10-601 Introduction to Machine Learning

Machine Learning Department
School of Computer Science
Carnegie Mellon University

Reinforcement Learning:
Value Iteration

Matt Gormley
Lecture 23
Apr. 10, 2019

Reminders

- **Homework 7: HMMs**
  - Out: Fri, Mar 29
  - Due: Mon, Apr 15 at 11:59pm
- **Homework 8: Reinforcement Learning**
  - Out: Wed, Apr 10
  - Due: Wed, Apr 24 at 11:59pm

- **Today's In-Class Poll**
  - http://p23.mlcourse.org

Q&A

MARKOV DECISION PROCESSES

Markov Decision Process

- For **supervised learning** the **PAC learning framework** provided assumptions about where our data came from:

$$\mathbf{x} \sim p^*(\cdot) \text{ and } y = c^*(\cdot)$$

- For **reinforcement learning** we assume our data comes from a **Markov decision process** (MDP)

Markov Decision Process

Whiteboard

- Components: states, actions, state transition probabilities, reward function
- Markovian assumption
- MDP Model
- MDP Goal: Infinite-horizon Discounted Reward
- deterministic vs. nondeterministic MDP
- deterministic vs. stochastic policy

Exploration vs. Exploitation

Whiteboard

– Explore vs. Exploit Tradeoff
– Ex: k-Armed Bandits
– Ex: Traversing a Maze

FIXED POINT ITERATION

Fixed Point Iteration for Optimization

- Fixed point iteration is a general tool for solving systems of equations
- It can also be applied to optimization.

$$J(\boldsymbol{\theta})$$

$$\frac{dJ(\boldsymbol{\theta})}{d\theta_i} = 0 = f(\boldsymbol{\theta})$$

$$0 = f(\boldsymbol{\theta}) \Rightarrow \theta_i = g(\boldsymbol{\theta})$$

$$\theta_i^{(t+1)} = g(\boldsymbol{\theta}^{(t)})$$

1. Given objective function:
2. Compute derivative, set to zero (call this function f).
3. Rearrange the equation s.t. one of parameters appears on the LHS.
4. Initialize the parameters.
5. For i in $\{1,...,K\}$, update each parameter and increment t:
6. Repeat #5 until convergence

Fixed Point Iteration for Optimization

- Fixed point iteration is a general tool for solving systems of equations
- It can also be applied to optimization.

$$J(x) = \frac{x^3}{3} + \frac{3}{2}x^2 + 2x$$

$$\frac{dJ(x)}{dx} = f(x) = x^2 - 3x + 2 = 0$$

$$\Rightarrow x = \frac{x^2 + 2}{3} = g(x)$$

$$x \leftarrow \frac{x^2 + 2}{3}$$

1. Given objective function:
2. Compute derivative, set to zero (call this function *f*).
3. Rearrange the equation s.t. one of parameters appears on the LHS.
4. Initialize the parameters.
5. For *i* in *{1,...,K}*, update each parameter and increment *t*:
6. Repeat #5 until convergence

Fixed Point Iteration for Optimization

We can implement our example in a few lines of python.

$$J(x) = \frac{x^3}{3} + \frac{3}{2}x^2 + 2x$$

$$\frac{dJ(x)}{dx} = f(x) = x^2 - 3x + 2 = 0$$

$$\Rightarrow x = \frac{x^2 + 2}{3} = g(x)$$

$$x \leftarrow \frac{x^2 + 2}{3}$$

```
def f1(x):
    '''f(x) = x^2 - 3x + 2'''
    return x**2 - 3.*x + 2.

def g1(x):
    '''g(x) = \frac{x^2 + 2}{3}'''
    return (x**2 + 2.) / 3.

def fpi(g, x0, n, f):
    '''Optimizes the 1D function g by fixed point iteration
    starting at x0 and stopping after n iterations. Also
    includes an auxiliary function f to test at each value.'''
    x = x0
    for i in range(n):
        print("i=%2d x=%.4f f(x)=%.4f" % (i, x, f(x)))
        x = g(x)
    i += 1
    print("i=%2d x=%.4f f(x)=%.4f" % (i, x, f(x)))
    return x

if __name__ == "__main__":
    x = fpi(g1, 0, 20, f1)
```

Fixed Point Iteration for Optimization

$$J(x) = \frac{x^3}{3} + \frac{3}{2}x^2 + 2x$$

$$\frac{dJ(x)}{dx} = f(x) = x^2 - 3x + 2 = 0$$

$$\Rightarrow x = \frac{x^2 + 2}{3} = g(x)$$

$$x \leftarrow \frac{x^2 + 2}{3}$$

```
$ python fixed-point-iteration.py
i= 0 x=0.0000 f(x)=2.0000
i= 1 x=0.6667 f(x)=0.4444
i= 2 x=0.8148 f(x)=0.2195
i= 3 x=0.8880 f(x)=0.1246
i= 4 x=0.9295 f(x)=0.0755
i= 5 x=0.9547 f(x)=0.0474
i= 6 x=0.9705 f(x)=0.0304
i= 7 x=0.9806 f(x)=0.0198
i= 8 x=0.9872 f(x)=0.0130
i= 9 x=0.9915 f(x)=0.0086
i=10 x=0.9944 f(x)=0.0057
i=11 x=0.9963 f(x)=0.0038
i=12 x=0.9975 f(x)=0.0025
i=13 x=0.9983 f(x)=0.0017
i=14 x=0.9989 f(x)=0.0011
i=15 x=0.9993 f(x)=0.0007
i=16 x=0.9995 f(x)=0.0005
i=17 x=0.9997 f(x)=0.0003
i=18 x=0.9998 f(x)=0.0002
i=19 x=0.9999 f(x)=0.0001
i=20 x=0.9999 f(x)=0.0001
```

VALUE ITERATION

Definitions for Value Iteration

Whiteboard

- State trajectory
- Value function
- Bellman equations
- Optimal policy
- Optimal value function
- Computing the optimal policy
- Ex: Path Planning

RL Terminology

Question: Match each term (on the left) to the corresponding statement or definition (on the right)

Terms:

A. a reward function
B. a transition probability
C. a policy
D. state/action/reward triples
E. a value function
F. transition function
G. an optimal policy
H. Matt's favorite statement

Statements:

1. gives the expected future discounted reward of a state
2. maps from states to actions
3. quantifies immediate success of agent
4. is a deterministic map from state/action pairs to states
5. quantifies the likelihood of landing a new state, given a state/action pair
6. is the desired output of an RL algorithm
7. can be influenced by trading off between exploitation/exploration

Value Iteration

Whiteboard

– Value Iteration Algorithm

Value Iteration

Algorithm 1 Value Iteration

procedure VALUEITERATION($R(s,a)$ reward function, $p(\cdot|s,a)$ transition probabilities)
  Initialize value function $V(s) = 0$ or randomly
  while not converged **do**
    for $s \in \mathcal{S}$ **do**
      for $a \in \mathcal{A}$ **do**
        $Q(s,a) = R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) V(s')$
      $V(s) = \max_a Q(s,a)$
  Let $\pi(s) = \text{argmax}_a Q(s,a), \ \forall s$
  return π

Value Iteration Convergence

very abridged

Theorem 1 (Bertsekas (1989))
V converges to V^*, if each state is visited infinitely often

Theorem 2 (Williams & Baird (1993))
if $max_s|V^{t+1}(s) - V^t(s)| < \epsilon$
then $max_s|V^{t+1}(s) - V^*(s)| < \frac{2\epsilon\gamma}{1-\gamma}, \forall s$

Theorem 3 (Bertsekas (1987))
greedy policy will be optimal in a finite number of steps (even if not converged to optimal value function!)

Holds for both asynchronous and sychronous updates

Provides reasonable stopping criterion for value iteration

Often greedy policy converges well before the value function