Name: ____________________________

LOGIC AND MECHANIZED REASONING

Final Exam

December 6, 2021

Write your answers in the space provided, using the back of the page if necessary. You may use additional scratch paper. Justify your answers, and provide clear, readable explanations.

Problem	**Points**	**Score**
1	10	
2	8	
3	9	
4	10	
5	8	
6	9	
7	8	
8	10	
Total	**72**	

Good luck!

Problem 1

Suppose we color the natural numbers red, green, and blue, and design a language with predicates Red(x), Green(x), and Blue(x) and a binary relation $<$ to talk about the coloring.

Part a) (3 points) Write down a first-order sentence in the language that says that every natural number has exactly one color.

$$\forall x.\,(\mathrm{Red}(x) \vee \mathrm{Green}(x) \vee \mathrm{Blue}(x)) \wedge$$
$$\neg(\mathrm{Red}(x) \wedge \mathrm{Green}(x)) \wedge \neg(\mathrm{Red}(x) \wedge \mathrm{Blue}(x)) \wedge \neg(\mathrm{Green}(x) \wedge \mathrm{Blue}(x))$$

Part b) (3 points) Write down a sentence that says between any two numbers that are colored red there is a number that is colored green or blue.

$$\forall x, y.\, x < y \wedge \mathrm{Red}(x) \wedge \mathrm{Red}(\mathrm{y}) \to \exists z.\,(x < z \wedge z < y \wedge (\mathrm{Green}(z) \vee \mathrm{Blue}(z)))$$

Part c) (4 points) Write down a sentence that says that there are three numbers in a row that are colored blue. (Hint: it takes some thought to say that x, y, and z are consecutive numbers using only $<$ and $=$.)

$$\exists x, y, z.\, x < y \wedge y < z \wedge$$
$$\forall w.\,(x < w \wedge w < z \to w = y) \wedge \mathrm{Blue}(x) \wedge \mathrm{Blue}(y) \wedge \mathrm{Blue}(z)$$

Problem 2

Part a) (4 points) Prove the following or provide a counterexample: for any two formulas $A(x)$ and $B(x)$ in first-order logic, if $\exists x.\, A(x) \wedge B(x)$ is satisfiable, then so are $\exists x.\, A(x)$ and $\exists x.\, B(x)$.

Let $\mathfrak{M}$ be any model of $\exists x.\, A(x) \wedge B(x)$. Then there is an a in $|\mathfrak{M}|$ such that A and B are both true in $\mathfrak{M}$ when x is interpreted by a. This means that $\mathfrak{M}$ is a model of $\exists x.\, A(x)$ and $\exists x.\, B(x)$ as well.

Part b) (4 points) Prove the following or provide a counterexample: for any two formulas $A(x)$ and $B(x)$ in first-order logic, if $\exists x.\, A(x)$ and $\exists x.\, B(x)$ are satisfiable, then so is $\exists x.\, A(x) \wedge B(x)$.

This is false. Let $A(x)$ be $\mathrm{Even}(x)$ and let $B(x)$ be $\neg\mathrm{Even}(x)$. Then $\exists x.\, A(x)$ and $\exists x.\, B(x)$ are both true of the natural numbers with the usual interpretation of Even, but $\exists x.\, \mathrm{Even}(x) \wedge \neg\mathrm{Even}(x)$ is false in every model.

Problem 3

In each of the following cases, find a most general unifier of the pair of expressions or explain why no such unifier exists. In all these problems, $x, y, z, \ldots$ are variables and $a, b, c, \ldots$ are constants.

Part a) (3 points) $P(a, f(a, x), g(y))$ and $P(a, f(a, f(g(b), a)), x)$.

These cannot be unified. To unify $f(a, x)$ with $f(a, f(g(b), a))$, x must unify with $f(g(b), a)$. But we also have to unify x with $g(y)$. There is no way to unify $f(g(b), a)$ with $g(y)$ because the first starts with f and the second starts with g.

Part b) (3 points) $P(f(a), g(y))$ and $P(x, g(x))$.

$$x \mapsto f(a), y \mapsto f(a).$$

Part c) (3 points) $P(f(x), g(x))$ and $P(f(f(a)), g(f(y)))$.

$$x \mapsto f(a), y \mapsto a.$$

Problem 4 (10 points).

Consider the following first-order formula with equality:

$$f(a) = g^3(a) \ \wedge \ f(a) = g^5(a) \ \wedge \ f(a) \neq a \ \wedge \ f(a) \neq g(a).$$

Here $g^3(a)$ abbreviates $g(g(g(a)))$, and similarly for $g^5(a)$.

Compute the congruence closure and list the equivalence classes. In case the formula is unsatisfiable, list the conflict. In case the formula is satisfiable, construct a model.

The classes are as follows:

- $[a]$
- $[g(a)]$
- $[g^2(a)]$
- $[g^3(a)] = \{f(a), g^3(a), g^5(a)\}$
- $[g^4(a)]$

The formula is satisfiable. Add a new element b to the list of classes above, and define

$$\begin{aligned}
&a^{\mathfrak{M}} = [a],\\
&f^{\mathfrak{M}}([a]) = [g^3(a)], f^{\mathfrak{M}}([g(a)]) = b, f^{\mathfrak{M}}([g^2(a)]) = b\\
&f^{\mathfrak{M}}([g^3(a)]) = b, f^{\mathfrak{M}}([g^4(a)]) = b, f^{\mathfrak{M}}(b) = b,\\
&g^{\mathfrak{M}}([a]) = [g(a)], g^{\mathfrak{M}}([g(a)]) = [g^2(a)], g^{\mathfrak{M}}([g^2(a)]) = [g^3(a)],\\
&g^{\mathfrak{M}}([g^3(a)]) = [g^4(a)], g^{\mathfrak{M}}([g^4(a)]) = [g^3(a)], g^{\mathfrak{M}}(b) = b.
\end{aligned}$$

Problem 5 (8 points).

Use the Fourier–Motzkin procedure (the decision procedure for linear arithmetic that you helped implement for homework) to determine whether the following set of inequalities is satisfiable:

1. $x + 2y - 3z < -8$
2. $2x - 4y + 2z < 7$
3. $-x + z < 2$

Adding the first and the third, we get

$$2y - 2z < -6.$$

Adding twice the third to the second we get

$$-4y + 4z < 11.$$

Adding twice the first of these to the second, we get

$$0 < -1.$$

This is a contradiction, so the inequalities are unsatisfiable.

Problem 6.

For this problem, we consider quantifier-free bitvector formulas (QF_BV) with unsigned bitvectors of length 4. We use the following notation: $|$ for logical or, & for logical and, $\sim$ for logical negation (all bits are negated), $>_u$, $\geq_u$, $<_u$, and $\leq_u$, for unsigned greater than, greater or equal, less than, and less or equal. Furthermore, $\ll k$ denotes left shift by k bits, while $\gg k$ denotes a right shift by k bits. Determine for each of them whether the formula is satisfiable or unsatisfiable. In case it is satisfiable, provide a satisfying assignment (a bitvector of length 4).

Part a) (3 points)

$$(a \mid \sim a) <_u a$$

This is unsatisfiable. A logical or can only set more bits to 1, which increases the value.

Part b) (3 points)

$$(a \ll 2) >_u a$$

Satisfiable: set $a = 0001$. Then $0100 >_u 0001$.

Part c) (3 points)

$$(a \;\&\; (a \gg 1)) >_u a$$

This is unsatisfiable. A logical and can only decrease the value.

Problem 7.

The Green Bridge of Wales is a famous rock formation. However you need to be lucky to see it: *If it rains during a day, you can't see the bridge.* Unfortunately it rains a lot in Wales: *On any two consecutive days, it rains on at least one of them.* That is why the tourist guide states: *If you can see the bridge, then it will rain tomorrow.*

Part a) (4 points) Express in first-order logic the two axioms and the conclusion (the text shown in italics) using the predicates $\mathrm{Rains}(x)$ and $\mathrm{Visible}(x)$ and the function $\mathrm{nextDay}(x)$. ($\mathrm{Visible}(x)$ means that the bridge is visible on day x.) Note that the variables range over days; you do not need to refer to any other kinds of objects.

- $\forall x.\, \mathrm{Rains}(x) \to \neg\mathrm{Visible}(x)$
- $\forall x.\, \mathrm{Rains}(x) \lor \mathrm{Rains}(\mathrm{nextDay}(x))$.
- $\forall x.\, \mathrm{Visible}(x) \to \mathrm{Rains}(\mathrm{nextDay}(x))$.

Part b) (4 points) Show that the conclusion follows from the two axioms using resolution for first-order logic.

From the hypotheses and negated conclusion, we get 1–4.

1. $\forall x.\, \neg\mathrm{Rains}(x) \lor \neg\mathrm{Visible}(x)$
2. $\forall x.\, \mathrm{Rains}(x) \lor \mathrm{Rains}(\mathrm{nextDay}(x))$
3. $\mathrm{Visible}(a)$
4. $\neg\mathrm{Rains}(\mathrm{nextDay}(a))$
5. $\neg\mathrm{Rains}(a)$, from 1 and 3
6. $\mathrm{Rains}(a)$, from 2 and 4
7. $\perp$ from 5 and 6.

Problem 8. For these problems, use $S(x)$ for "x is a student," $H(x, y)$ for "x has y," $C(y)$ for "y is a car," and $D(y)$ for "y is a driver's license."

Part a) (2 points) Write down a formalization of the statement "every student that has a car has a driver's license."

$$\forall x, y.\, S(x) \wedge H(x, y) \wedge C(y) \rightarrow \exists z.\, H(x, z) \wedge D(z)$$

The y can also be moved into the antecedent and expressed as $\exists$, with the scope limited:

$$\forall x.\, S(x) \wedge (\exists y.\, H(x, y) \wedge C(y)) \rightarrow \exists z.\, H(x, z) \wedge D(z)$$

Part b) (3 points) Skolemize the previous statement and transform it to an equisatisfiable set of (one or more) universally quantified clauses.

$$\forall x, y.\, \neg S(x) \vee \neg H(x, y) \vee \neg C(y) \vee H(x, f(x, y))$$

and

$$\forall x, y.\, \neg S(x) \vee \neg H(x, y) \vee \neg C(y) \vee D(f(x, y)).$$

Part c) (2 points) Write down a formalization of the statement "not every student that has a driver's license has a car."

$$\exists x, y.\, S(x) \wedge H(x, y) \wedge D(y) \wedge \forall z.\, C(z) \rightarrow \neg H(x, z)$$

Part d) (3 points) Skolemize the previous statement and transform it to an equisatisfiable set of universally quantified clauses.

$$S(b), \quad H(b, c), \quad D(c), \quad \forall z.\, \neg C(z) \vee \neg H(b, z).$$