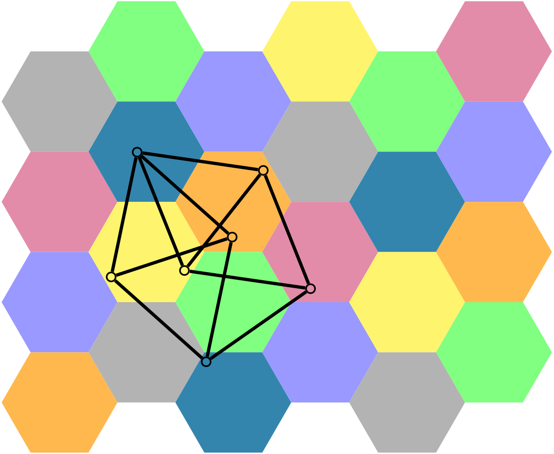
Graph Coloring: Chromatic Number of the Plane

The Hadwiger-Nelson problem:
How many colors are required to color the plane such that each pair of points that are exactly 1 apart are colored differently?

The answer must be three or more because three points can be mutually 1 apart—and thus must be colored differently.



Graph Coloring: Bounds since the 1950s



- The Moser Spindle graph shows the lower bound of 4
- A coloring of the plane showing the upper bound of 7

Graph Coloring: First progress in decades

Recently enormous progress:

- Lower bound of 5 [DeGrey '18] based on a 1581-vertex graph
- This breakthrough started a polymath project
- Improved bounds of the fractional chromatic number of the plane

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We found smaller graphs with SAT:

- 874 vertices on April 14, 2018
- 803 vertices on April 30, 2018
- 610 vertices on May 14, 2018

Record by Proof Minimization: 510 Vertices [Heule 2021]