10-301/601: Introduction to Machine Learning Lecture 3 – Decision Trees

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1/22/25

Front Matter

- Announcements:
  - HW1 released 1/13, due 1/22 (today!) at 11:59 PM
    - Reminder: we will grant (basically) any extension requests for this assignment
  - HW2 released 1/22 (today!), due 2/3 at 11:59 PM
    - Unlike HW1, you will only have:
      - 1 submission for the written portion
      - 10 submissions of the programming portion to our autograder

Q & A:

How do these in-class polls work?

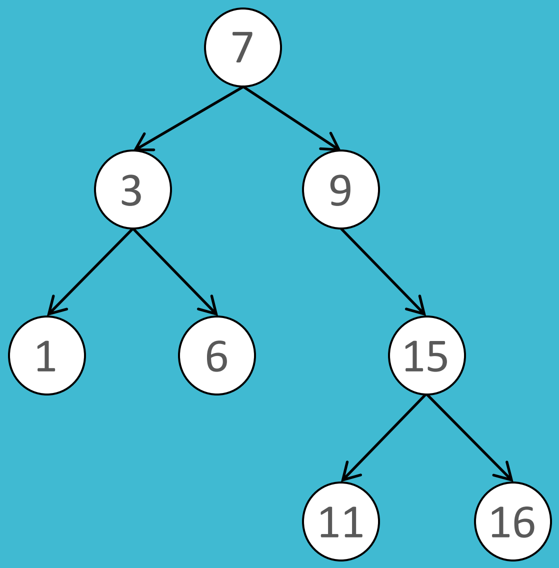
- Open the poll, either by clicking the [Poll] link on the schedule page of our course website or going to http://poll.mlcourse.org
- Sign into Google Forms using your **Andrew email**
- Answer all poll questions **during lecture for full credit** or **within 24 hours for half credit**
    - **Exception: for today's polls only, any response within 24 hours will receive full credit**
- Avoid the **toxic option** (will be clearly specified in lecture) which gives **negative poll points**
- You have 8 free "poll points" for the semester that will excuse you from all polls from a single lecture; you cannot use more than 3 poll points consecutively.

Poll Question 1:

Which of the following did you bring to class today? Select all that apply

A. A tablet
B. A smartphone
C. A payphone
D. A laptop
E. None of the above

Background: Recursion

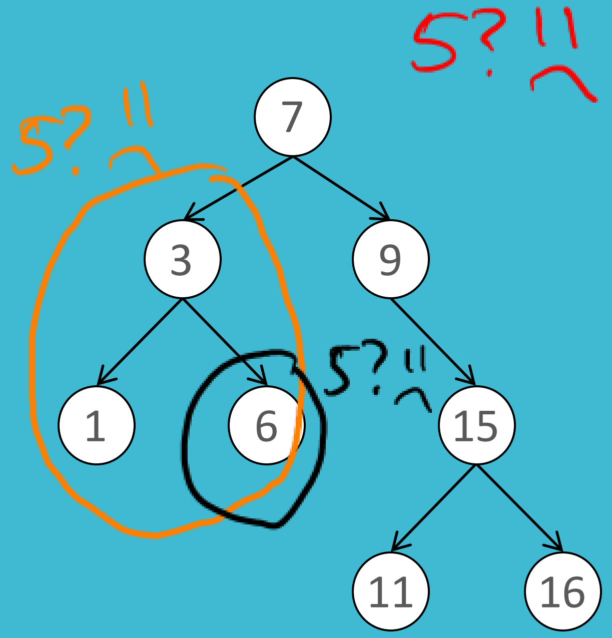


- A **binary search tree** (BST) consists of nodes, where each node:
    - has a value, v
    - up to 2 children, a left descendant and a right descendant
    - all its left descendants have values less than v and its right descendants have values greater than v
- We like BSTs because they permit search in O(log(n)) time, assuming n nodes in the tree

root

```
def contains_iterative(node, key):
    curr = node
    while (true):
        if key < curr.value & curr.left != null:
            curr = curr.left
        else if key > curr.value & curr.right != null:
            curr = curr.right
        else: break
    return key == curr.value
```

Background: Recursion



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- We like BSTs because they permit search in O(log(n)) time, assuming n nodes in the tree

root

```
def contains_recursive(node, key):
    if key < node.value & node.left != null:
        return contains_recursive(node.left, key)
    else if key > node.value & node.right != null:
        return contains_recursive(node.right, key)
    else:
        return key == node.value
```

Recall: Decision Stumps

Algorithm 3: based on a single feature, x_d, predict the most common label in the training dataset among all data points that have the same value for x_d

	y	x_1	x_2	x_3	x_4
predictions	**allergic?**	**hives?**	**sneezing?**	**red eye?**	**has cat?**
$-$	$-$	Y	N	N	N
$+$	$-$	N	Y	N	N
$+$	$+$	Y	Y	N	N
$-$	$-$	Y	N	Y	Y
$+$	$+$	N	Y	Y	N

Nonzero training error, but perhaps still better than the memorizer

Example decision stump:

$$h(\mathbf{x}) = \begin{cases} + \text{ if sneezing} = \text{Y} \\ - \text{ otherwise} \end{cases}$$

But why would we only use just one feature?

Algorithm 3: based on a single feature, x_d, predict the most common label in the training dataset among all data points that have the same value for x_d

	y	x_1	x_2	x_3	x_4
predictions	**allergic?**	**hives?**	**sneezing?**	**red eye?**	**has cat?**
−	−	Y	N	N	N
+	−	N	Y	N	N
+	+	Y	Y	N	N
−	−	Y	N	Y	Y
+	+	N	Y	Y	N

Nonzero training error, but perhaps still better than the memorizer

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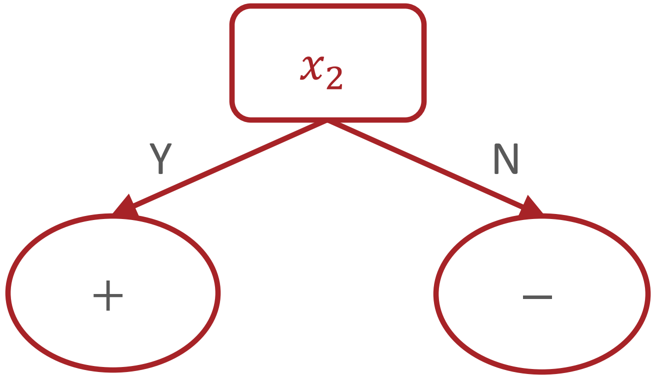
$$h(\mathbf{x}) = \begin{cases} + \text{ if sneezing} = \text{Y} \\ - \text{ otherwise} \end{cases}$$

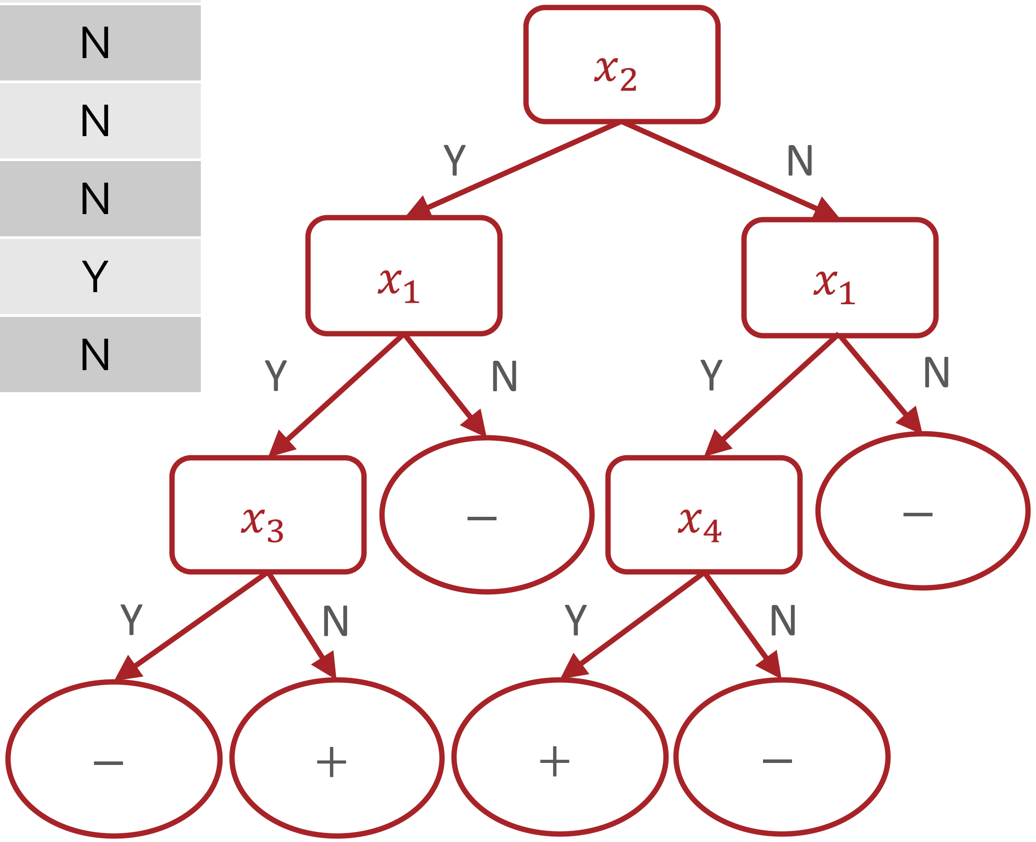
From Decision Stump …

y allergic?	x_1 hives?	x_2 sneezing?	x_3 red eye?	x_4 has cat?
$-$	Y	N	N	N
$-$	N	Y	N	N
$+$	Y	Y	N	N
$-$	Y	N	Y	Y
$+$	N	Y	Y	N



From Decision Stump to Decision Tree

y allergic?	x_1 hives?	x_2 sneezing?	x_3 red eye?	x_4 has cat?
−	Y	N	N	N
−	N	Y	N	N
+	Y	Y	N	N
−	Y	N	Y	Y
+	N	Y	Y	N



Decision Tree: In-class Activity

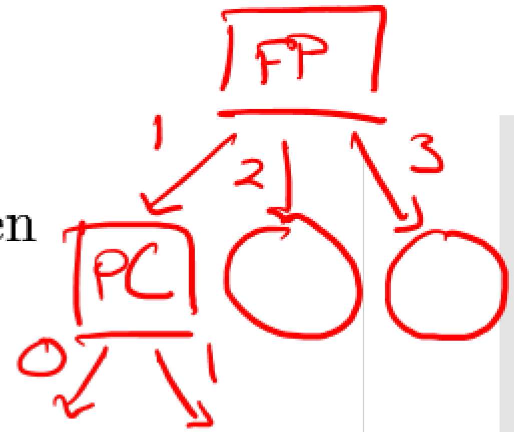
1. Group 1: Answer the questions to determine which leaf node corresponds to your feature values
2. Group 2:
   a) Take a blue sticky note if you prefer dogs to cats; otherwise, take a red sticky note
   b) Answer the questions to determine which leaf node corresponds to your feature values and place your sticky note there
   c) Answer the new question to determine which new leaf node to move your sticky note to
3. Group 3: Answer the questions to determine which leaf node corresponds to your feature values

Decision Tree: Example

Learned from medical records of 1000 women

Negative examples are C-sections

```
[833+,167-] .83+ .17-
Fetal_Presentation = 1: [822+,116-] .88+ .12-
| Previous_Csection = 0: [767+,81-] .90+ .10-
| | Primiparous = 0: [399+,13-] .97+ .03-
| | Primiparous = 1: [368+,68-] .84+ .16-
| | | Fetal_Distress = 0: [334+,47-] .88+ .12-
| | | Fetal_Distress = 1: [34+,21-] .62+ .38-
| Previous_Csection = 1: [55+,35-] .61+ .39-
Fetal_Presentation = 2: [3+,29-] .11+ .89-
Fetal_Presentation = 3: [8+,22-] .27+ .73-
```



Decision Tree: Pseudocode

$[x'_1, x'_2, \ldots, x'_D]$

def $h(\boldsymbol{x}')$:

- walk from the root node to a leaf

while (true):

    if current_node is "internal" (not a leaf):

        check the associated feature, x_d and go down the branch corresponding to x'_d

    if current_node is a leaf:

        return label stored at current_node

Decision Tree Questions

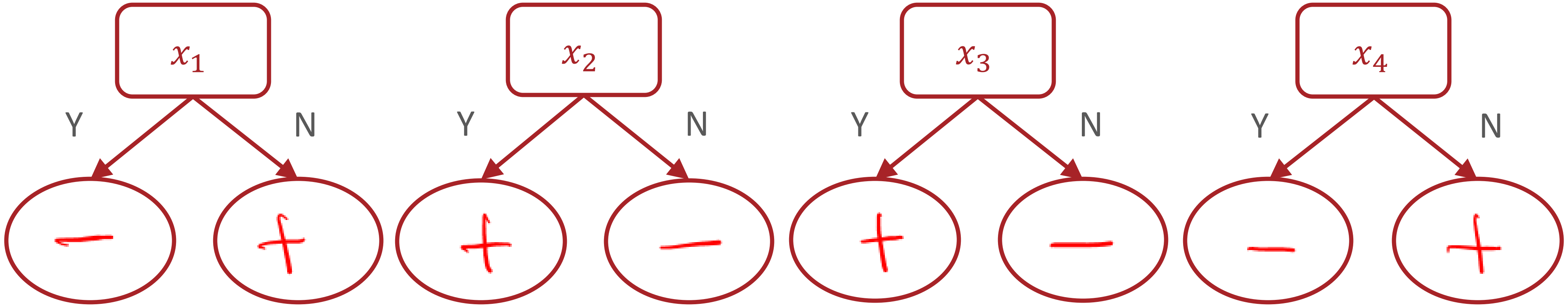
1. How can we pick which feature to split on?
2. Why stop at just one feature?

Splitting Criterion

- A **splitting criterion** is a function that measures how good or useful splitting on a particular feature is *for a specified dataset*
- Idea: when deciding which feature to split on, use the one that optimizes the splitting criterion

Training Error Rate as a Splitting Criterion

y allergic?	x_1 hives?	x_2 sneezing?	x_3 red eye?	x_4 has cat?
–	Y	N	N	N
–	N	Y	N	N
+	Y	Y	N	N
–	Y	N	Y	Y
+	N	Y	Y	N



Training error rate: 2/5

$\boxed{\text{Training error rate: } 1/5}$

Training error rate: 2/5

Training error rate: 2/5

Poll Question 2:

Which feature would you split on using training error rate as the splitting criterion?

x_1	x_2	y
1	0	0
1	0	0
1	0	1
1	0	1
1	1	1
1	1	1
1	1	1
1	1	1

A. x_1

B. x_2

C. Either x_1 or x_2

D. Neither x_1 nor x_2

Poll Question 2:

Which feature would you split on using training error rate as the splitting criterion?

x_1	x_2	y
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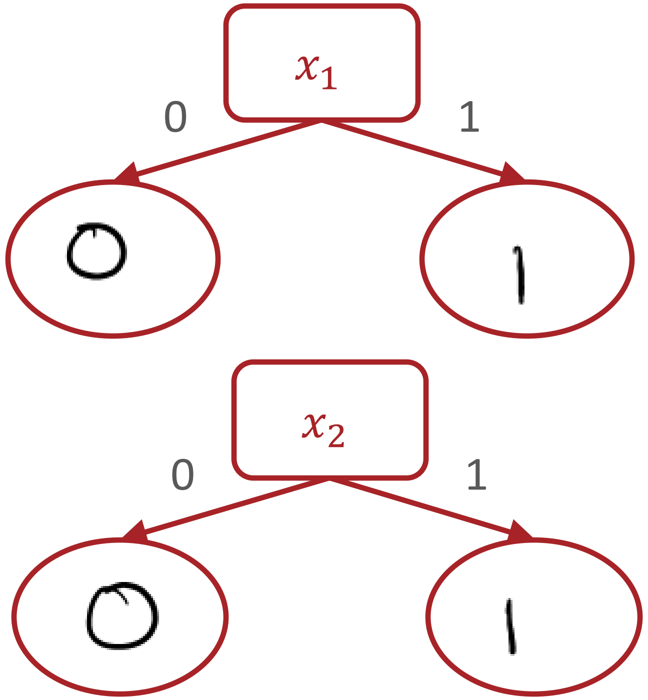
1	0	0
1	0	1
1	0	1
1	1	1
1	1	1
1	1	1
1	1	1



Training error rate: 2/8

Splitting Criterion

- A **splitting criterion** is a function that measures how good or useful splitting on a particular feature is *for a specified dataset*
- Idea: when deciding which feature to split on, use the one that optimizes the splitting criterion
- Potential splitting criteria:
  - Training error rate (minimize)
  - Gini impurity (minimize) → CART algorithm
  - Mutual information (maximize) → ID3 algorithm

Splitting Criterion

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- Potential splitting criteria:
  - Training error rate (minimize)
  - Gini impurity (minimize) → CART algorithm
  - **Mutual information** (maximize) → ID3 algorithm

Entropy

- The **entropy** of a *random variable* describes the uncertainty of its outcome: the higher the entropy, the less certain we are about what the outcome will be.

$$H(X) = -\sum_{v \in V(X)} P(X = v) \log_2\big(P(X = v)\big)$$

where X is a (discrete) random variable

$V(X)$ is the set of possible values X can take on

Entropy

- The **entropy** of a *set* describes how uniform or pure it is: the higher the entropy, the more impure or "mixed-up" the set is

$$H(S) = -\sum_{v \in V(S)} \frac{|S_v|}{|S|} \log_2\left(\frac{|S_v|}{|S|}\right)$$

1.1

"size of"

where S is a collection of values,

$V(S)$ is the set of unique values in S

S_v is the collection of elements in S with value v

- If all the elements in S are the same, then

$$H(S) = -\frac{|S|}{|S|}\log_2\frac{|S|}{|S|} = -1\log_2 1 = 0$$

Entropy

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$$H(S) = -\sum_{v \in V(S)} \frac{|S_v|}{|S|} \log_2 \left(\frac{|S_v|}{|S|} \right)$$

where S is a collection of values,

$V(S)$ is the set of unique values in S

S_v is the collection of elements in S with value v

- If S is split fifty-fifty between two values, then

$$H(S) = -\left(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2}\right)$$
$$= -\left(\frac{1}{2}(-1) + \frac{1}{2}(-1)\right) = 1$$

Mutual Information

- The **mutual information** between *two random variables* describes how much clarity knowing the value of one random variables provides about the other

$$I(Y;X) = H(Y) - H(Y|X)$$

$$= H(Y) - \sum_{v \in V(X)} P(X = v) H(Y|X = v)$$

where X and Y are random variables

$V(X)$ is the set of possible values X can take on

$H(Y|X = v)$ is the conditional entropy of Y given $X = v$

Mutual Information

- The **mutual information** between *a feature and the label* describes how much clarity knowing the feature provides about the label

$$I(y; x_d) = H(y) - H(y|x_d)$$

$$= H(y) - \sum_{v \in V(x_d)} f_v * H\left(Y_{x_d=v}\right)$$

where x_d is a feature and y is the set of all labels

$V(x_d)$ is the set of possible values x_d can take on

f_v is the fraction of data points where $x_d = v$

$Y_{x_d=v}$ is the set of all labels where $x_d = v$

Mutual Information: Example

x_d	y
1	1
1	1
0	0
0	0

$$I(x_d, Y) = H(Y) - \sum_{v \in V(x_d)} f_v * H\left(Y_{x_d=v}\right)$$

$$= 1 - \left(\frac{1}{2}\underbrace{H(Y_{x_d=1})}_{0} + \frac{1}{2}\underbrace{H(Y_{x_d=0})}_{0}\right)$$

$$= 1$$

Mutual Information: Example

x_d	y
1	1
0	1
1	0
0	0

$$I(x_d, Y) = H(Y) - \sum_{v \in V(x_d)} f_v * H\left(Y_{x_d=v}\right)$$

$$= 1 - \left(\tfrac{1}{2} H(Y_{x_d=1}) + \tfrac{1}{2} H(Y_{x_d=0})\right)$$

1 1

$= 0$:(

Poll Question 3:

Which feature would you split on using mutual information as the splitting criterion?

x_1	x_2	y
1	0	0
1	0	0
1	0	1
1	0	1
1	1	1
1	1	1
1	1	1
1	1	1

A. x_1

B. x_2

C. Either x_1 or x_2

D. Neither x_1 nor x_2

Poll Question 3:

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x_1	x_2	y
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1	0	0
1	0	1
1	0	1
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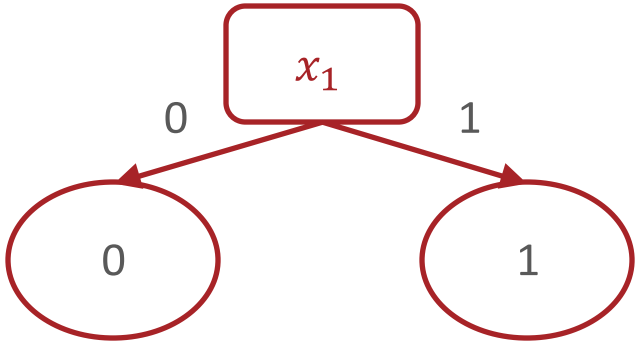
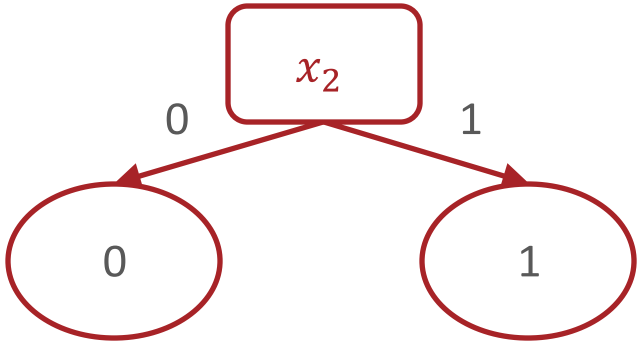


Mutual Information: 0



Mutual Information: $-\frac{2}{8}\log_2\frac{2}{8} - \frac{6}{8}\log_2\frac{6}{8} - \frac{1}{2}(1) - \frac{1}{2}(0) \approx 0.31$

Decision Tree Questions

1. How can we pick which feature to split on?

   Mutual Information

2. Why stop at just one feature?

   a) If we split on more than one feature, how do we decide the order to spilt on?

      Recursion!

Decision Tree: Pseudocode

```
def train(𝒟):
    store root = tree_recurse(𝒟)
def tree_recurse(𝒟'):
    q = new node()
    base case – if (SOME CONDITION):
    recursion – else:
        Find the best attribute to split on, x_d
        q.split = x_d
        for v in V(x_d), all possible values of x_d:
            D_v = {(x^(n), y^(n)) ∈ D' | x_d^(n) = v}
            q.children(v) = tree_recurse(D_v)
    return q
```

Decision Tree: Pseudocode

```
def train(𝒟):
    store root = tree_recurse(𝒟)
def tree_recurse(𝒟′):
    q = new node()
    base case – if ( all labels in D′ are the
                     same OR
                     D′ is empty OR
                     all feature vectors in D′
                    are the same ):
                q.label = majority_vote(D′)
    recursion – else:
    return q
```