Assignment 9

due 6pm Thursday, November 11, 2021

Problem 1 (7 points)

Fill in the proofs of the propositional formulas found in `assignment9.lean`.

Problem 2 (2 points)

Let $\mathcal{A}$ be the structure consisting of "all objects on the planet Earth" with relations *Cow(x)*, *EatsGrass(x)*, *Car(x)*, etc. Give reasonable formalizations of the following sentences:

1. All cows eat grass.
2. There is a car that is blue and old.
3. No car is not pink.
4. All cars that are old must be inspected annually.

Problem 3 (2 points)

Consider a first-order language, with relation symbols $<$ and $=$. The intended interpretation is the natural numbers, with the usual less-than relation and equality. Formalize the following statements:

1. 0 is the smallest number.
2. There is a smallest number.
3. There is no largest number.
4. Every number has an immediate successor. (In other words, for every number, there is another one that is the "next largest.")

Problem 4 (3 points)

Consider a language designed to talk about numbers, with symbols $+$, $\times$, and $<$. Consider these four interpretations of the language:

1. $\mathfrak{N} = (\mathbb{N}, +, \times, <)$
2. $\mathfrak{Z} = (\mathbb{Z}, +, \times, <)$
3. $\mathfrak{Q} = (\mathbb{Q}, +, \times, <)$
4. $\mathfrak{R} = (\mathbb{R}, +, \times, <)$

In other words, we consider the natural numbers, the integers, the rational numbers, and the real numbers, all with the usual interpretations of the symbols.

1. Write down a sentence in the language that is satisfied by $\mathfrak{Z}$ but not $\mathfrak{N}$.
2. Write down a sentence in the language that is satisfied by $\mathfrak{Q}$ but not $\mathfrak{Z}$.
3. Write down a sentence in the language that is satisfied by $\mathfrak{R}$ but not $\mathfrak{Q}$. (Hint: This is tricky. One option is to say that every sufficiently large number has a square root.)

Problem 5 (2 points)

Remember that substitution $t[s/x]$ for terms is defined recursively, and there is a similar definition of substitution $A[s/x]$ for formulas. As we did for propositional logic, we can prove the following, using the semantic definitions in Section 10.3:

$$\mathfrak{M} \models_{\sigma} A[t/x] \quad \text{if and only iff} \quad \mathfrak{M} \models_{\sigma[x \mapsto [\![t]\!]_{\mathfrak{M},\sigma}]} A.$$

Use this fact (you don't have to prove it), and the semantic definitions, to show that for every formula A, every model $\mathfrak{M}$, every term t, and every assignment σ, we have

$$\mathfrak{M} \models_{\sigma} (\forall x.\, A) \to A[t/x].$$