Machine Learning Department
School of Computer Science
Carnegie Mellon University

Stochastic Gradient Descent

+

Probabilistic Learning

(Binary Logistic Regression)

Matt Gormley
Lecture 9
Feb. 12, 2020

Reminders

- **Homework 3: KNN, Perceptron, Lin.Reg.**
  - Out: Wed, Feb. 05 (+ 1 day)
  - Due: **Fri, Feb. 14 at 11:59pm**
    last possible moment to submit HW3:
    Sun, Feb. 16 at 11:59pm
- **Exam 1 Practice Problems**
  - problems + solutions released: Wed, Feb. 12
- **Midterm Exam 1**
  - Tue, Feb. 18, 7:00pm – 9:00pm
- **Today's In-Class Poll**
  - http://p9.mlcourse.org

MIDTERM EXAM LOGISTICS

Midterm Exam

- **Time / Location**
  - **Time:** Evening Exam
    Tue, Feb. 18, 7:00pm – 9:00pm
  - **Room:** We will contact each student individually with **your room assignment.** The rooms are **not** based on section.
  - **Seats:** There will be **assigned seats.** Please arrive early.
  - Please watch Piazza carefully for announcements regarding room / seat assignments.
- **Logistics**
  - Covered material: Lecture 1 – Lecture 8
  - Format of questions:
    - Multiple choice
    - True / False (with justification)
    - Derivations
    - Short answers
    - Interpreting figures
    - Implementing algorithms on paper
  - No electronic devices
  - You are allowed to **bring** one 8½ x 11 sheet of notes (front and back)

Midterm Exam

- **How to Prepare**
  - Attend the midterm review lecture (right now!)
  - Review exam practice problems (we'll post them)
  - Review this year's homework problems
  - Consider whether you have achieved the "learning objectives" for each lecture / section

Midterm Exam

- **Advice (for during the exam)**
  - Solve the easy problems first (e.g. multiple choice before derivations)
    - if a problem seems extremely complicated you're likely missing something
  - Don't leave any answer blank!
  - If you make an assumption, write it down
  - If you look at a question and don't know the answer:
    - we probably haven't told you the answer
    - but we've told you enough to work it out
    - imagine arguing for some answer and see if you like it

Topics for Midterm 1

- Foundations
  - Probability, Linear Algebra, Geometry, Calculus
  - Optimization
- Important Concepts
  - Overfitting
  - Experimental Design
- Classification
  - Decision Tree
  - KNN
  - Perceptron
- Regression
  - Linear Regression

SAMPLE QUESTIONS

Sample Questions

1.4 Probability

Assume we have a sample space Ω. Answer each question with **T** or **F**.

(a) [1 pts.] **T or F:** If events A, B, and C are disjoint then they are independent.

(b) [1 pts.] **T or F:** $P(A|B) \propto \frac{P(A)P(B|A)}{P(A|B)}$. (The sign '$\propto$' means 'is proportional to')

Sample Questions

5.2 Constructing decision trees

Consider the problem of predicting whether the university will be closed on a particular day. We will assume that the factors which decide this are whether there is a snowstorm, whether it is a weekend or an official holiday. Suppose we have the training examples described in the Table 5.2.

Snowstorm	Holiday	Weekend	Closed
T	T	F	F
T	T	F	T
F	T	F	F
T	T	F	F
F	F	F	F
F	F	F	T
T	F	F	T
F	F	F	T

Table 1: Training examples for decision tree

- **[2 points]** What would be the effect of the Weekend attribute on the decision tree if it were made the root? Explain in terms of information gain.

- **[8 points]** If we cannot make Weekend the root node, which attribute should be made the root node of the decision tree? Explain your reasoning and show your calculations. (You may use $\log_2 0.75 = -0.4$ and $\log_2 0.25 = -2$)

Sample Questions

4 K-NN [12 pts]

Now we will apply K-Nearest Neighbors using Euclidean distance to a binary classification task. We assign the class of the test point to be the class of the majority of the k nearest neighbors. A point can be its own neighbor.

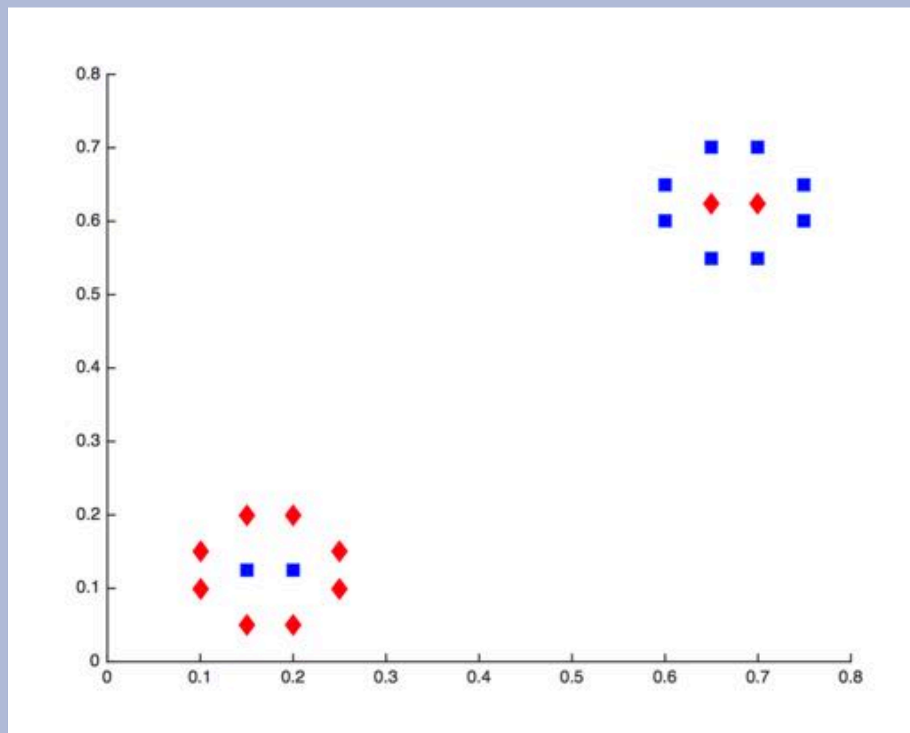


Figure 5

3. **[2 pts]** What value of k minimizes leave-one-out cross-validation error for the dataset shown in Figure 5? What is the resulting error?

Sample Questions

4.1 True or False

Answer each of the following questions with **T** or **F** and **provide a one line justification**.

(a) [2 pts.] Consider two datasets $D^{(1)}$ and $D^{(2)}$ where $D^{(1)} = \{(x_1^{(1)}, y_1^{(1)}), ..., (x_n^{(1)}, y_n^{(1)})\}$ and $D^{(2)} = \{(x_1^{(2)}, y_1^{(2)}), ..., (x_m^{(2)}, y_m^{(2)})\}$ such that $x_i^{(1)} \in \mathbb{R}^{d_1}$, $x_i^{(2)} \in \mathbb{R}^{d_2}$. Suppose $d_1 > d_2$ and $n > m$. Then the maximum number of mistakes a perceptron algorithm will make is higher on dataset $D^{(1)}$ than on dataset $D^{(2)}$.

Sample Questions

3.1 Linear regression

Consider the dataset S plotted in Fig. 1 along with its associated regression line. For each of the altered data sets S^{new} plotted in Fig. 3, indicate which regression line (relative to the original one) in Fig. 2 corresponds to the regression line for the new data set. Write your answers in the table below.

Dataset	(a)	(b)	(c)	(d)	(e)
Regression line					

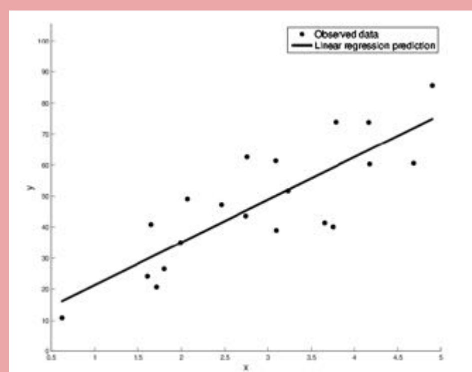
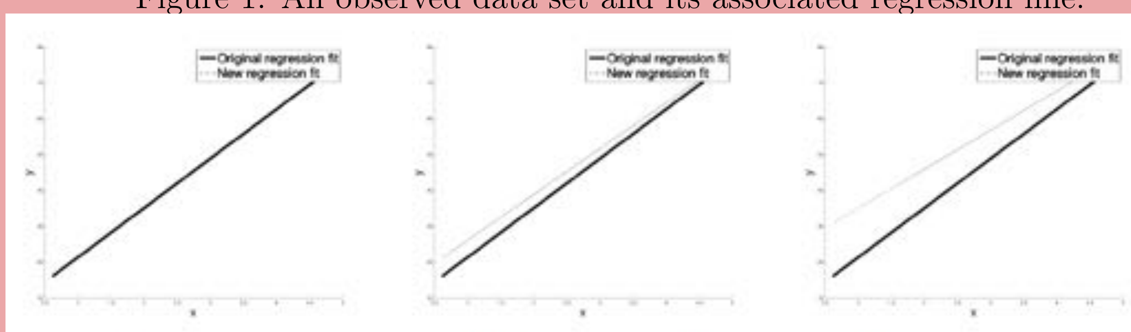


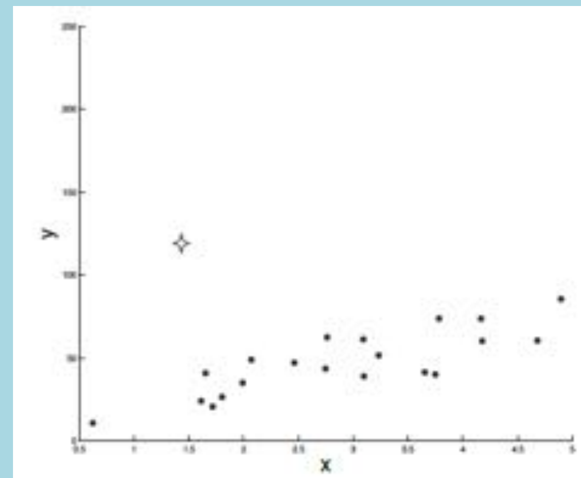
Figure 1: An observed data set and its associated regression line.



(a) Old and new regression lines. (b) Old and new regression lines. (c) Old and new regression lines.

Figure 2: New regression lines for altered data sets S^{new}.

Dataset



(a) Adding one outlier to the original data set.

Sample Questions

3.1 Linear regression

Consider the dataset S plotted in Fig. 1 along with its associated regression line. For each of the altered data sets S^{new} plotted in Fig. 3, indicate which regression line (relative to the original one) in Fig. 2 corresponds to the regression line for the new data set. Write your answers in the table below.

Dataset	(a)	(b)	(c)	(d)	(e)
Regression line					

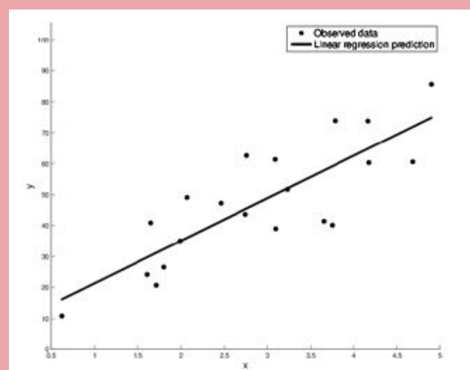
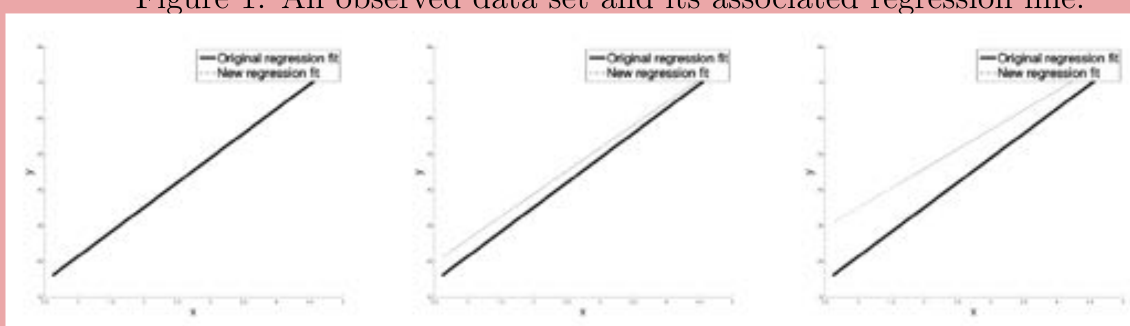


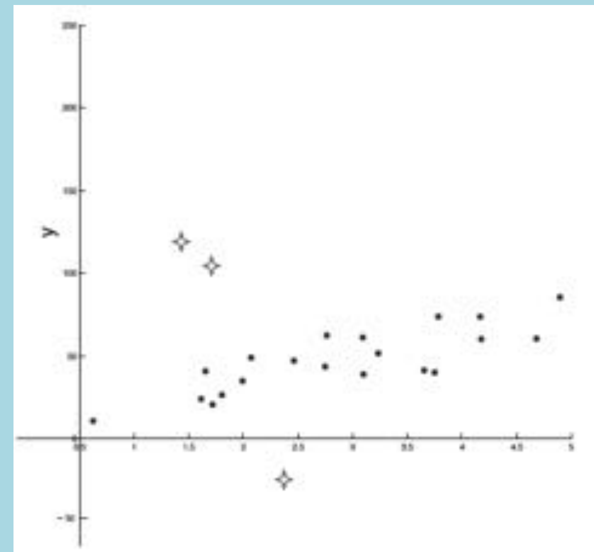
Figure 1: An observed data set and its associated regression line.



(a) Old and new regression lines. (b) Old and new regression lines. (c) Old and new regression lines.

Figure 2: New regression lines for altered data sets S^{new}.

Dataset



(c) Adding three outliers to the original data set. Two on one side and one on the other side.

Sample Questions

3.1 Linear regression

Consider the dataset S plotted in Fig. 1 along with its associated regression line. For each of the altered data sets S^{new} plotted in Fig. 3, indicate which regression line (relative to the original one) in Fig. 2 corresponds to the regression line for the new data set. Write your answers in the table below.

Dataset	(a)	(b)	(c)	(d)	(e)
Regression line					

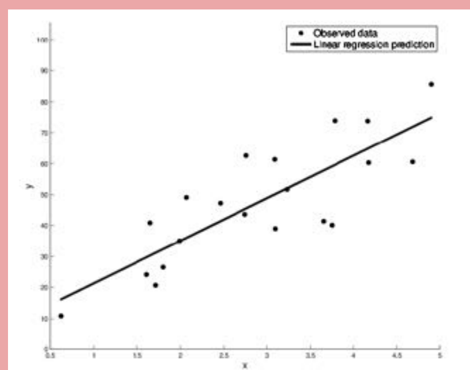
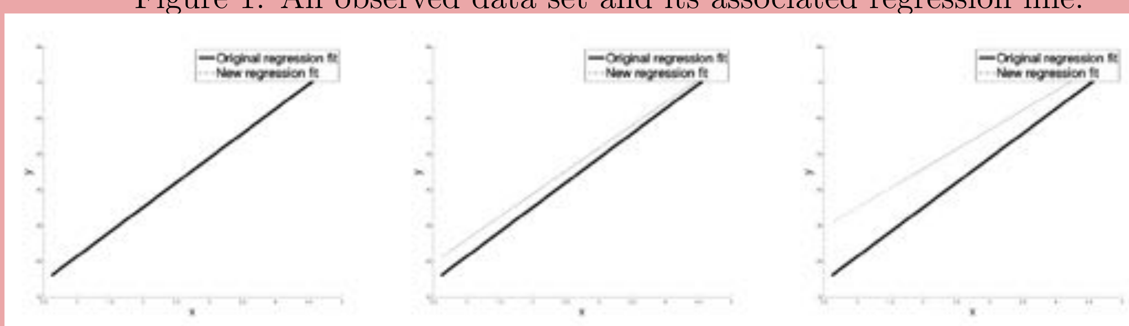


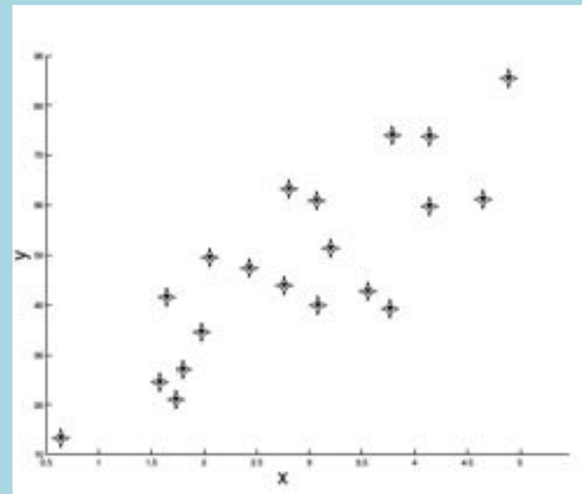
Figure 1: An observed data set and its associated regression line.



(a) Old and new regression lines. (b) Old and new regression lines. (c) Old and new regression lines.

Figure 2: New regression lines for altered data sets S^{new}.

Dataset



(d) Duplicating the original data set.

Sample Questions

3.1 Linear regression

Consider the dataset S plotted in Fig. 1 along with its associated regression line. For each of the altered data sets S^{new} plotted in Fig. 3, indicate which regression line (relative to the original one) in Fig. 2 corresponds to the regression line for the new data set. Write your answers in the table below.

Dataset	(a)	(b)	(c)	(d)	(e)
Regression line					

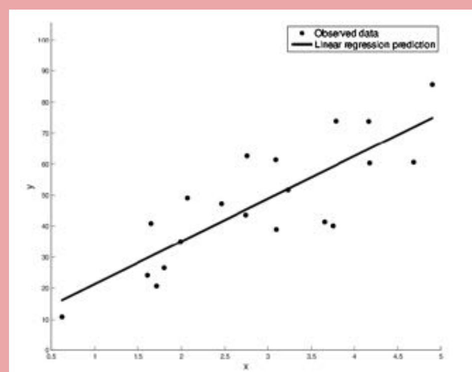
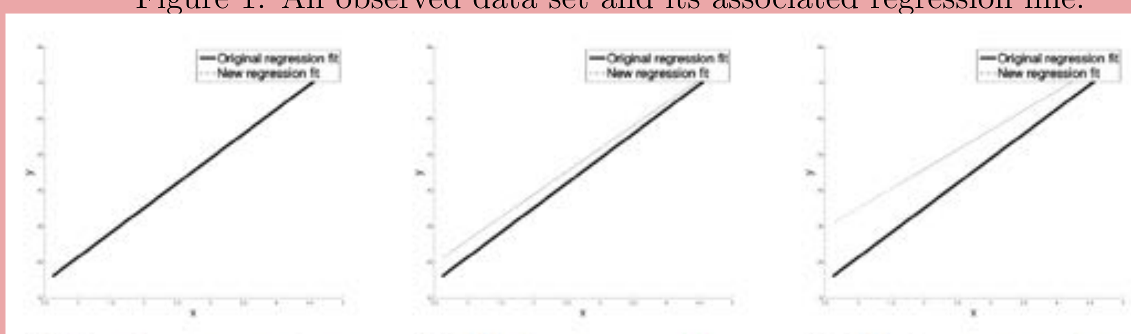


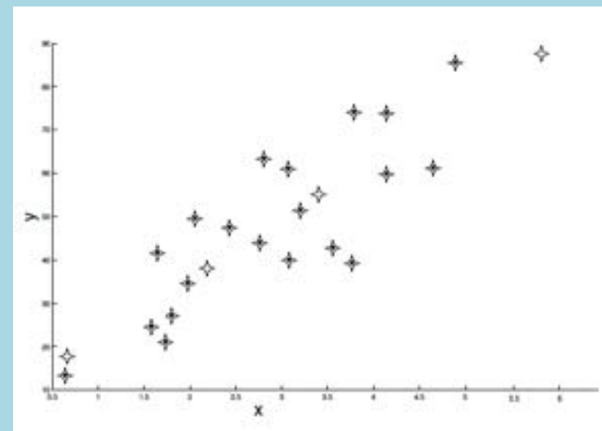
Figure 1: An observed data set and its associated regression line.



(a) Old and new regression lines. (b) Old and new regression lines. (c) Old and new regression lines.

Figure 2: New regression lines for altered data sets S^{new}.

Dataset



(e) Duplicating the original data set and adding four points that lie on the trajectory of the original regression line.

Matching Game

Goal: Match the Algorithm to its Update Rule

1. SGD for Logistic Regression $h_{\boldsymbol{\theta}}(\mathbf{x}) = p(y=1\|\mathbf{x})$	**4.** $\theta_k \leftarrow \theta_k + (h_{\boldsymbol{\theta}}(\mathbf{x}^{(i)}) - y^{(i)})$
2. Least Mean Squares $h_{\boldsymbol{\theta}}(\mathbf{x}) = \boldsymbol{\theta}^T\mathbf{x}$	**5.** $\theta_k \leftarrow \theta_k + \frac{1}{1+\exp \lambda(h_{\boldsymbol{\theta}}(\mathbf{x}^{(i)}) - y^{(i)})}$
3. Perceptron $h_{\boldsymbol{\theta}}(\mathbf{x}) = \text{sign}(\boldsymbol{\theta}^T\mathbf{x})$	**6.** $\theta_k \leftarrow \theta_k + \lambda(h_{\boldsymbol{\theta}}(\mathbf{x}^{(i)}) - y^{(i)})x_k^{(i)}$

A. 1=5, 2=4, 3=6
B. 1=5, 2=6, 3=4
C. 1=6, 2=4, 3=4
D. 1=5, 2=6, 3=6
E. 1=6, 2=6, 3=6
F. 1=6, 2=5, 3=5
G. 1=5, 2=5, 3=5
H. 1=4, 2=5, 3=6

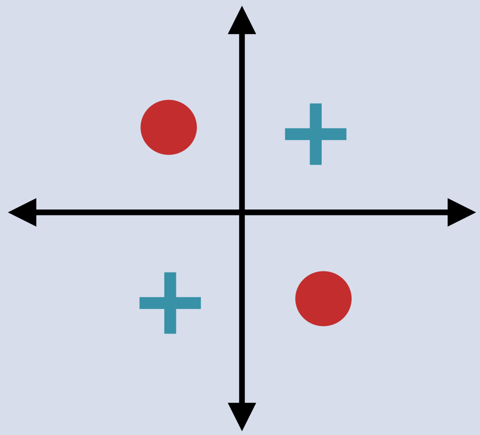
Q&A

Q&A

Q: Why did we focus mostly on the Perceptron mistake bound for **linearly separable data;** isn't that an unrealistic setting?

A: Not at all! Even if your data isn't linearly separable to begin with, we can often add features to make it so.

x_1	x_2	y
+1	+1	+
+1	-1	-
-1	+1	-
-1	-1	+



Exercise: Add another feature to transform this nonlinearly separable data into linearly separable data.

OPTIMIZATION METHOD #3: STOCHASTIC GRADIENT DESCENT

Gradient Descent

Algorithm 1 Gradient Descent

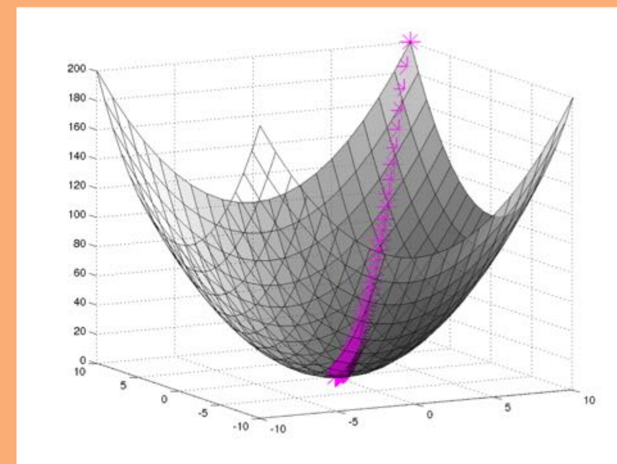
procedure GD($\mathcal{D}$, $\boldsymbol{\theta}^{(0)}$)

  $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta}^{(0)}$

  while not converged **do**

    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \gamma \nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$

  return $\boldsymbol{\theta}$

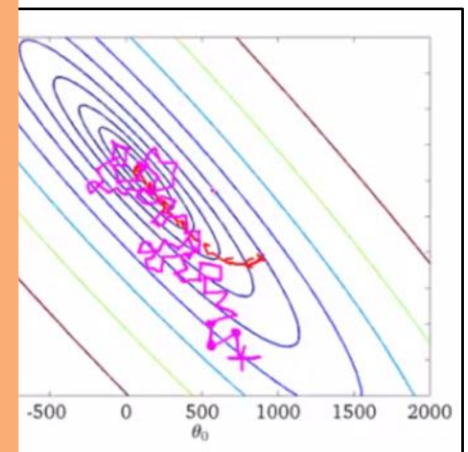


M

Stochastic Gradient Descent (SGD)

Algorithm 2 Stochastic Gradient Descent (SGD)

procedure SGD($\mathcal{D}, \boldsymbol{\theta}^{(0)}$)
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta}^{(0)}$
    while not converged **do**
        $i \sim$ Uniform($\{1, 2, \ldots, N\}$)
        $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \lambda \nabla_{\boldsymbol{\theta}} J^{(i)}(\boldsymbol{\theta})$
    return $\boldsymbol{\theta}$



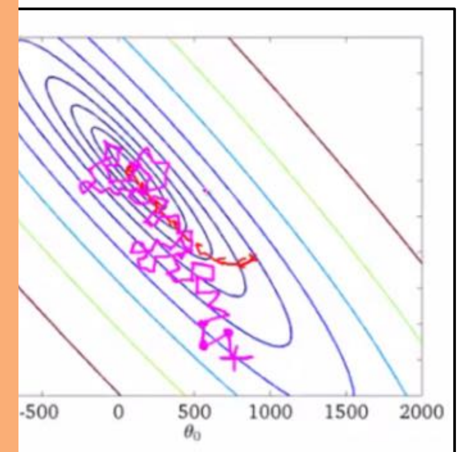
We need a per-example objective:

$$\text{Let } J(\boldsymbol{\theta}) = \sum_{i=1}^{N} J^{(i)}(\boldsymbol{\theta})$$

Stochastic Gradient Descent (SGD)

Algorithm 2 Stochastic Gradient Descent (SGD)

procedure SGD($\mathcal{D}, \boldsymbol{\theta}^{(0)}$)
  $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta}^{(0)}$
  while not converged **do**
    for $i \in$ shuffle($\{1, 2, \ldots, N\}$) **do**
      $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \gamma \nabla_{\boldsymbol{\theta}} J^{(i)}(\boldsymbol{\theta})$
  return $\boldsymbol{\theta}$

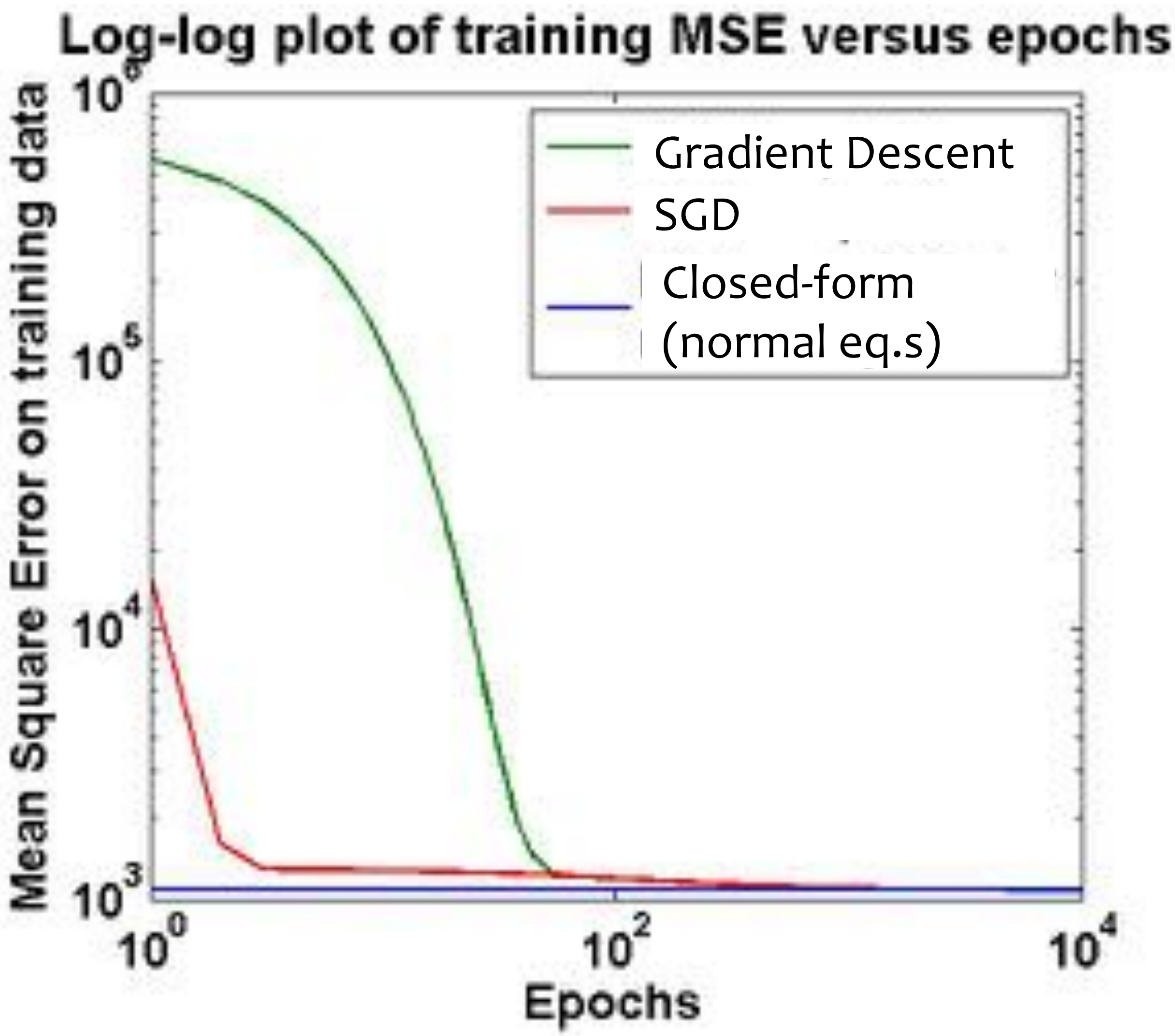


In practice, it is common to implement SGD using sampling **without** replacement (i.e. shuffle({1,2,… N}), even though most of the theory is for sampling **with** replacement (i.e. Uniform({1,2,… N}).

We need a per-example objective:

$$\text{Let } J(\boldsymbol{\theta}) = \sum_{i=1}^{N} J^{(i)}(\boldsymbol{\theta})$$

Convergence Curves

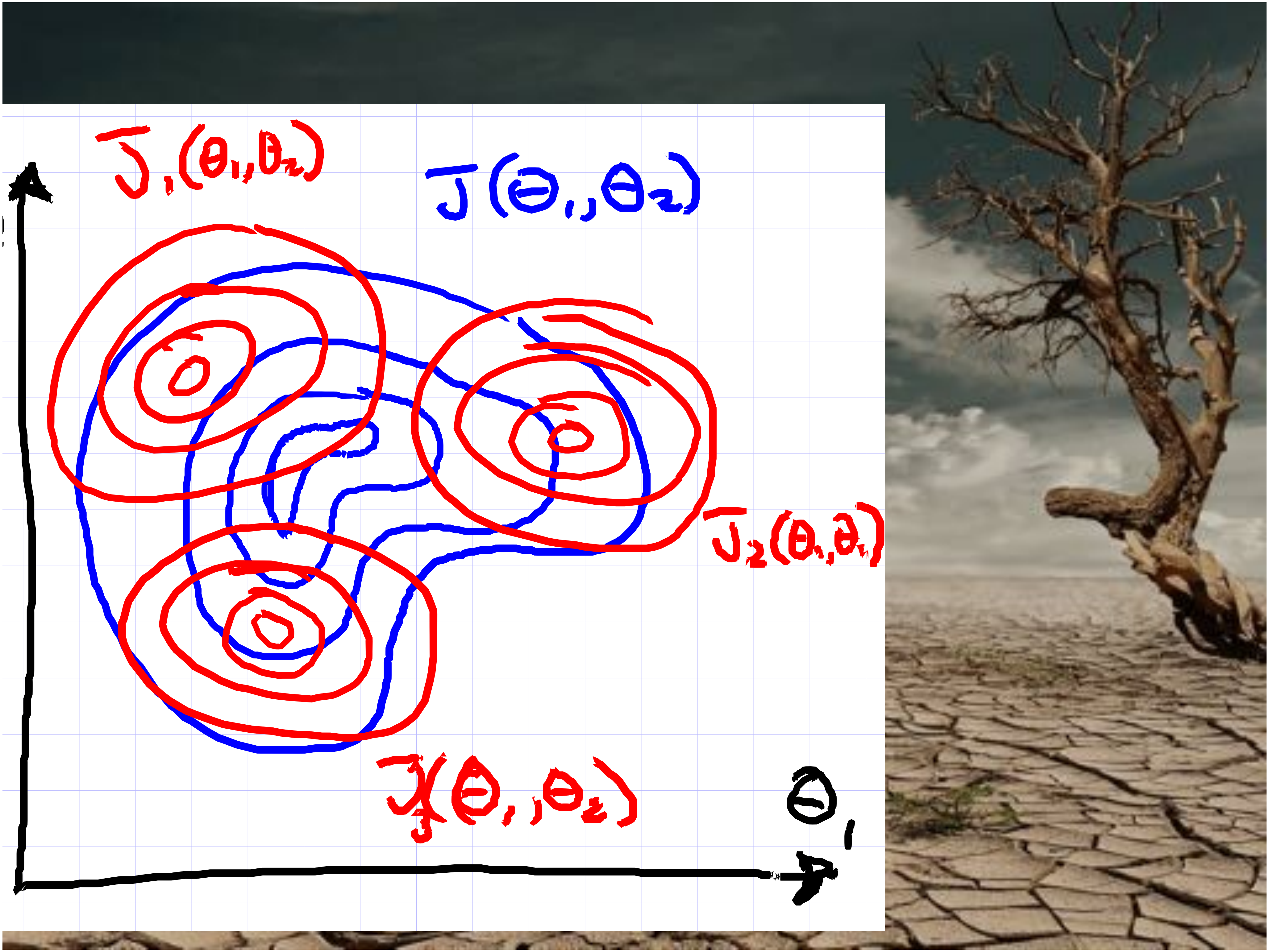


- *Def*: an **epoch** is a single pass through the training data

1. For GD, only **one update** per epoch
2. For SGD, ***N* updates** per epoch
   N = (# train examples)

- SGD reduces MSE much more rapidly than GD
- For GD / SGD, training MSE is initially large due to uninformed initialization

Figure adapted from Eric P. Xing


$J_1(\theta_1,\theta_2)$
$J(\theta_1,\theta_2)$
$J_2(\theta_1,\theta_2)$
$J_3(\theta_1,\theta_2)$
θ_1

Expectations of Gradients

$$\frac{dJ(\vec{\theta})}{d\theta_j} = \frac{1}{N}\sum_{i=1}^{N} \frac{d}{d\theta_j}\left(J_i(\vec{\theta})\right)$$

$$\nabla J(\vec{\theta}) = \begin{array}{r}\\ j^{th}\\ \\ \end{array}\begin{bmatrix} \vdots \\ \hline \\ \vdots \end{bmatrix} = \frac{1}{N}\sum_{i=1}^{N} \nabla J_i(\vec{\theta})$$

Recall: for any discrete r.v. X

$$E_X[f(X)] \triangleq \sum_x P(X=x) f(x)$$

Q: What is the expected value of a randomly chosen $\nabla J_i(\theta)$?

Let $I \sim \text{Uniform}(\{1, \ldots, N\})$

$\Rightarrow P(I=i) = \frac{1}{N}$ if $i \in \{1 \ldots N\}$

$$E_I[\nabla J_I(\vec{\theta})] = \sum_{i=1}^{N} P(I=i) \nabla J_i(\vec{\theta})$$

$$= \frac{1}{N}\sum_{i=1}^{N} \nabla J_i(\vec{\theta})$$

$$= \nabla J(\vec{\theta})$$

Convergence of Optimizers

Convergence Analysis:

true unknown min

Def: convergence is when $J(\vec{\theta}) - J(\vec{\theta}^*) < \epsilon$

not covered

Methods	Steps to Converge	Computation per iteration
Newton's Method	$O(\ln \ln 1/\epsilon)$	$\nabla J(\theta)$ $\nabla^2 J(\theta)$ ← $O(NM^2)$
GD	$O(\ln 1/\epsilon)$	$\nabla J(\theta)$ ← $O(NM)$
SGD	$O(1/\epsilon)$	$\nabla J_i(\theta)$ ← $O(M)$

"almost sure" convergence

lots of caveats and conditions

way less computation

Takeaway: SGD has much slower asymptotic convergence, but is often faster in practice.

SGD FOR LINEAR REGRESSION

Linear Regression as Function Approximation

$\mathcal{D} = \{\mathbf{x}^{(i)}, y^{(i)}\}_{i=1}^{N}$
where $\mathbf{x} \in \mathbb{R}^M$ and $y \in \mathbb{R}$

1. Assume $\mathcal{D}$ generated as:

$$\mathbf{x}^{(i)} \sim p^*(\cdot)$$
$$y^{(i)} = h^*(\mathbf{x}^{(i)})$$

2. Choose hypothesis space, $\mathcal{H}$:
   all linear functions in M-dimensional space

$$\mathcal{H} = \{h_{\boldsymbol{\theta}} : h_{\boldsymbol{\theta}}(\mathbf{x}) = \boldsymbol{\theta}^T \mathbf{x}, \boldsymbol{\theta} \in \mathbb{R}^M\}$$

3. Choose an objective function:
   mean squared error (MSE)

$$\begin{aligned} J(\boldsymbol{\theta}) &= \frac{1}{N} \sum_{i=1}^{N} e_i^2 \\ &= \frac{1}{N} \sum_{i=1}^{N} \left(y^{(i)} - h_{\boldsymbol{\theta}}(\mathbf{x}^{(i)})\right)^2 \\ &= \frac{1}{N} \sum_{i=1}^{N} \left(y^{(i)} - \boldsymbol{\theta}^T \mathbf{x}^{(i)}\right)^2 \end{aligned}$$

4. Solve the unconstrained optimization problem via favorite method:
   - *gradient descent*
   - *closed form*
   - *stochastic gradient descent*
   - ...

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}}{\operatorname{argmin}}\, J(\boldsymbol{\theta})$$

5. Test time: given a new $\mathbf{x}$, make prediction $\hat{y}$

$$\hat{y} = h_{\hat{\boldsymbol{\theta}}}(\mathbf{x}) = \hat{\boldsymbol{\theta}}^T \mathbf{x}$$

Gradient Calculation for Linear Regression

Derivative of $J^{(i)}(\boldsymbol{\theta})$:

$$\begin{aligned}\frac{d}{d\theta_k} J^{(i)}(\boldsymbol{\theta}) &= \frac{d}{d\theta_k} \frac{1}{2}(\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)})^2 \\ &= \frac{1}{2}\frac{d}{d\theta_k}(\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)})^2 \\ &= (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)})\frac{d}{d\theta_k}(\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) \\ &= (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)})\frac{d}{d\theta_k}\left(\sum_{j=1}^{K} \theta_j x_j^{(i)} - y^{(i)}\right) \\ &= (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) x_k^{(i)}\end{aligned}$$

Gradient of $J^{(i)}(\boldsymbol{\theta})$ [used by SGD]

$$\nabla_{\boldsymbol{\theta}} J^{(i)}(\boldsymbol{\theta}) = \begin{bmatrix} \frac{d}{d\theta_1} J^{(i)}(\boldsymbol{\theta}) \\ \frac{d}{d\theta_2} J^{(i)}(\boldsymbol{\theta}) \\ \vdots \\ \frac{d}{d\theta_M} J^{(i)}(\boldsymbol{\theta}) \end{bmatrix} = \begin{bmatrix} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) x_1^{(i)} \\ (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) x_2^{(i)} \\ \vdots \\ (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) x_N^{(i)} \end{bmatrix}$$
$$= (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) \mathbf{x}^{(i)}$$

Derivative of $J(\boldsymbol{\theta})$:

$$\begin{aligned}\frac{d}{d\theta_k} J(\boldsymbol{\theta}) &= \sum_{i=1}^{N} \frac{d}{d\theta_k} J^{(i)}(\boldsymbol{\theta}) \\ &= \sum_{i=1}^{N} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) x_k^{(i)}\end{aligned}$$

Gradient of $J(\boldsymbol{\theta})$ [used by Gradient Descent]

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \begin{bmatrix} \frac{d}{d\theta_1} J(\boldsymbol{\theta}) \\ \frac{d}{d\theta_2} J(\boldsymbol{\theta}) \\ \vdots \\ \frac{d}{d\theta_M} J(\boldsymbol{\theta}) \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) x_1^{(i)} \\ \sum_{i=1}^{N} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) x_2^{(i)} \\ \vdots \\ \sum_{i=1}^{N} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) x_N^{(i)} \end{bmatrix}$$
$$= \sum_{i=1}^{N} (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)}) \mathbf{x}^{(i)}$$

SGD for Linear Regression

SGD applied to Linear Regression is called the "Least Mean Squares" algorithm

---

Algorithm 1 Least Mean Squares (LMS)

---

procedure LMS($\mathcal{D}$, $\boldsymbol{\theta}^{(0)}$)
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta}^{(0)}$ ▷ Initialize parameters
    while not converged **do**
        for $i \in$ shuffle($\{1, 2, \ldots, N\}$) **do**
            $\mathbf{g} \leftarrow (\boldsymbol{\theta}^T \mathbf{x}^{(i)} - y^{(i)})\mathbf{x}^{(i)}$ ▷ Compute gradient
            $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \gamma \mathbf{g}$ ▷ Update parameters
    return $\boldsymbol{\theta}$

GD for Linear Regression

Gradient Descent for Linear Regression repeatedly takes steps opposite the gradient of the objective function

---

Algorithm 1 GD for Linear Regression

1: **procedure** GDLR($\mathcal{D}$, $\boldsymbol{\theta}^{(0)}$)
2: $\quad$ $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta}^{(0)}$ $\qquad$ ▷ Initialize parameters
3: $\quad$ **while** not converged **do**
4: $\quad\quad$ $\mathbf{g} \leftarrow \sum_{i=1}^{N}(\boldsymbol{\theta}^T\mathbf{x}^{(i)} - y^{(i)})\mathbf{x}^{(i)}$ $\qquad$ ▷ Compute gradient
5: $\quad\quad$ $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \gamma\mathbf{g}$ $\qquad$ ▷ Update parameters
6: $\quad$ **return** $\boldsymbol{\theta}$

Optimization Objectives

You should be able to…

- Apply gradient descent to optimize a function
- Apply stochastic gradient descent (SGD) to optimize a function
- Apply knowledge of zero derivatives to identify a closed-form solution (if one exists) to an optimization problem
- Distinguish between convex, concave, and nonconvex functions
- Obtain the gradient (and Hessian) of a (twice) differentiable function

Linear Regression Objectives

You should be able to…

- Design k-NN Regression and Decision Tree Regression
- Implement learning for Linear Regression using three optimization techniques: (1) closed form, (2) gradient descent, (3) stochastic gradient descent
- Choose a Linear Regression optimization technique that is appropriate for a particular dataset by analyzing the tradeoff of computational complexity vs. convergence speed
- Distinguish the three sources of error identified by the bias-variance decomposition: bias, variance, and irreducible error.

PROBABILISTIC LEARNING

Probabilistic Learning

Function Approximation

Previously, we assumed that our output was generated using a **deterministic target function:**

$$\mathbf{x}^{(i)} \sim p^*(\cdot)$$

$$y^{(i)} = c^*(\mathbf{x}^{(i)})$$

Our goal was to learn a hypothesis h(**x**) that best approximates $c^*(\mathbf{x})$

Probabilistic Learning

Today, we assume that our output is **sampled** from a conditional **probability distribution:**

$$\mathbf{x}^{(i)} \sim p^*(\cdot)$$

$$y^{(i)} \sim p^*(\cdot|\mathbf{x}^{(i)})$$

Our goal is to learn a probability distribution p(y|**x**) that best approximates $p^*(y|\mathbf{x})$

Robotic Farming

	Deterministic	**Probabilistic**
Classification (binary output)	Is this a picture of a wheat kernel?	Is this plant drought resistant?
Regression (continuous output)	How many wheat kernels are in this picture?	What will the yield of this plant be?

Bayes Optimal Classifier

Whiteboard

- Bayes Optimal Classifier
- Reducible / irreducible error
- Ex: Bayes Optimal Classifier for 0/1 Loss

Maximum Likelihood Estimation

The principle of Maximum likelihood estimator (MLE):

Choose parameters that make the data "most likely".

Assumptions: Data generated iid from distribution $p^*(x|\vec{\theta}^*)$ and comes from a family of distn parameterized $\theta \in \Theta$ ← set of possible parameters

Formally:

$$\theta_{MLE} = \underset{\theta \in \Theta}{\operatorname{argmax}}\ p(D|\theta)$$

$$= \underset{\theta \in \Theta}{\operatorname{argmax}}\ \log p(D|\theta)$$

$$= \underset{\theta \in \Theta}{\operatorname{argmax}}\ \ell(\theta)$$

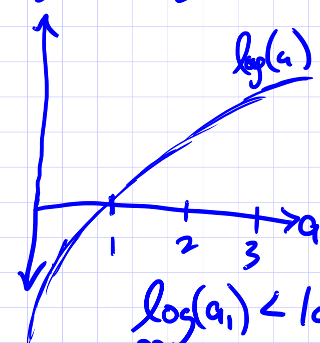
usually a continuous optimization

where $\ell(\theta) \triangleq \log p(D|\theta)$

"log-likelihood"

treat as function of θ where D is constant

since log is monotonic



$\log(a_1) < \log(a_2)$ iff $a_1 < a_2$

$\Rightarrow \log(f(a_1)) < \log(f(a_2))$ iff $f(a_1) < f(a_2)$

MLE

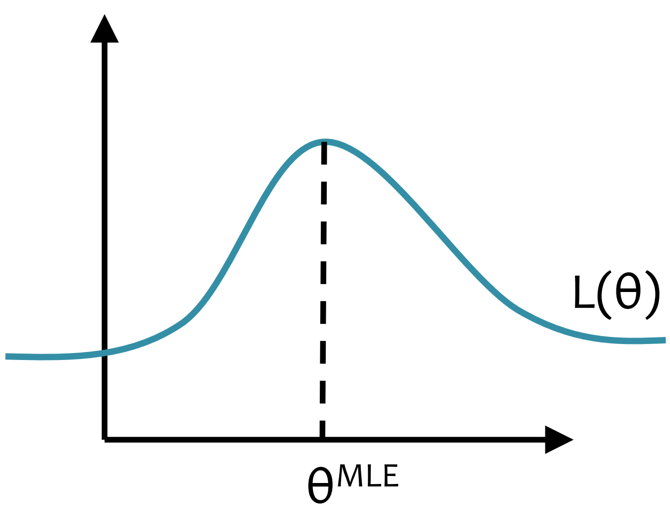
Suppose we have data $\mathcal{D} = \{x^{(i)}\}_{i=1}^N$

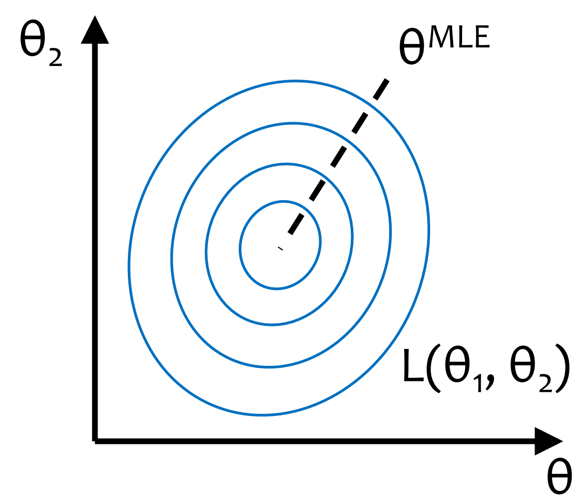
Principle of Maximum Likelihood Estimation:

Choose the parameters that maximize the likelihood of the data.

$$\boldsymbol{\theta}^{\text{MLE}} = \underset{\boldsymbol{\theta}}{\text{argmax}} \prod_{i=1}^{N} p(\mathbf{x}^{(i)}|\boldsymbol{\theta})$$

Maximum Likelihood Estimate (MLE)





MLE

What does maximizing likelihood accomplish?

- There is only a finite amount of probability mass (i.e. sum-to-one constraint)
- MLE tries to allocate **as much** probability mass **as possible** to the things we have observed…

  … **at the expense** of the things we have **not** observed

Learning from Data (Frequentist)

Whiteboard

- Principle of Maximum Likelihood Estimation (MLE)
- Strawmen:
  - Example: Bernoulli
  - Example: Gaussian
  - Example: Conditional #1
    (Bernoulli conditioned on Gaussian)
  - Example: Conditional #2
    (Gaussians conditioned on Bernoulli)