Machine Learning Department
School of Computer Science
Carnegie Mellon University

Neural Networks

Matt Gormley
Lecture 12
Feb. 25, 2019

Reminders

- Homework 4: Logistic Regression
  - Out: Fri, Feb 15
  - Due: Fri, Mar 1 at 11:59pm
- Homework 5: Neural Networks
  - Out: Fri, Mar 1
  - Due: Fri, Mar 22 at 11:59pm

Q&A

NEURAL NETWORKS

Background

A Recipe for Machine Learning

1. Given training data:

$$\{\boldsymbol{x}_i, \boldsymbol{y}_i\}_{i=1}^N$$

Face

Face

Not a face

2. Choose each of these:
   – Decision function

$$\hat{\boldsymbol{y}} = f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)$$

   – Loss function

$$\ell(\hat{\boldsymbol{y}}, \boldsymbol{y}_i) \in \mathbb{R}$$

Examples: Linear regression, Logistic regression, Neural Network

Examples: Mean-squared error, Cross Entropy

Background

A Recipe for Machine Learning

1. Given training data:

$$\{\boldsymbol{x}_i, \boldsymbol{y}_i\}_{i=1}^N$$

2. Choose each of these:

– Decision function

$$\hat{\boldsymbol{y}} = f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)$$

– Loss function

$$\ell(\hat{\boldsymbol{y}}, \boldsymbol{y}_i) \in \mathbb{R}$$

3. Define goal:

$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^N \ell(f_{\boldsymbol{\theta}}(\boldsymbol{x}_i), \boldsymbol{y}_i)$$

4. Train with SGD:
(take small steps opposite the gradient)

$$\boldsymbol{\theta}^{(t+1)} = \boldsymbol{\theta}^{(t)} - \eta_t \nabla \ell(f_{\boldsymbol{\theta}}(\boldsymbol{x}_i), \boldsymbol{y}_i)$$

Background

A Recipe for

1. Given training dat

$$\{\boldsymbol{x}_i, \boldsymbol{y}_i\}_{i=1}^N$$

2. Choose each of th

– Decision functio

$$\hat{y} = f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)$$

– Loss function

$$\ell(\hat{\boldsymbol{y}}, \boldsymbol{y}_i) \in \mathbb{R}$$

$\boldsymbol{y}_i)$

opp site the gradient)

$$\boldsymbol{\theta}^{(t} \quad {}^{(t)} - \eta_t \nabla \ell(f_{\boldsymbol{\theta}}(\boldsymbol{x}_i), \boldsymbol{y}_i)$$

Gradients

Backpropagation can compute this gradient!

And it's a **special case of a more general algorithm** called reverse-mode automatic differentiation that can compute the gradient of any differentiable function efficiently!

Goals for Today's Lecture

1. Explore a **new class of decision functions** (Neural Networks)
2. Consider **variants of this recipe** for training

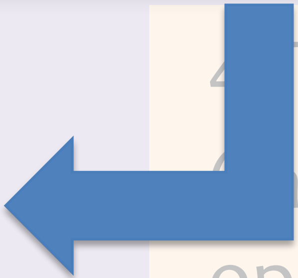
A Recipe for

2. Choose each of these:
   - Decision function

$$\hat{y} = f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)$$

   - Loss function

$$\ell(\hat{\boldsymbol{y}}, \boldsymbol{y}_i) \in \mathbb{R}$$



4. Train with SGD:
(take small steps opposite the gradient)

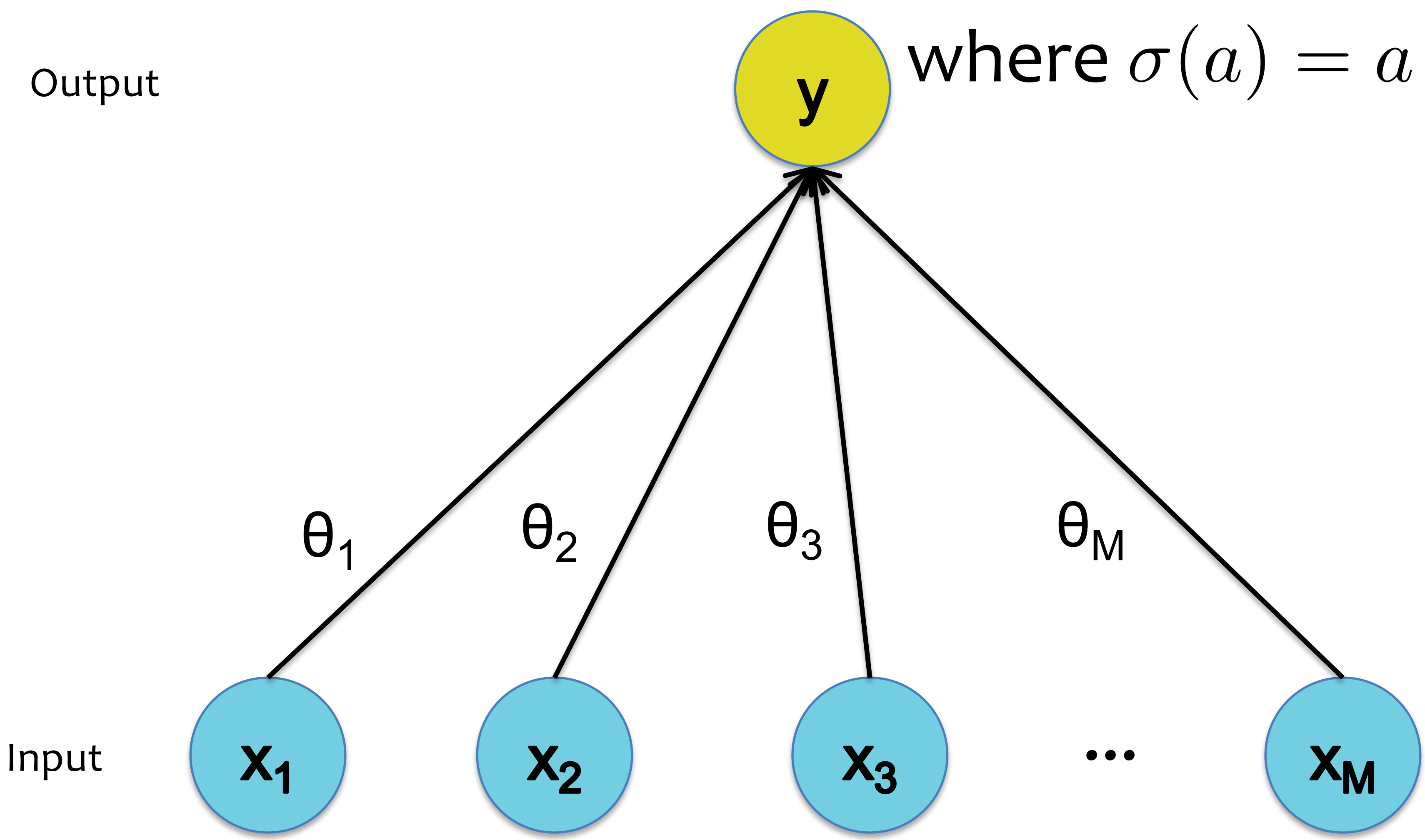
$$\boldsymbol{\theta}^{(t+1)} = \boldsymbol{\theta}^{(t)} - \eta_t \nabla \ell(f_{\boldsymbol{\theta}}(\boldsymbol{x}_i), \boldsymbol{y}_i)$$

Decision Functions

Linear Regression

$$y = h_{\boldsymbol{\theta}}(\boldsymbol{x}) = \sigma(\boldsymbol{\theta}^T \boldsymbol{x})$$

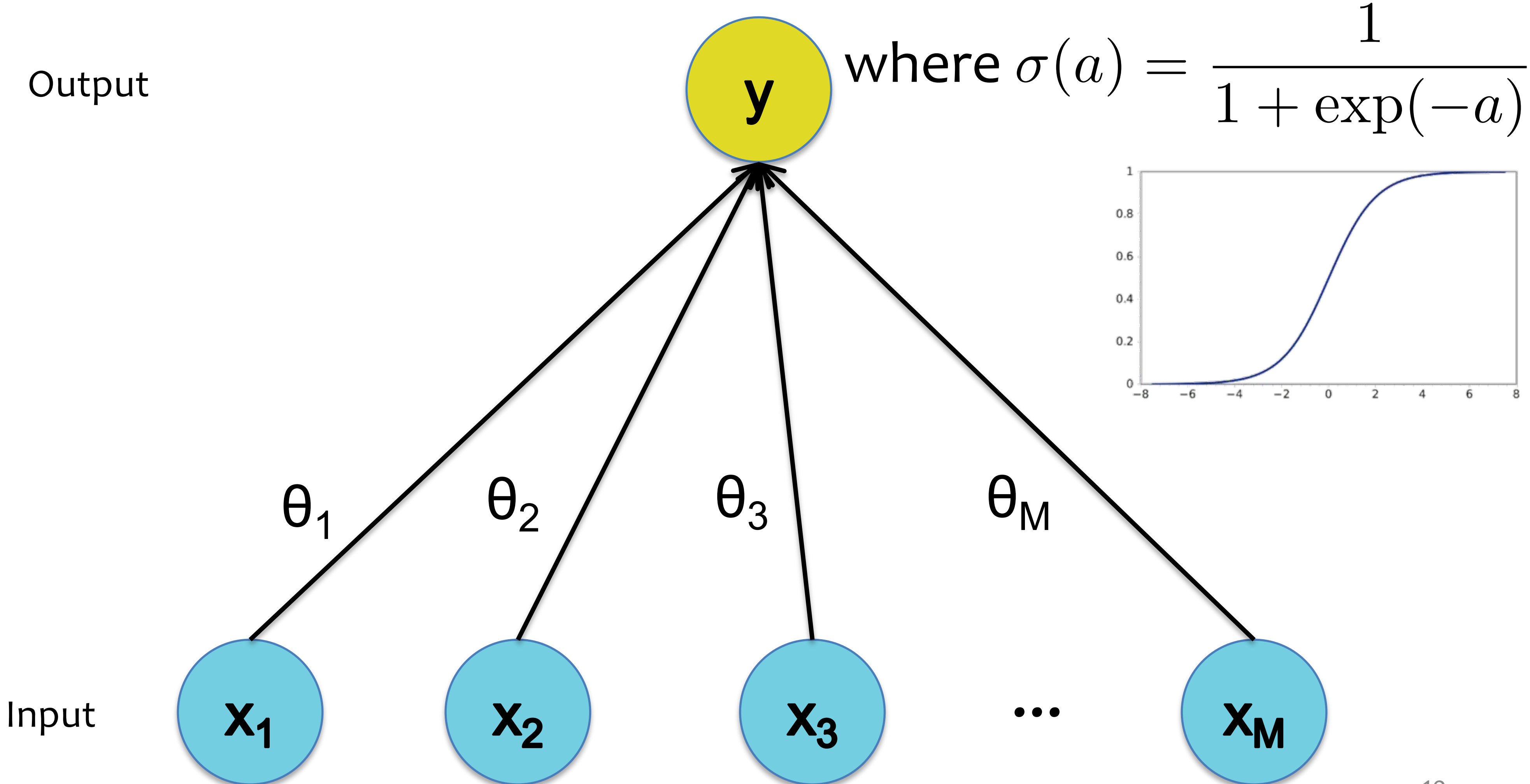
where $\sigma(a) = a$



Logistic Regression

$$y = h_{\boldsymbol{\theta}}(\boldsymbol{x}) = \sigma(\boldsymbol{\theta}^T \boldsymbol{x})$$

where $\sigma(a) = \dfrac{1}{1 + \exp(-a)}$

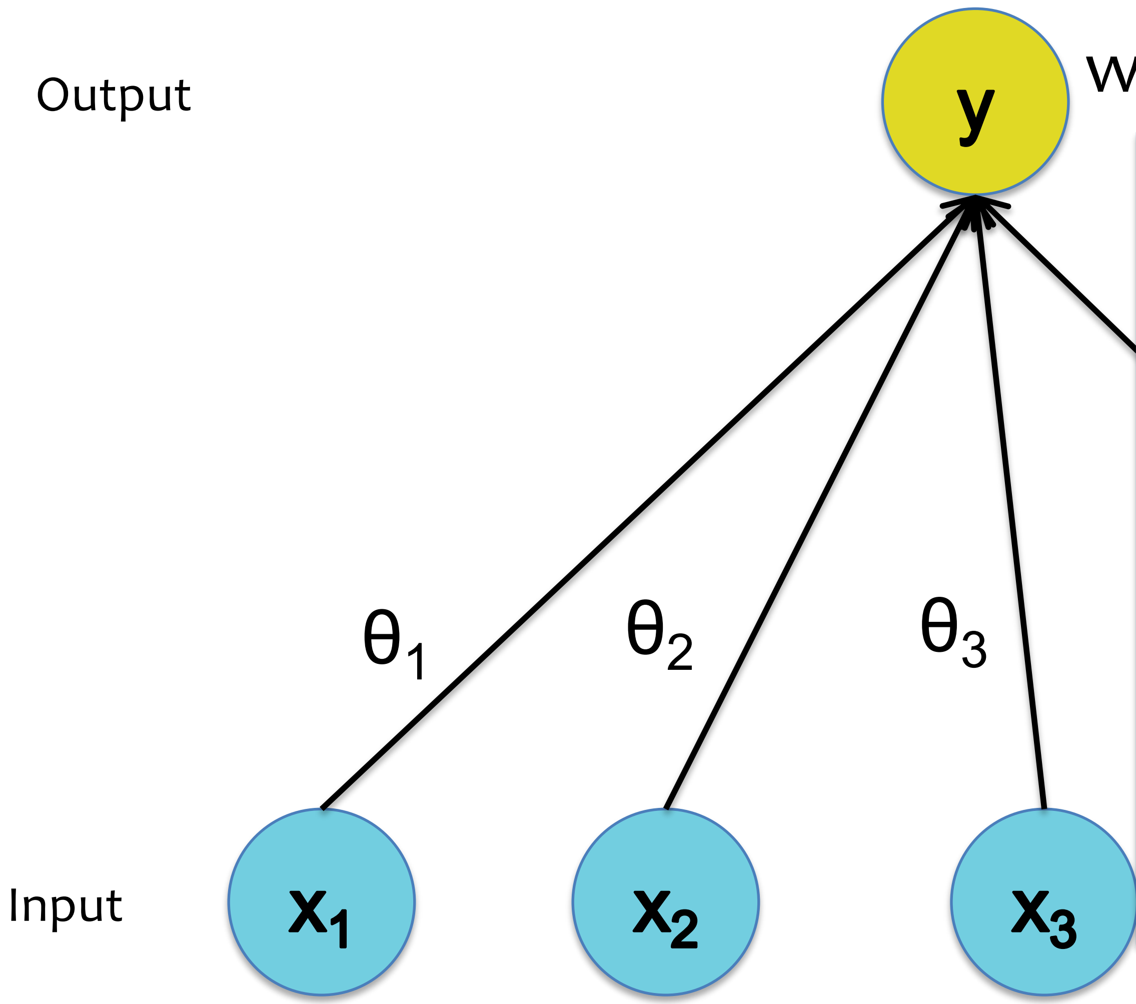


Decision Functions

Logistic Regression

$$y = h_{\boldsymbol{\theta}}(\boldsymbol{x}) = \sigma(\boldsymbol{\theta}^T \boldsymbol{x})$$

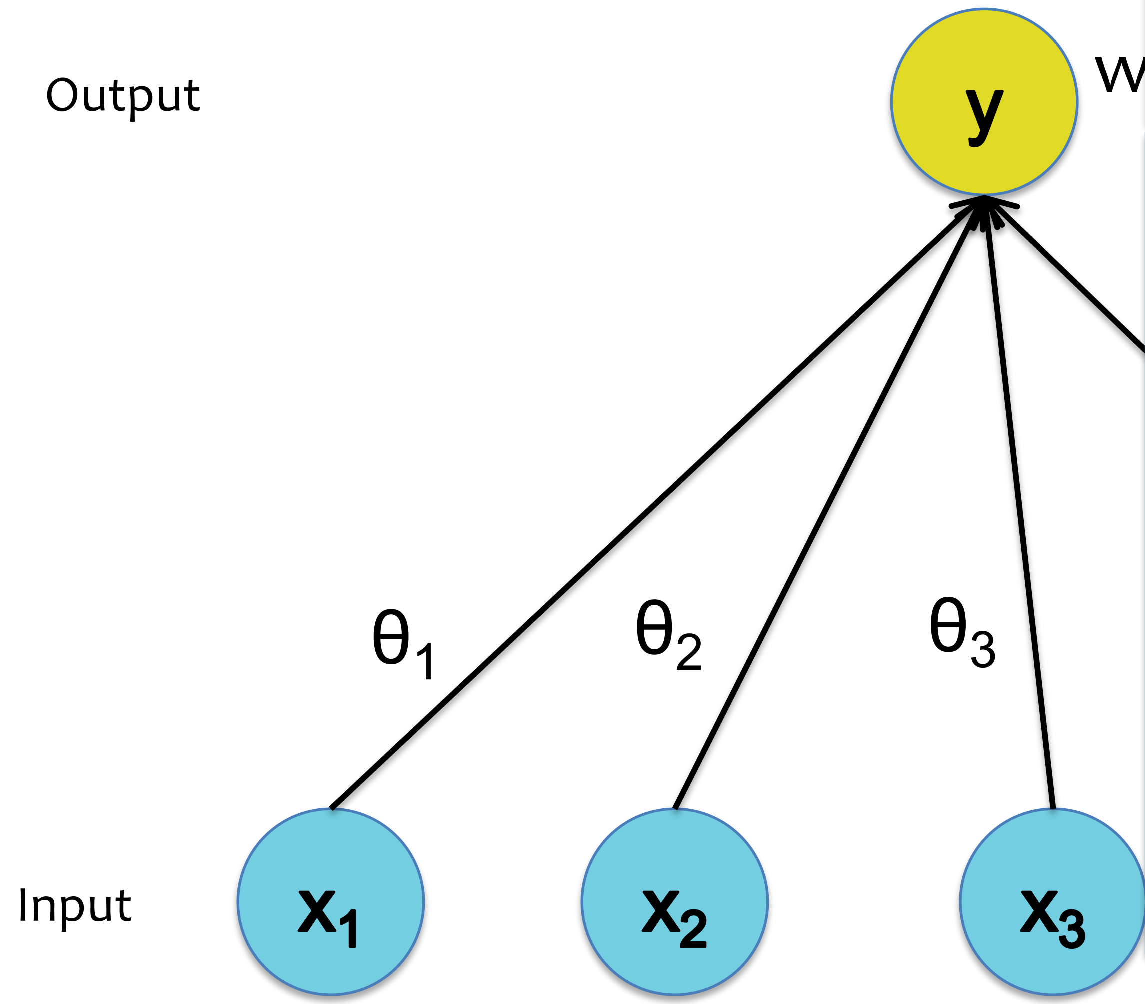
where $\sigma(a) = \frac{1}{1 + \exp(-a)}$



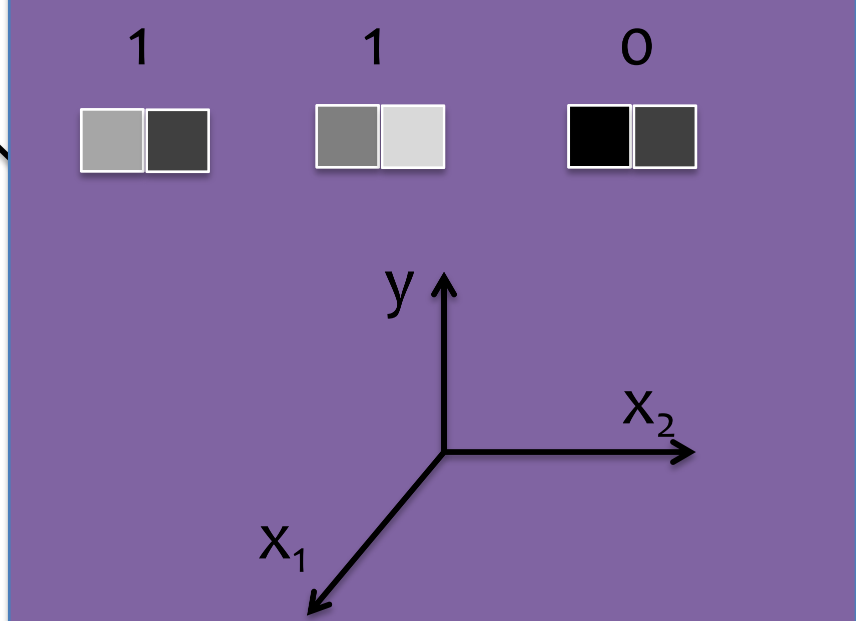
Decision Functions

Logistic Regression

$$y = h_{\boldsymbol{\theta}}(\boldsymbol{x}) = \sigma(\boldsymbol{\theta}^T \boldsymbol{x})$$



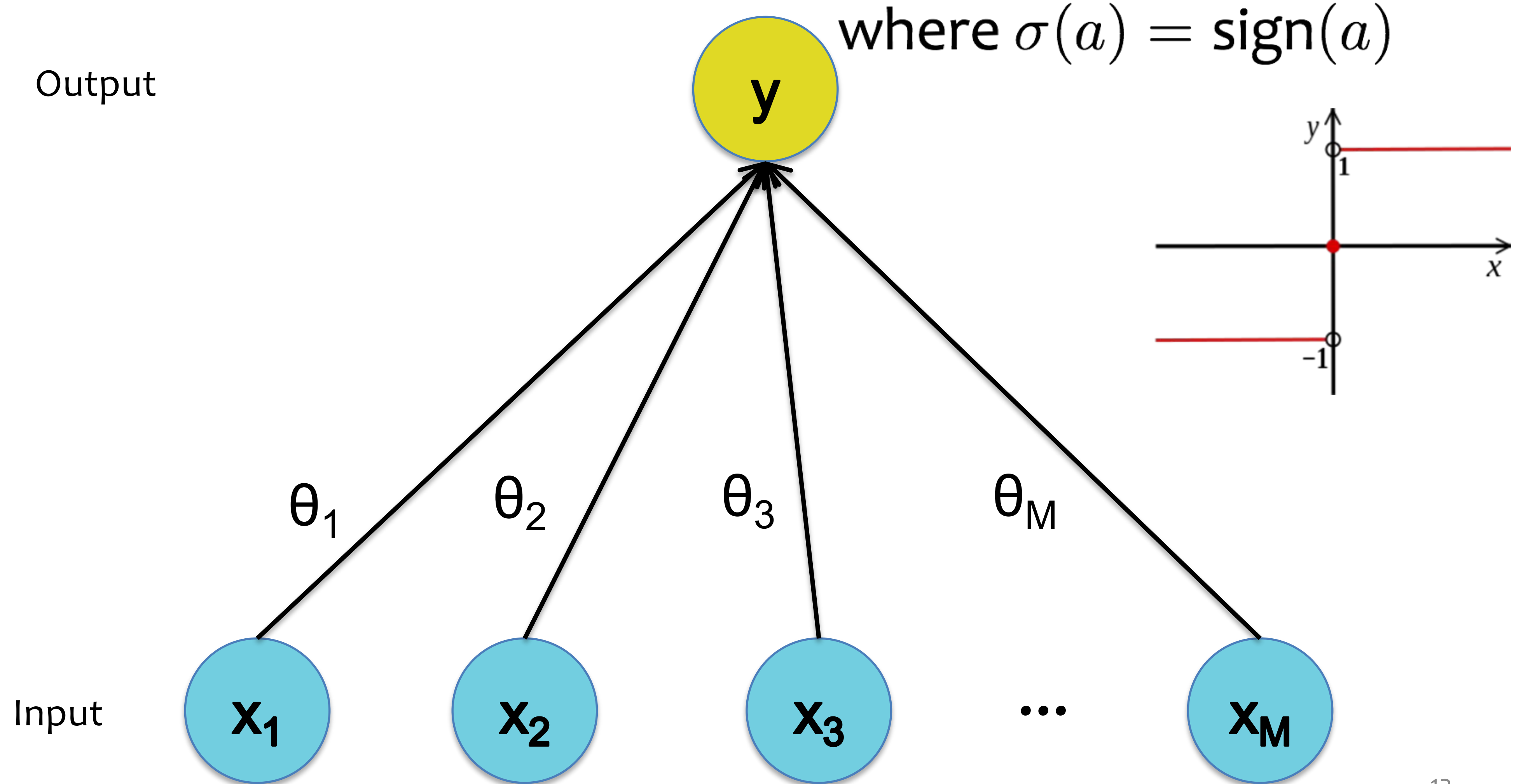
In-Class Example



Perceptron

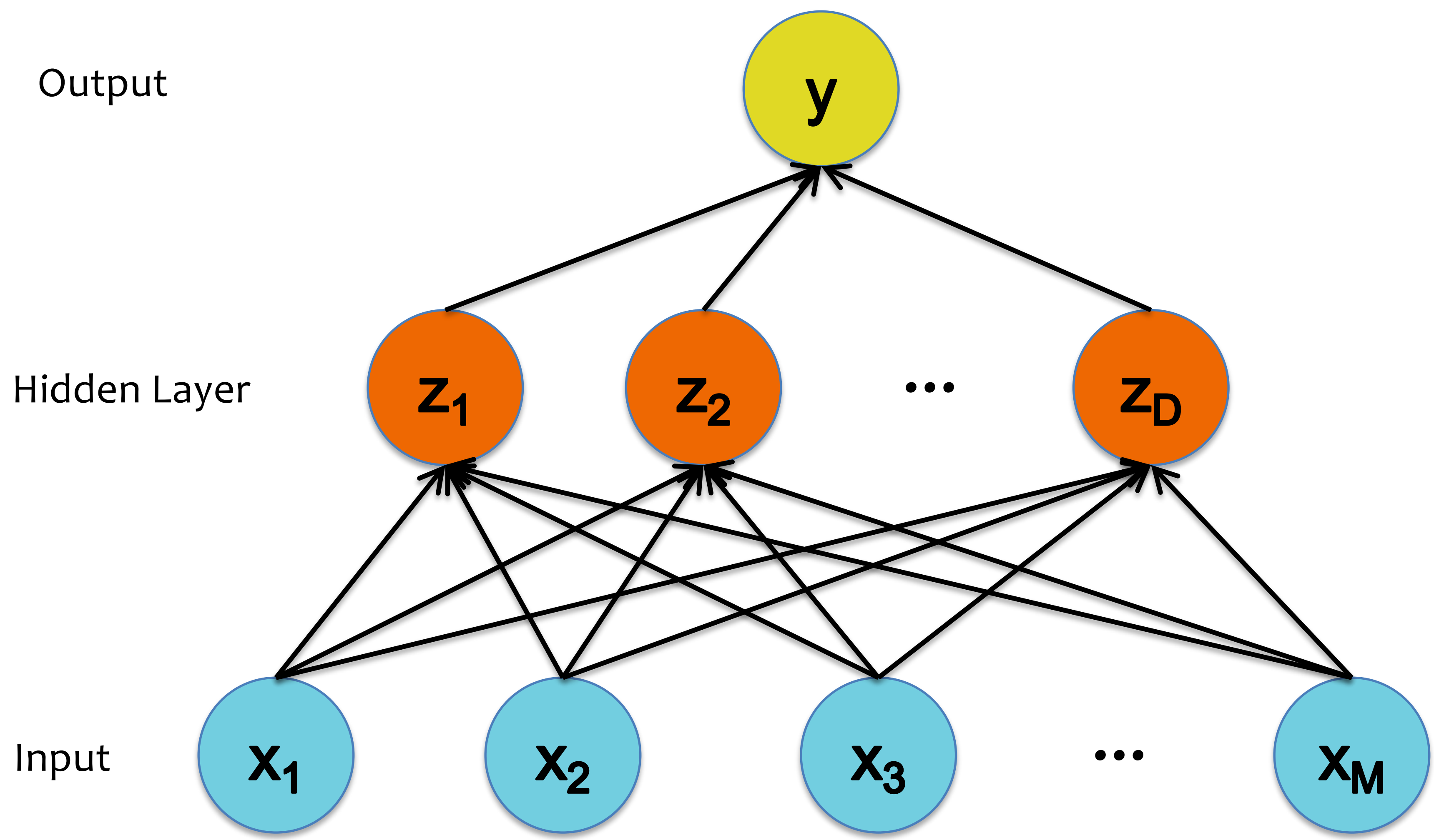
$$y = h_{\boldsymbol{\theta}}(\boldsymbol{x}) = \sigma(\boldsymbol{\theta}^T \boldsymbol{x})$$

where $\sigma(a) = \text{sign}(a)$

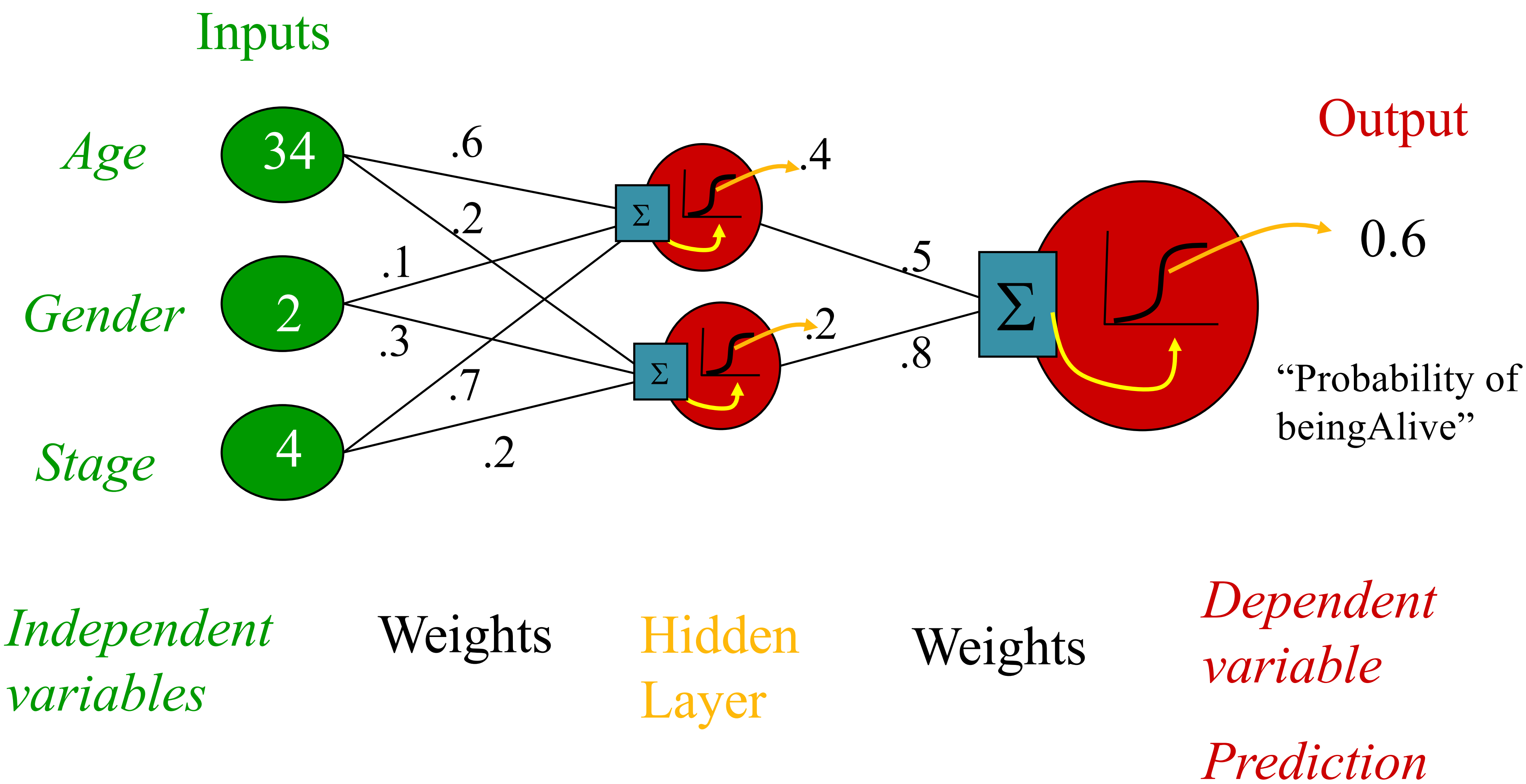


Decision Functions

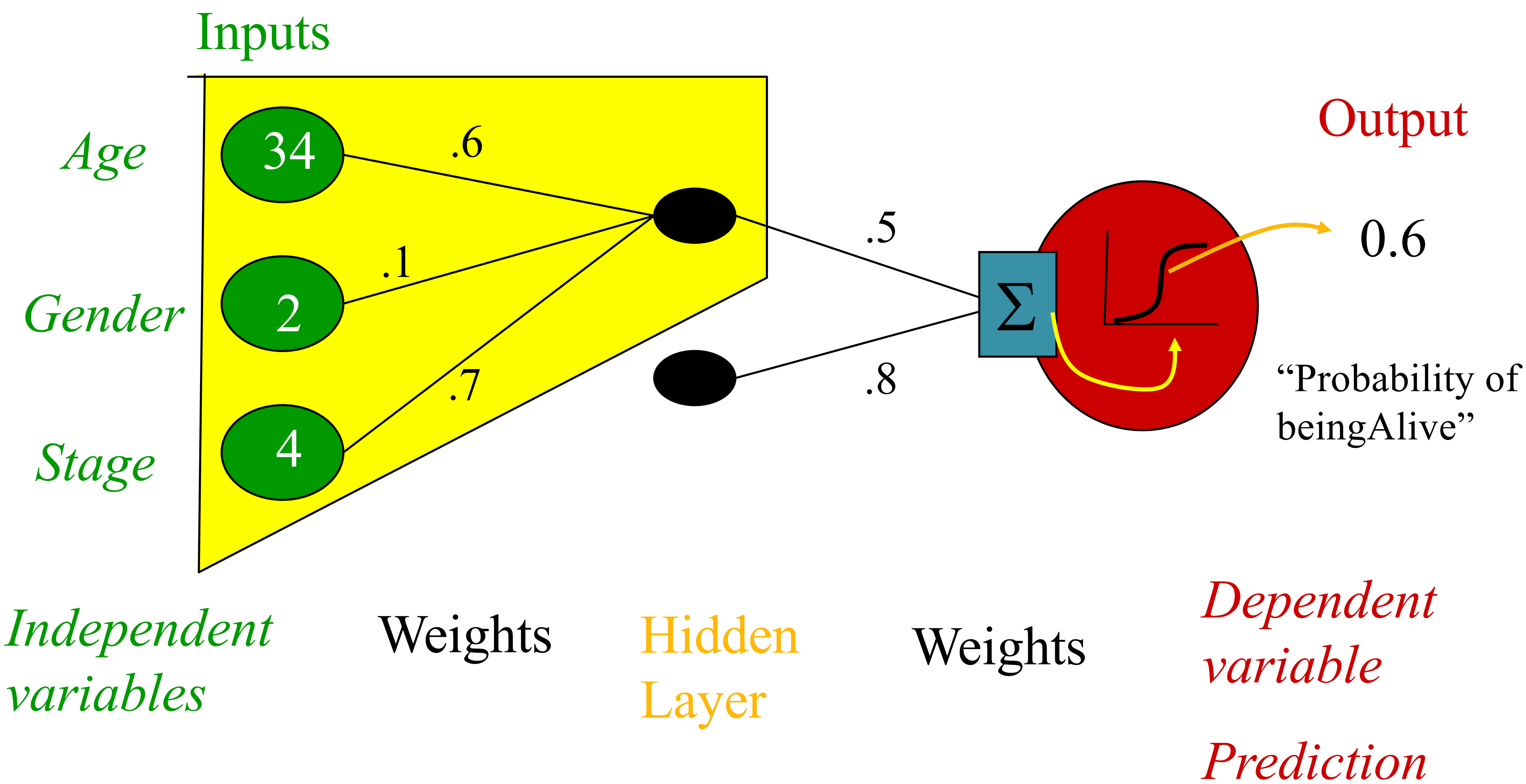
Neural Network

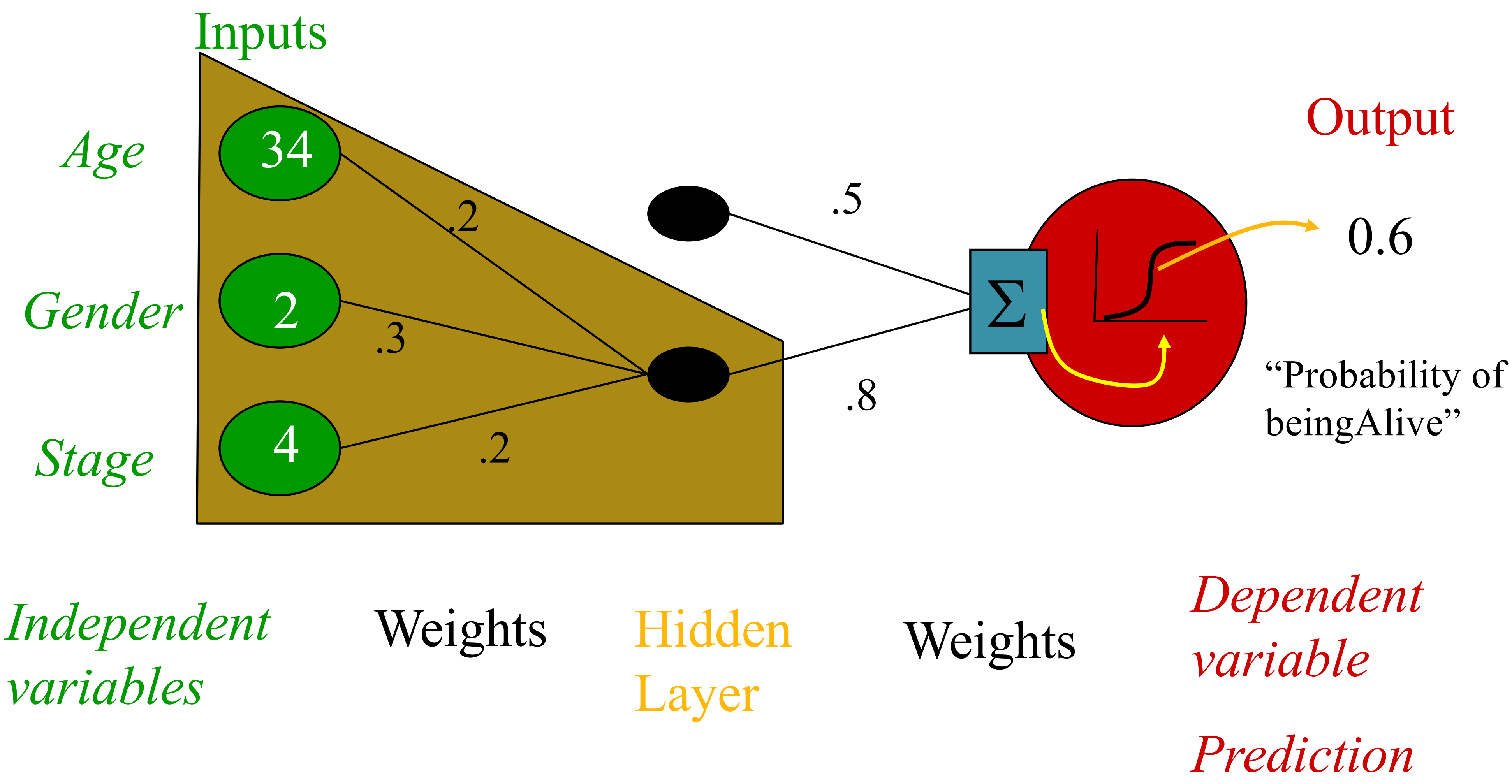


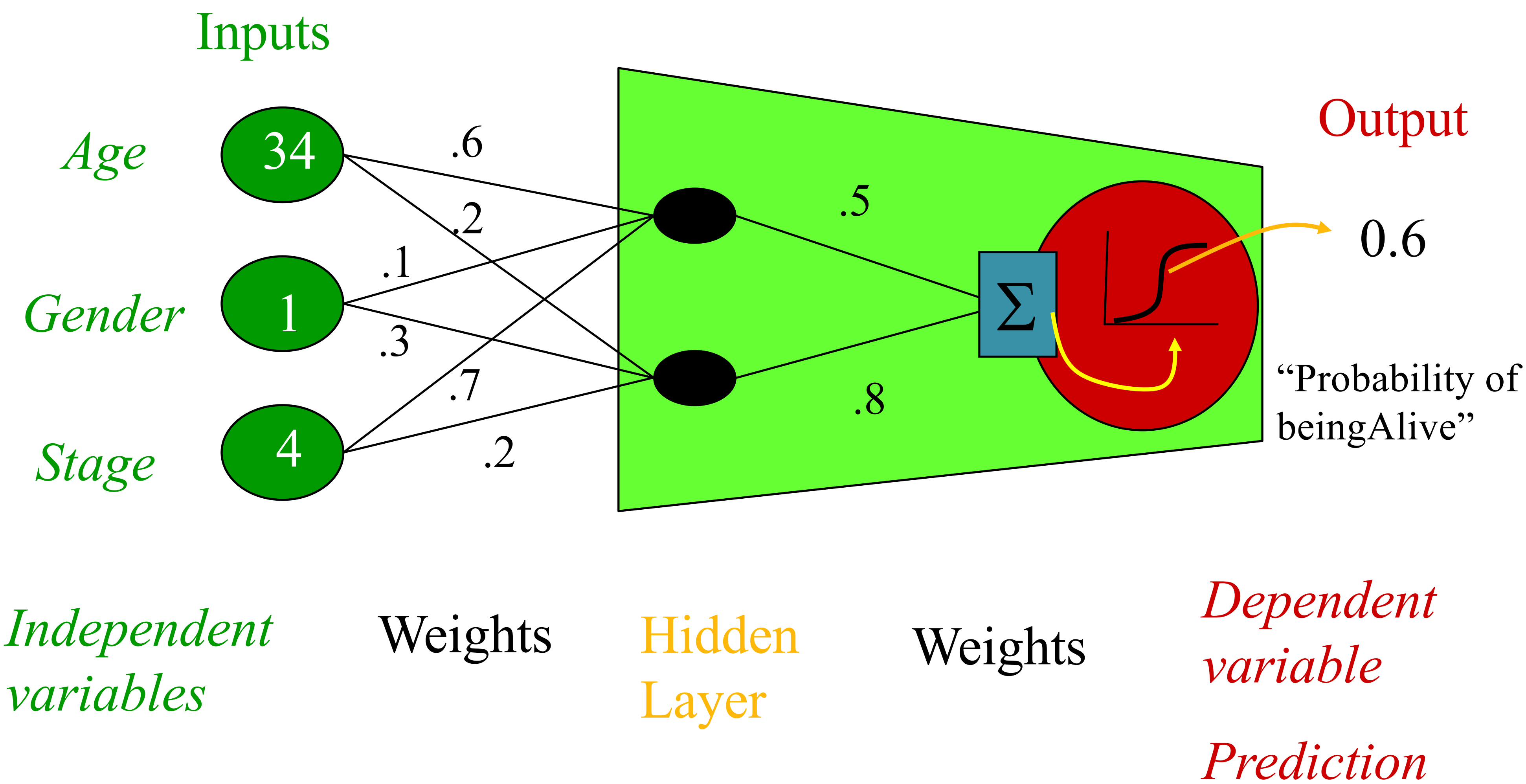
Neural Network Model



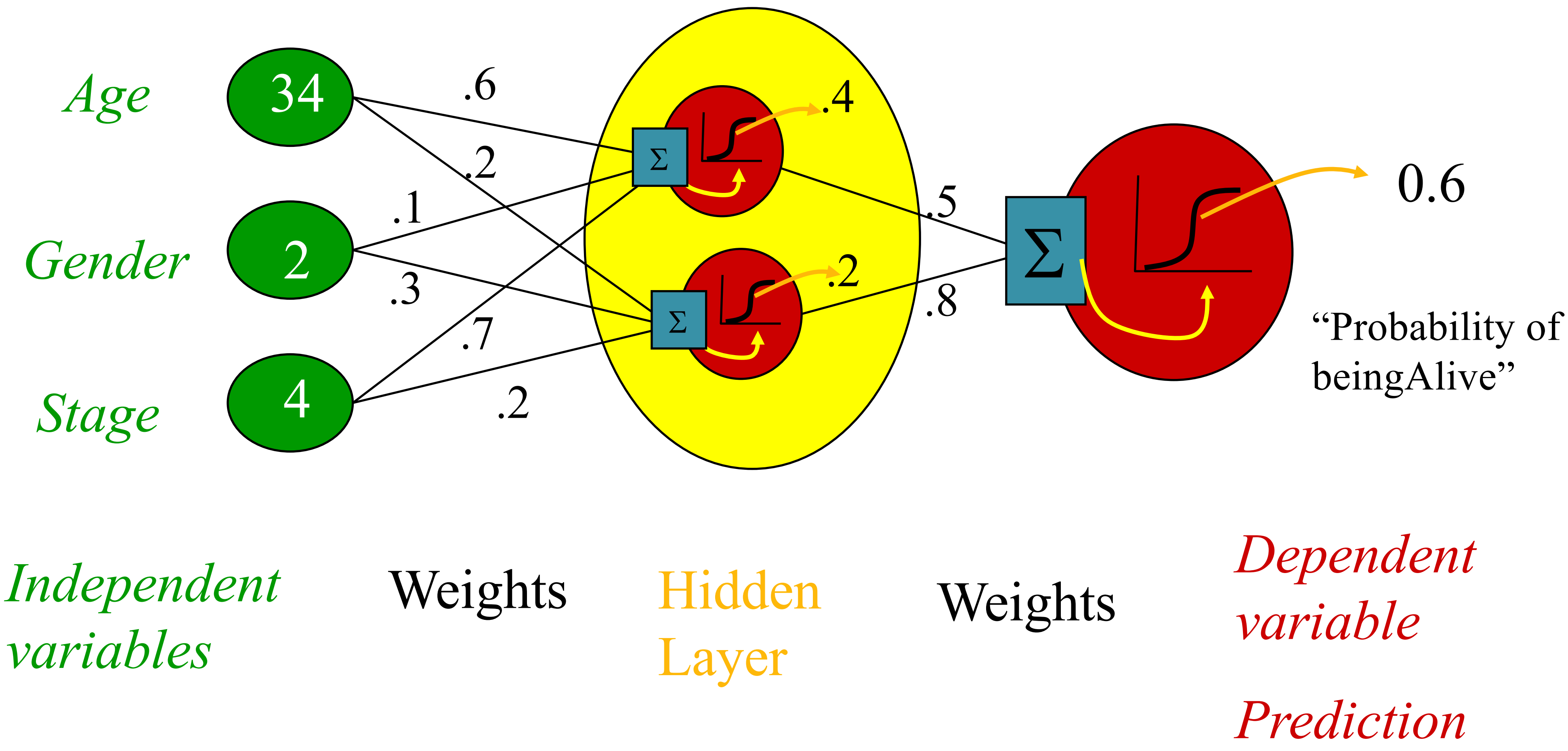
"Combined logistic models"




Inputs
Age
34
Gender
2
Stage
4
.2
.3
.2
.5
.8
Σ
Output
0.6
"Probability of beingAlive"
Independent variables
Weights
Hidden Layer
Weights
Dependent variable
Prediction

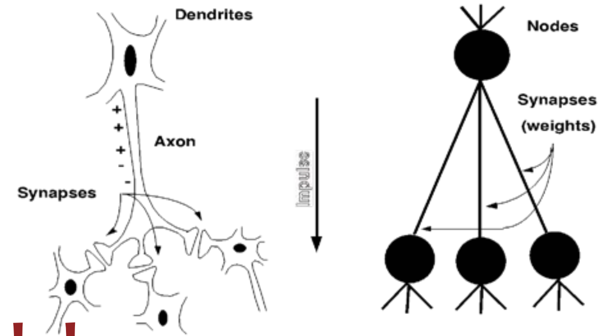

Inputs
Age
34
Gender
1
Stage
4
.6
.2
.1
.3
.7
.2
.5
.8
Σ
Output
0.6
"Probability of beingAlive"
Independent variables
Weights
Hidden Layer
Weights
Dependent variable
Prediction

Not really, no target for hidden units...



From Biological to Artificial

The motivation for Artificial Neural Networks comes from biology…



Biological "Model"

- **Neuron:** an excitable cell
- **Synapse:** connection between neurons
- A *neuron* sends an **electrochemical pulse** along its *synapses* when a sufficient voltage change occurs
- **Biological Neural Network:** collection of neurons along some pathway through the brain

Artificial Model

- **Neuron:** node in a directed acyclic graph (DAG)
- **Weight:** multiplier on each edge
- **Activation Function:** nonlinear thresholding function, which allows a neuron to "fire" when the input value is sufficiently high
- **Artificial Neural Network:** collection of neurons into a DAG, which define some differentiable function

Biological "Computation"

- Neuron switching time : ~ 0.001 sec
- Number of neurons: ~ 10^{10}
- Connections per neuron: ~ $10^{4\text{-}5}$
- Scene recognition time: ~ 0.1 sec

Artificial Computation

- Many neuron-like threshold switching units
- Many weighted interconnections among units
- Highly parallel, distributed processes

Slide adapted from Eric Xing

Neural Networks

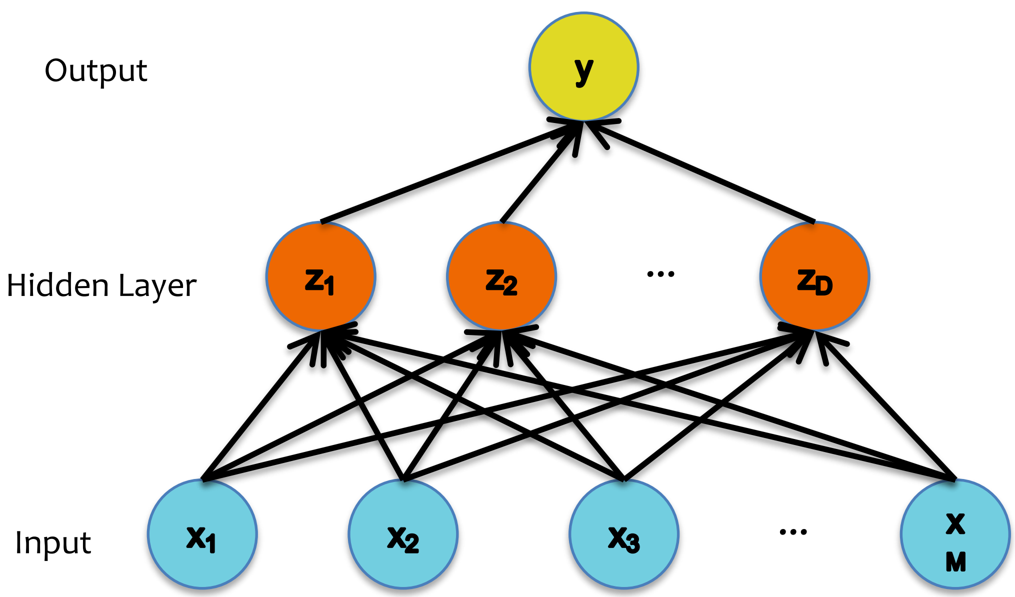
Chalkboard

– Example: Neural Network w/1 Hidden Layer
– Example: Neural Network w/2 Hidden Layers
– Example: Feed Forward Neural Network

Decision Functions

Neural Network

Neural Network for Classification



(E) **Output (sigmoid)**
$$y = \frac{1}{1+\exp(-b)}$$

(D) **Output (linear)**
$$b = \sum_{j=0}^{D} \beta_j z_j$$

(C) **Hidden (sigmoid)**
$$z_j = \frac{1}{1+\exp(-a_j)}, \ \forall j$$

(B) **Hidden (linear)**
$$a_j = \sum_{i=0}^{M} \alpha_{ji} x_i, \ \forall j$$

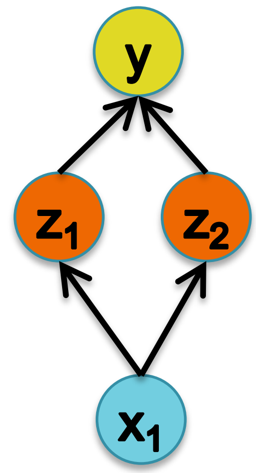
(A) **Input**
Given $x_i, \ \forall i$

Neural Network Parameters

Question:

Suppose you are training a one-hidden layer neural network with sigmoid activations for binary classification.

True or False: There is a unique set of parameters that maximize the likelihood of the dataset above.



Answer:

ARCHITECTURES

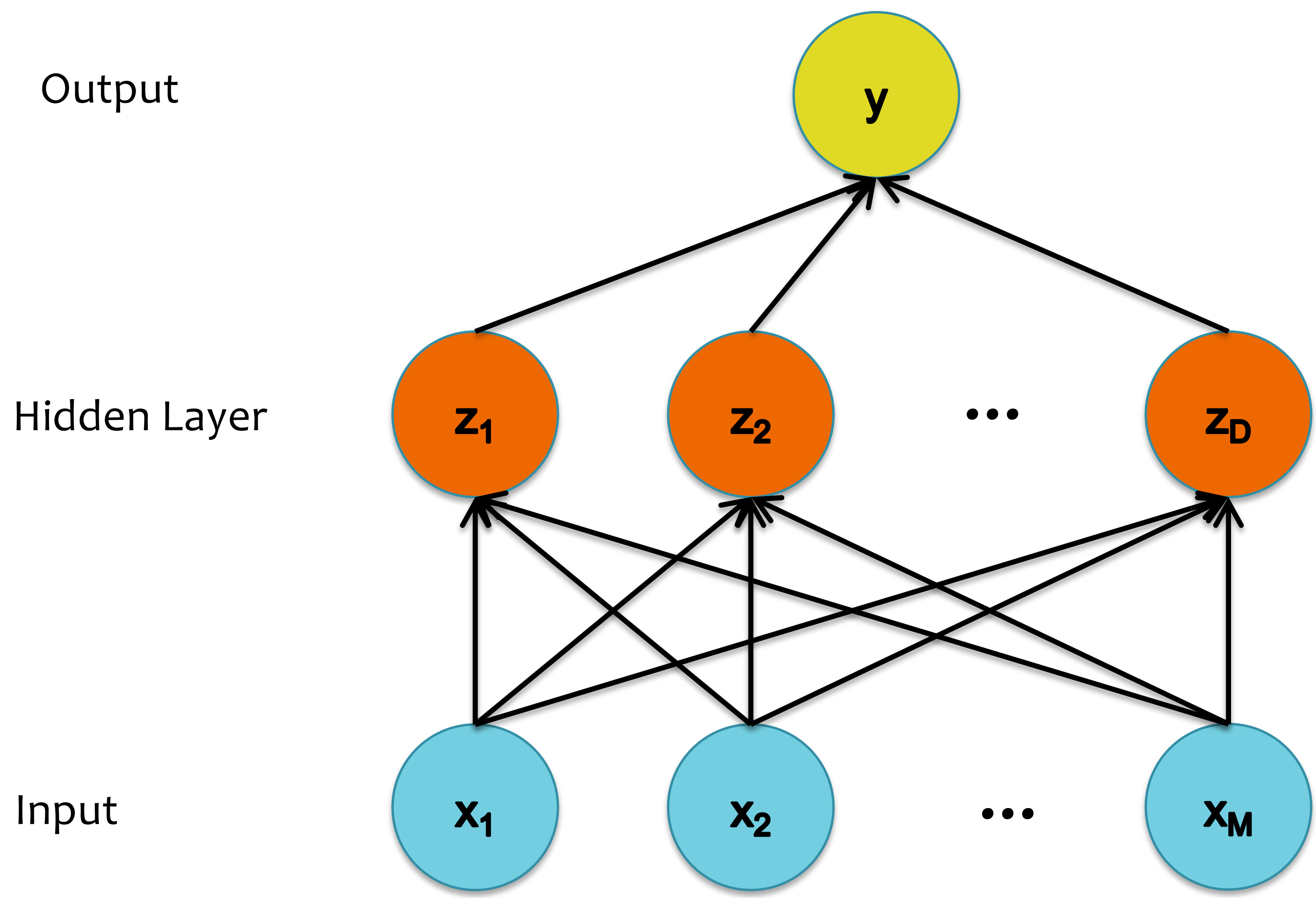
Neural Network Architectures

Even for a basic Neural Network, there are many design decisions to make:

1. # of hidden layers (depth)
2. # of units per hidden layer (width)
3. Type of activation function (nonlinearity)
4. Form of objective function

Building a Neural Net

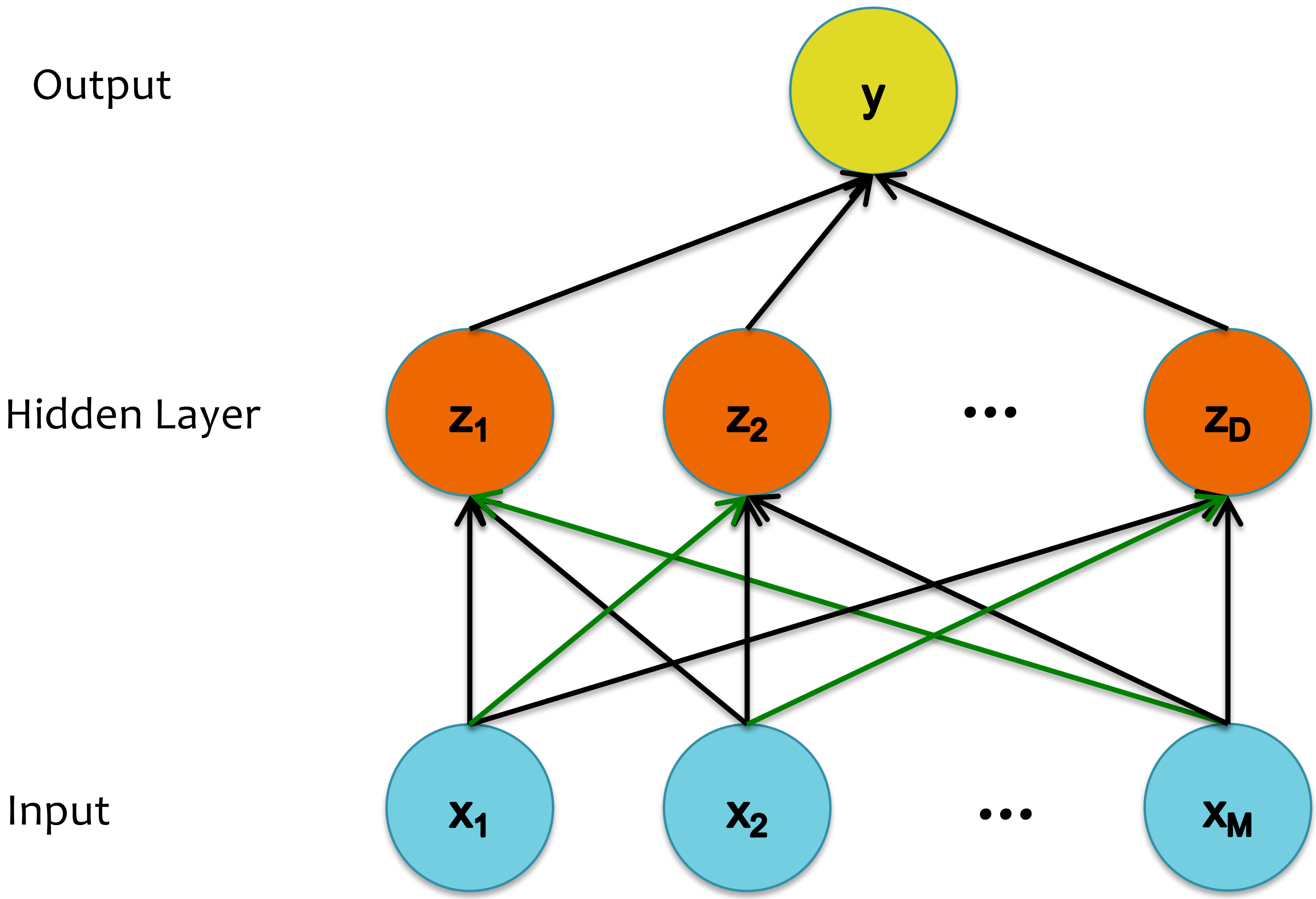
Q: How many hidden units, D, should we use?



D = M

Building a Neural Net

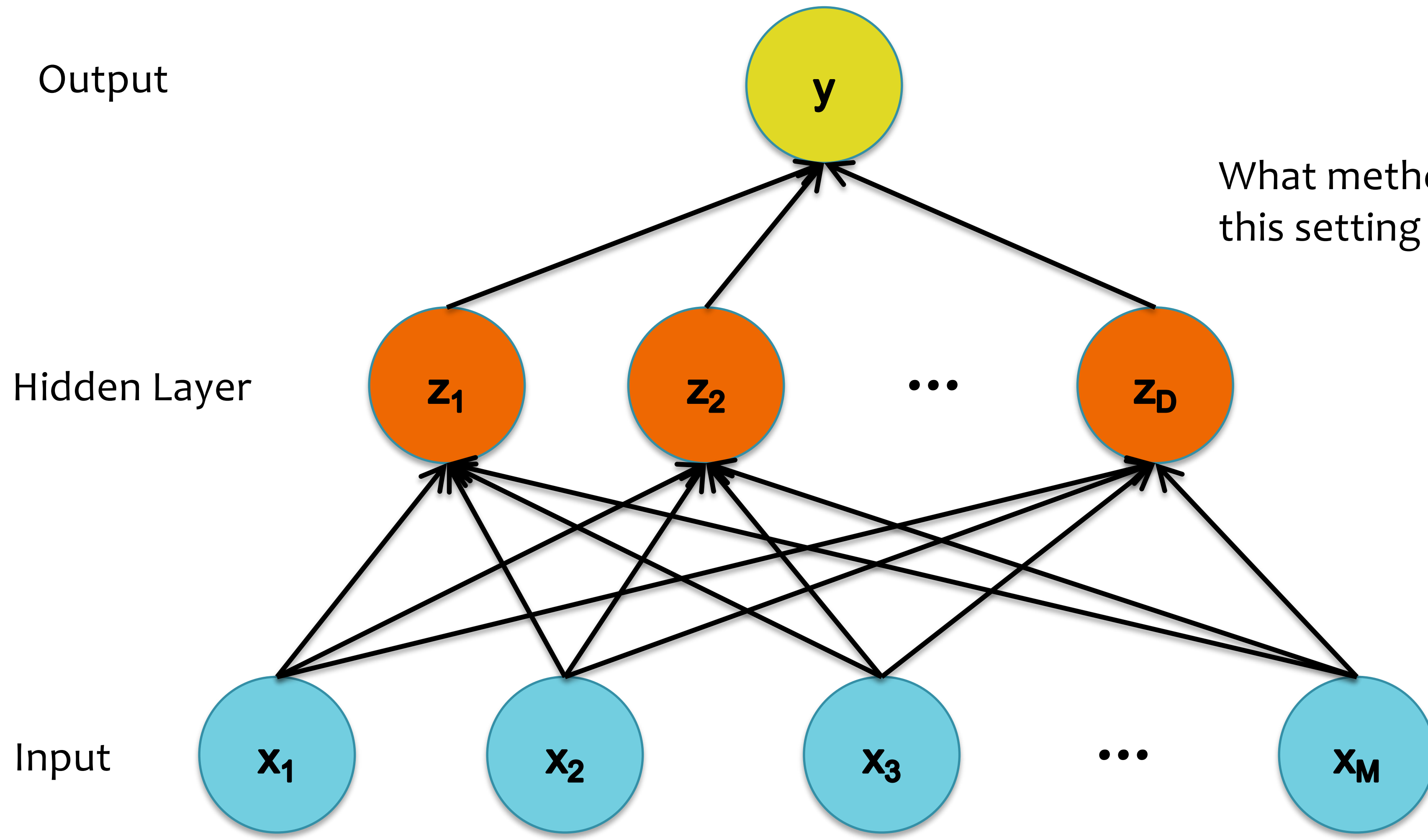
Q: How many hidden units, D, should we use?



D = M

Building a Neural Net

Q: How many hidden units, D, should we use?



What method(s) is this setting similar to?

D < M

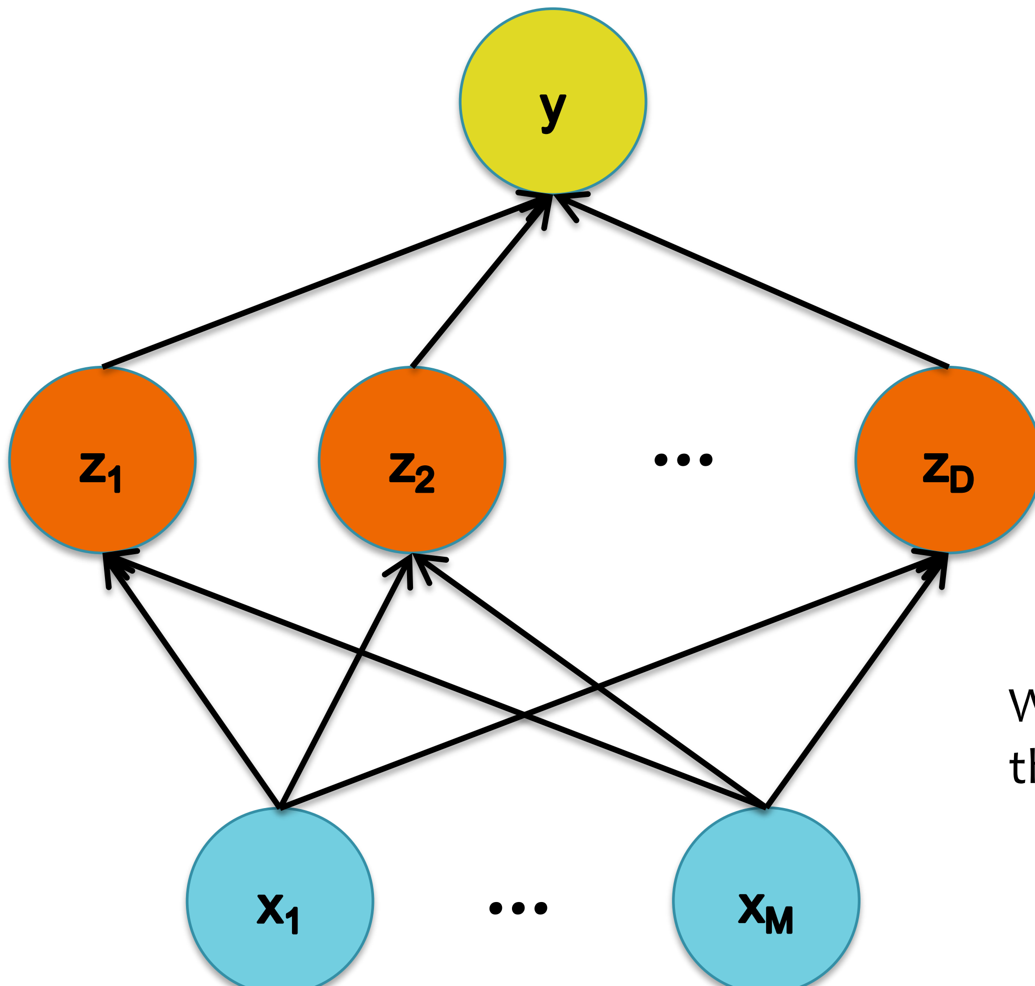
Building a Neural Net

Q: How many hidden units, D, should we use?

Output

Hidden Layer

Input

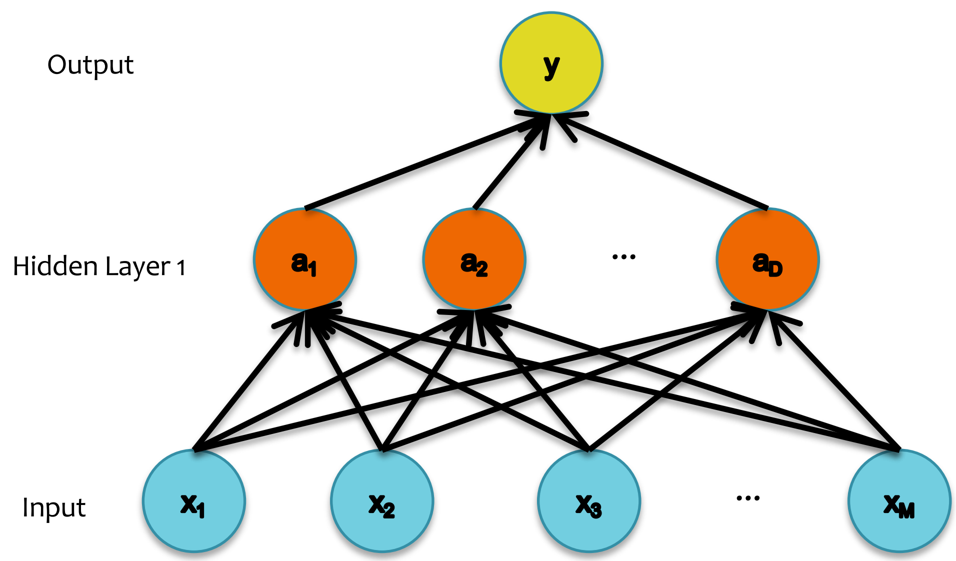


D > M

What method(s) is
this setting similar to?

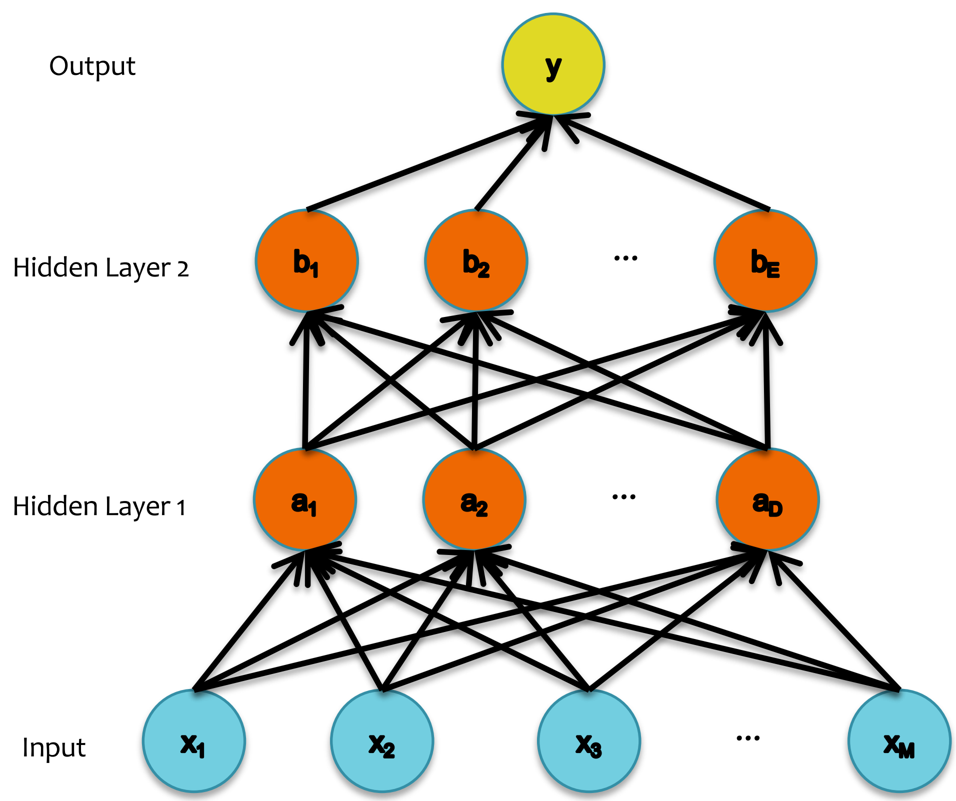
Deeper Networks

Q: How many layers should we use?



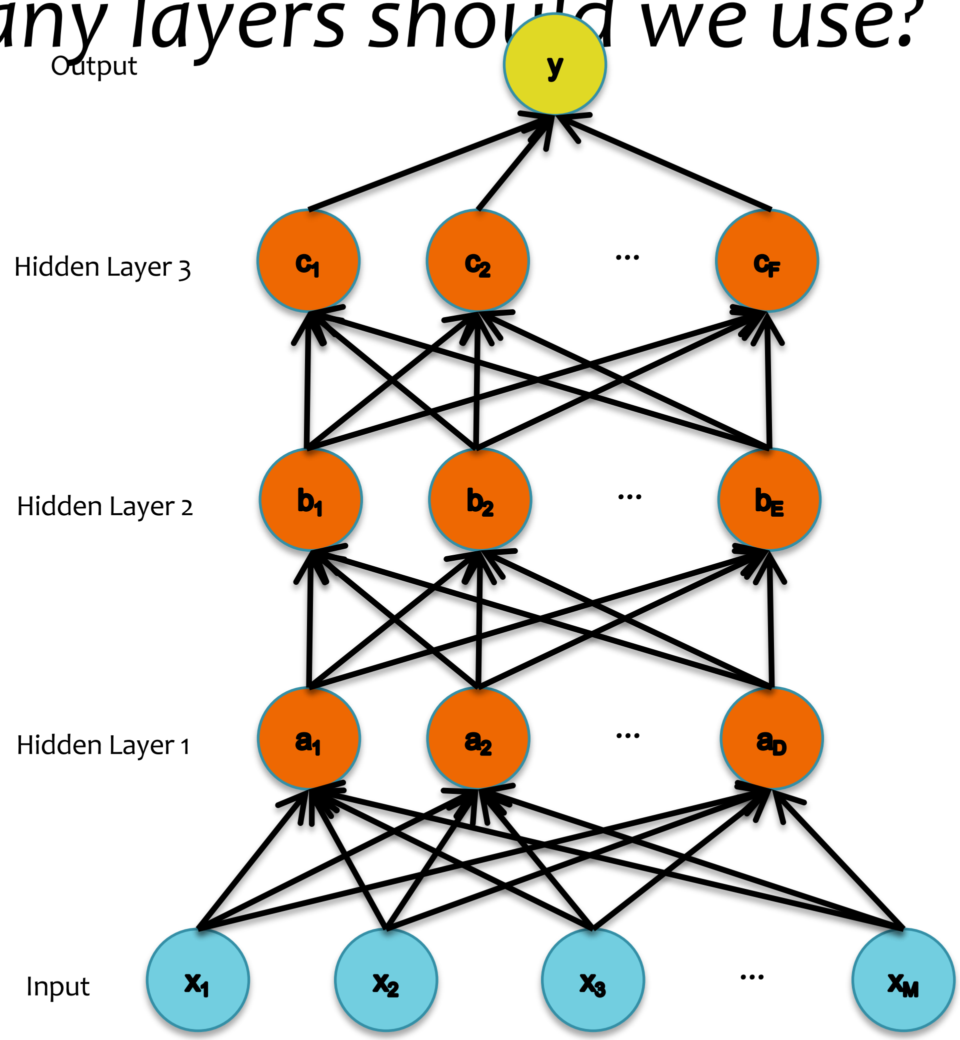
Deeper Networks

Q: How many layers should we use?



Deeper Networks

Q: How many layers should we use?



Deeper Networks

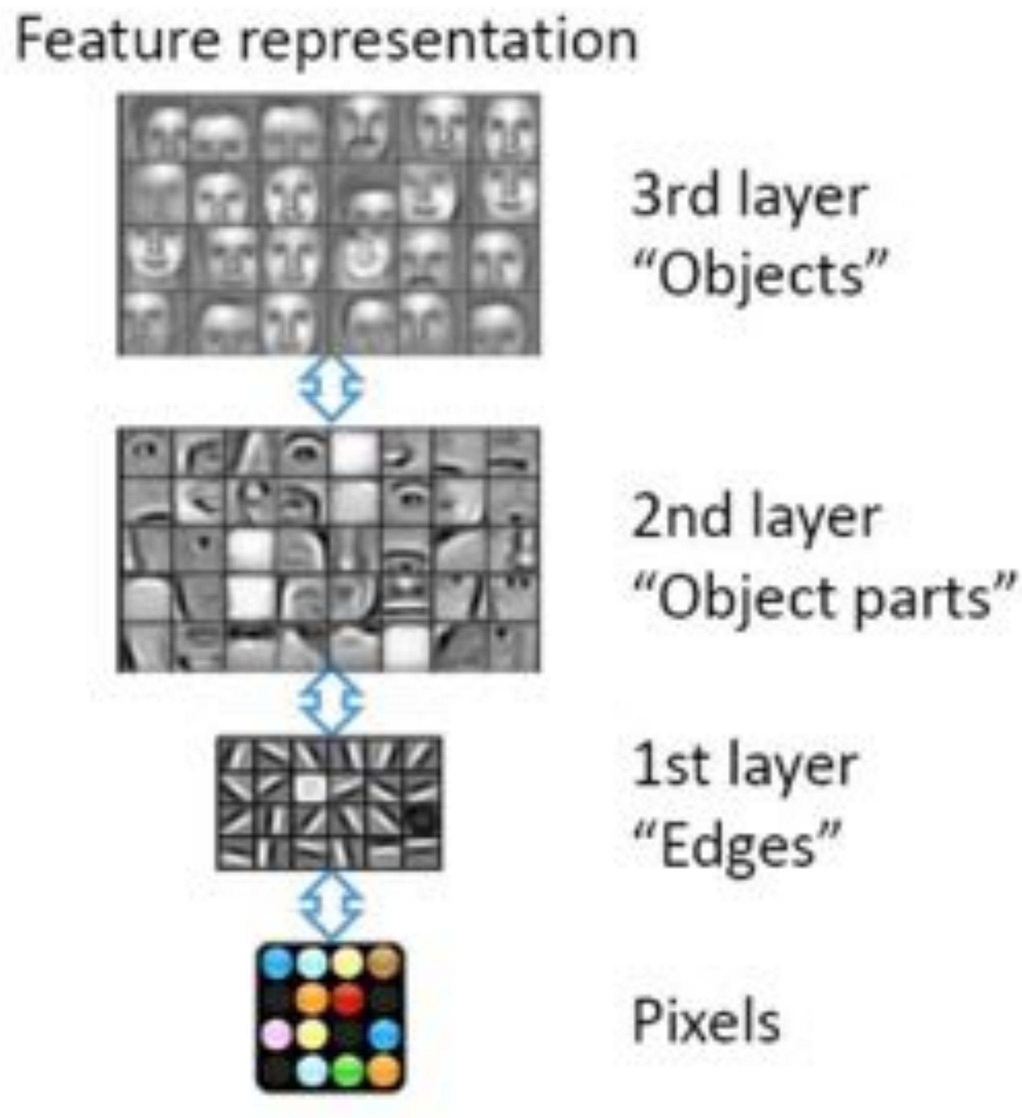
Q: How many layers should we use?

- **Theoretical answer:**
  - A neural network with 1 hidden layer is a **universal function approximator**
  - Cybenko (1989): For any continuous function $g(\mathbf{x})$, there exists a 1-hidden-layer neural net $h_\theta(\mathbf{x})$ s.t. $| h_\theta(\mathbf{x}) - g(\mathbf{x}) | < \epsilon$ for all $\mathbf{x}$, assuming sigmoid activation functions
- **Empirical answer:**
  - Before 2006: "Deep networks (e.g. 3 or more hidden layers) are too hard to train"
  - After 2006: "Deep networks are easier to train than shallow networks (e.g. 2 or fewer layers) for many problems"

    Big caveat: You need to know and use the right tricks.

Different Levels of Abstraction

- We don’t know the “right” levels of abstraction
- So let the model figure it out!

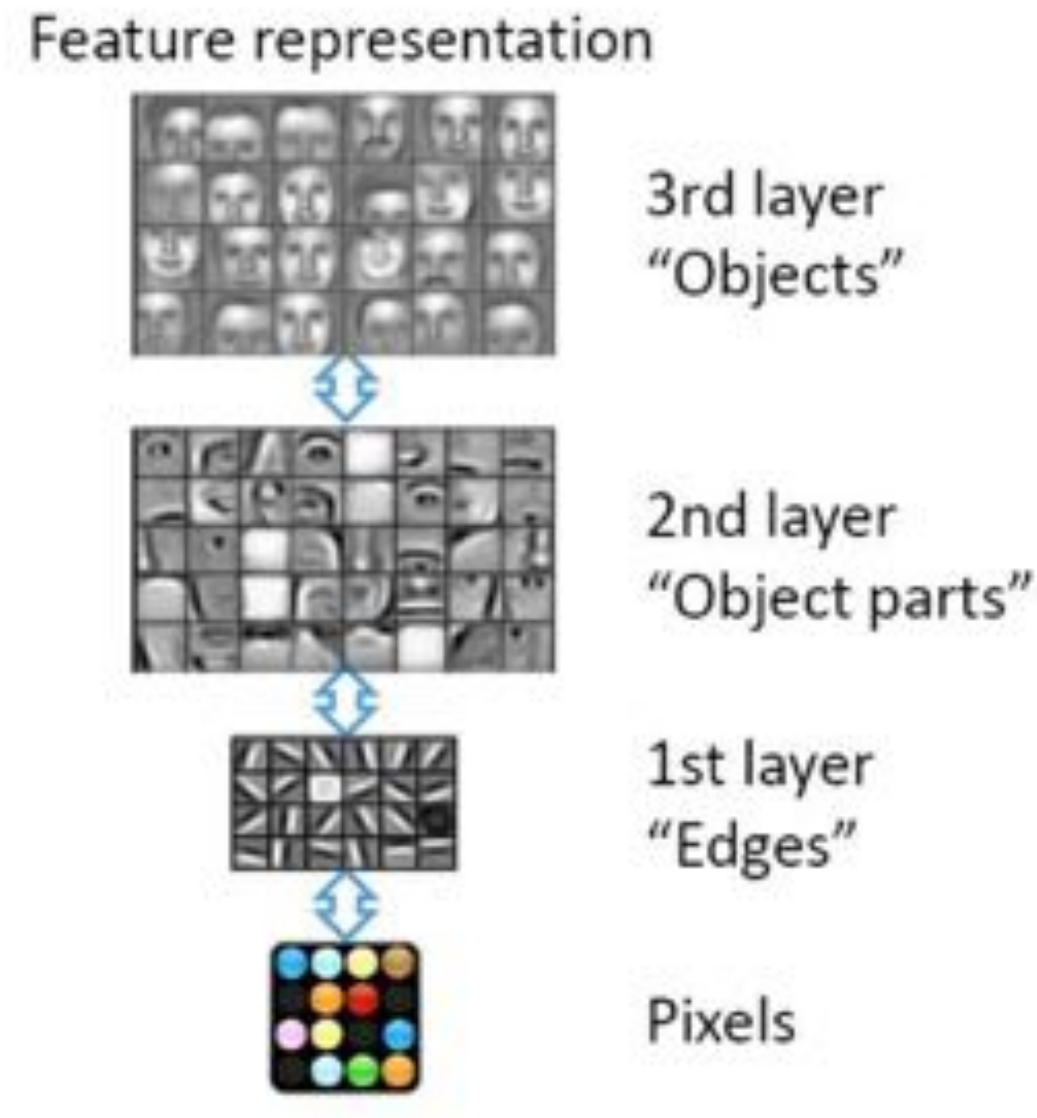


Example from Honglak Lee (NIPS 2010)

Different Levels of Abstraction

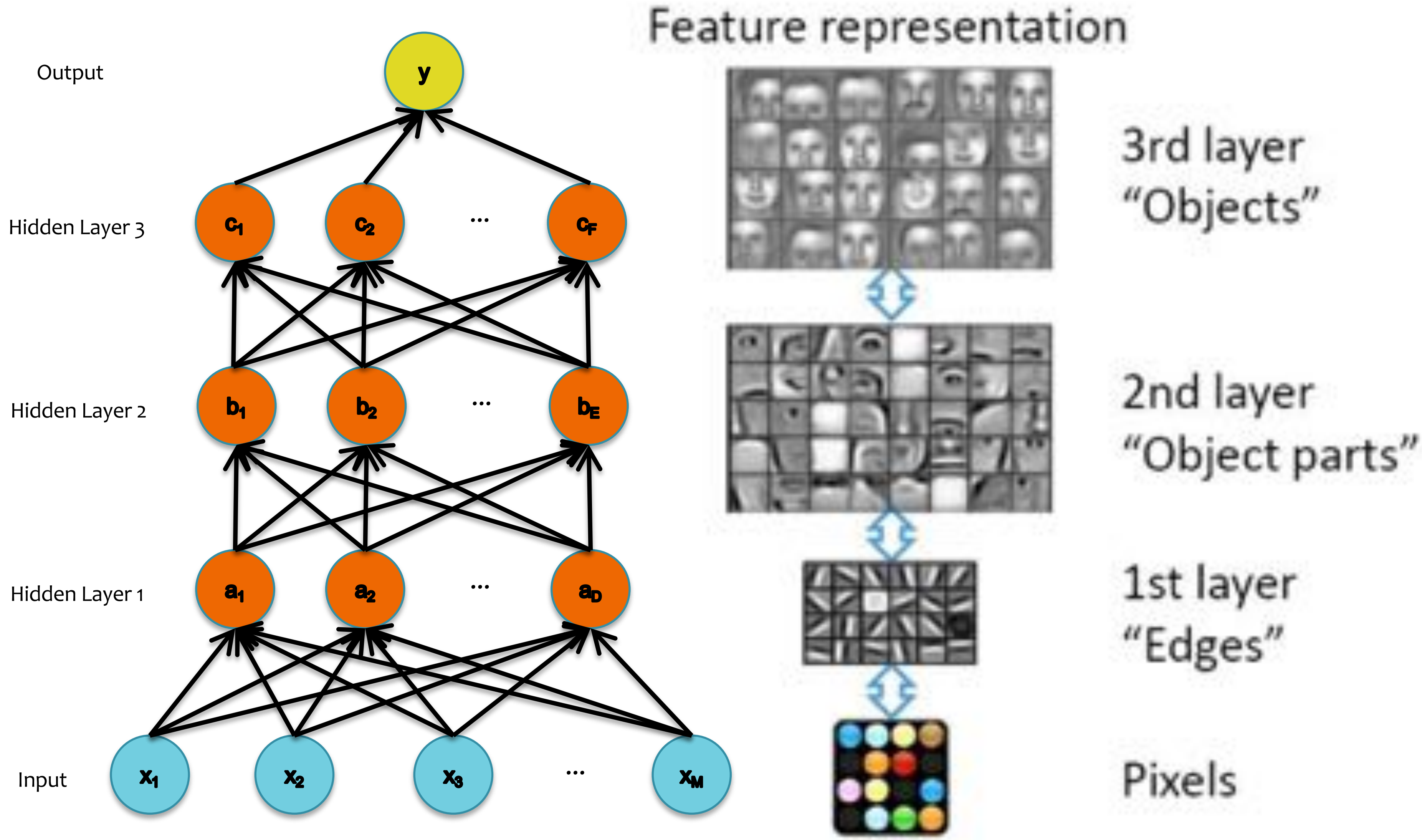
Face Recognition:

- Deep Network can build up increasingly higher levels of abstraction
- Lines, parts, regions



Example from Honglak Lee (NIPS 2010)

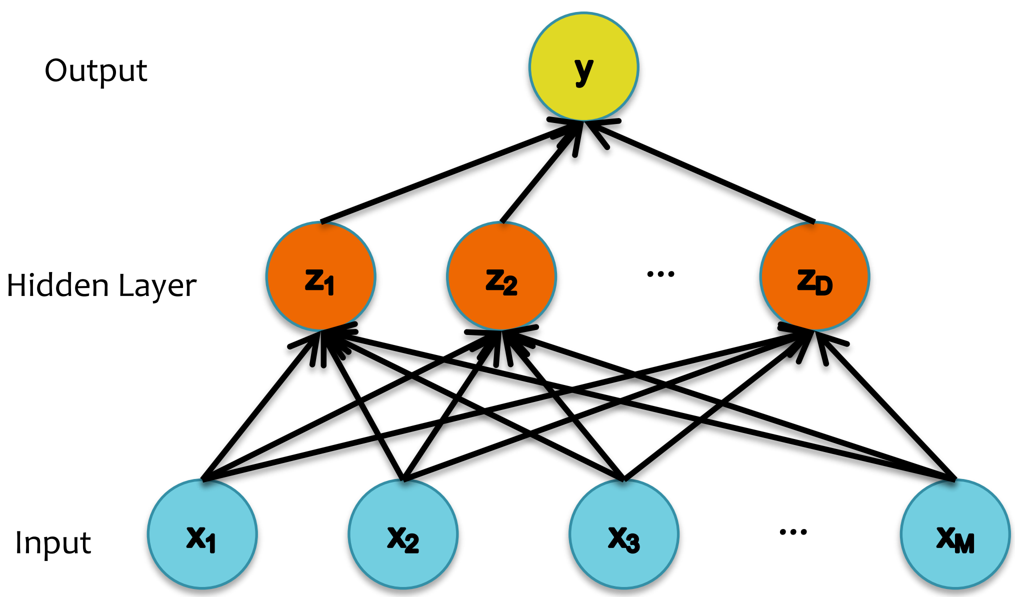
Different Levels of Abstraction



Example from Honglak Lee (NIPS 2010)

Activation Functions

Neural Network with sigmoid activation functions



(F) **Loss**
$J = \frac{1}{2}(y - y^*)^2$

(E) **Output (sigmoid)**
$y = \frac{1}{1+\exp(-b)}$

(D) **Output (linear)**
$b = \sum_{j=0}^{D} \beta_j z_j$

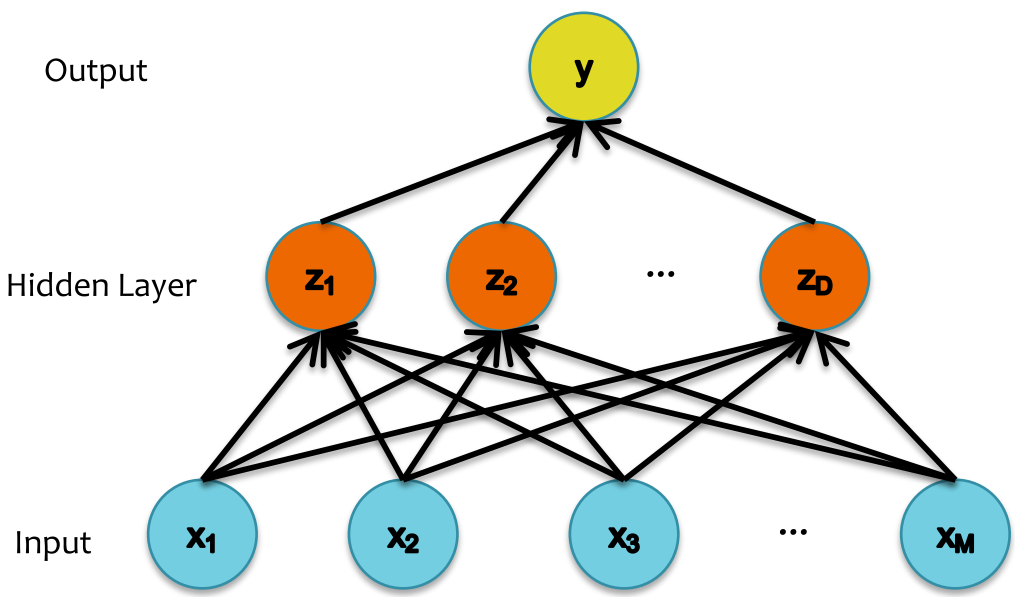
(C) **Hidden (sigmoid)**
$z_j = \frac{1}{1+\exp(-a_j)}, \ \forall j$

(B) **Hidden (linear)**
$a_j = \sum_{i=0}^{M} \alpha_{ji} x_i, \ \forall j$

(A) **Input**
Given $x_i, \ \forall i$

Activation Functions

Neural Network with arbitrary nonlinear activation functions



(F) **Loss**
$J = \frac{1}{2}(y - y^*)^2$

↑

(E) **Output (nonlinear)**
$y = \sigma(b)$

↑

(D) **Output (linear)**
$b = \sum_{j=0}^{D} \beta_j z_j$

↑

(C) **Hidden (nonlinear)**
$z_j = \sigma(a_j), \ \forall j$

↑

(B) **Hidden (linear)**
$a_j = \sum_{i=0}^{M} \alpha_{ji} x_i, \ \forall j$

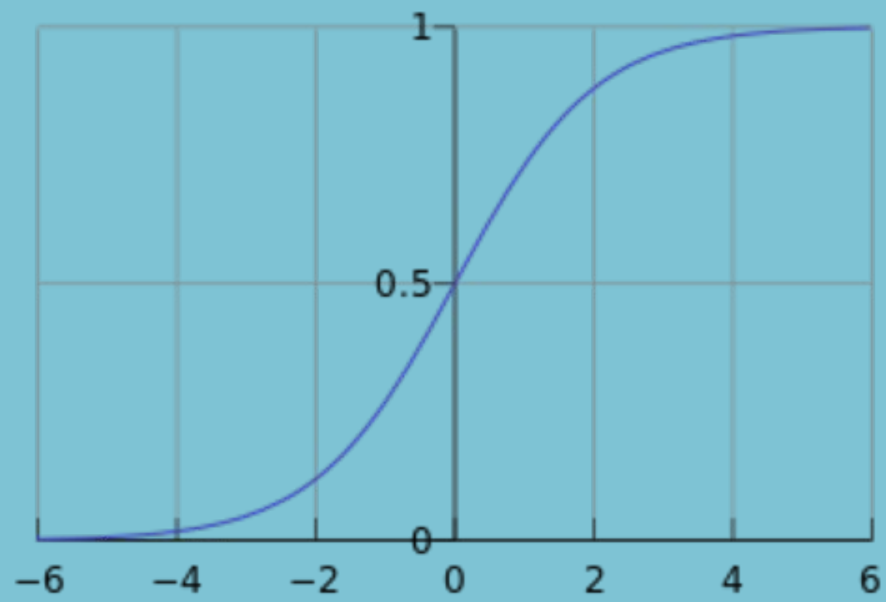
↑

(A) **Input**
Given $x_i, \ \forall i$

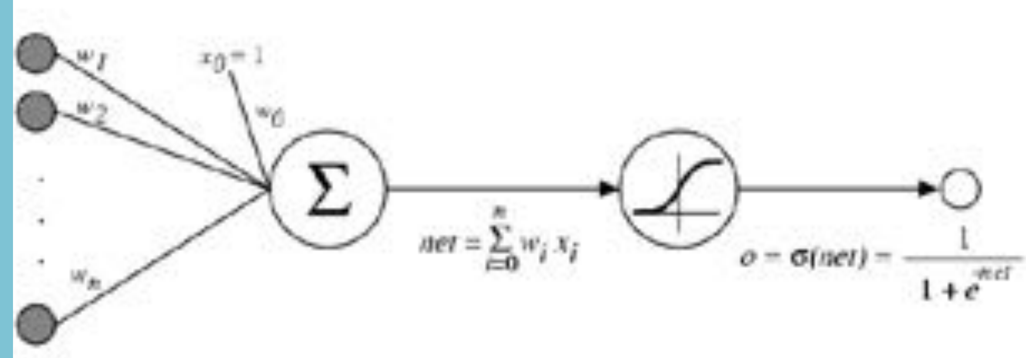
Activation Functions

Sigmoid / Logistic Function

$$\text{logistic}(u) \equiv \frac{1}{1+e^{-u}}$$

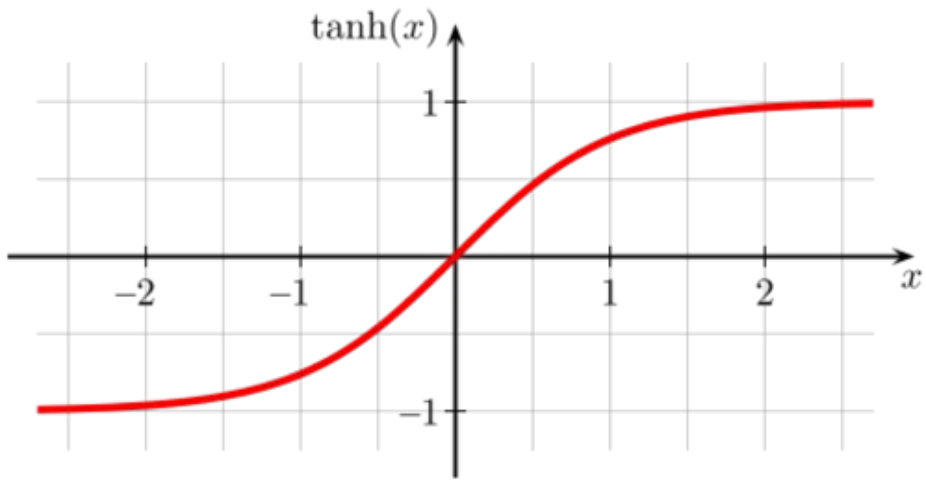


So far, we've assumed that the activation function (nonlinearity) is always the sigmoid function…



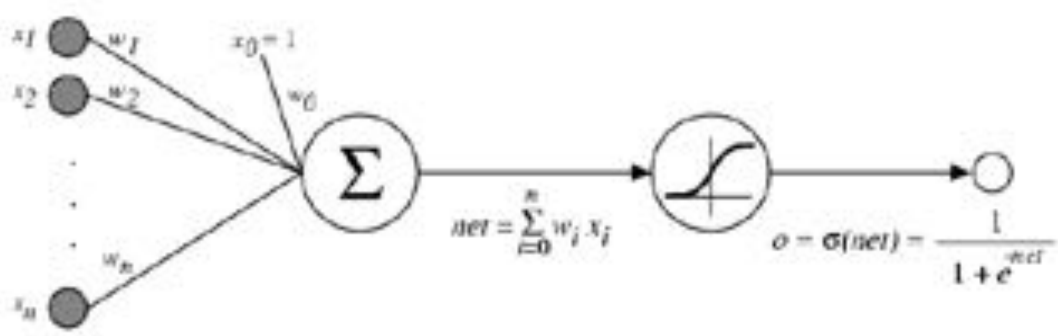
Activation Functions

- A new change: modifying the nonlinearity
  - The logistic is not widely used in modern ANNs



Alternate 1: tanh

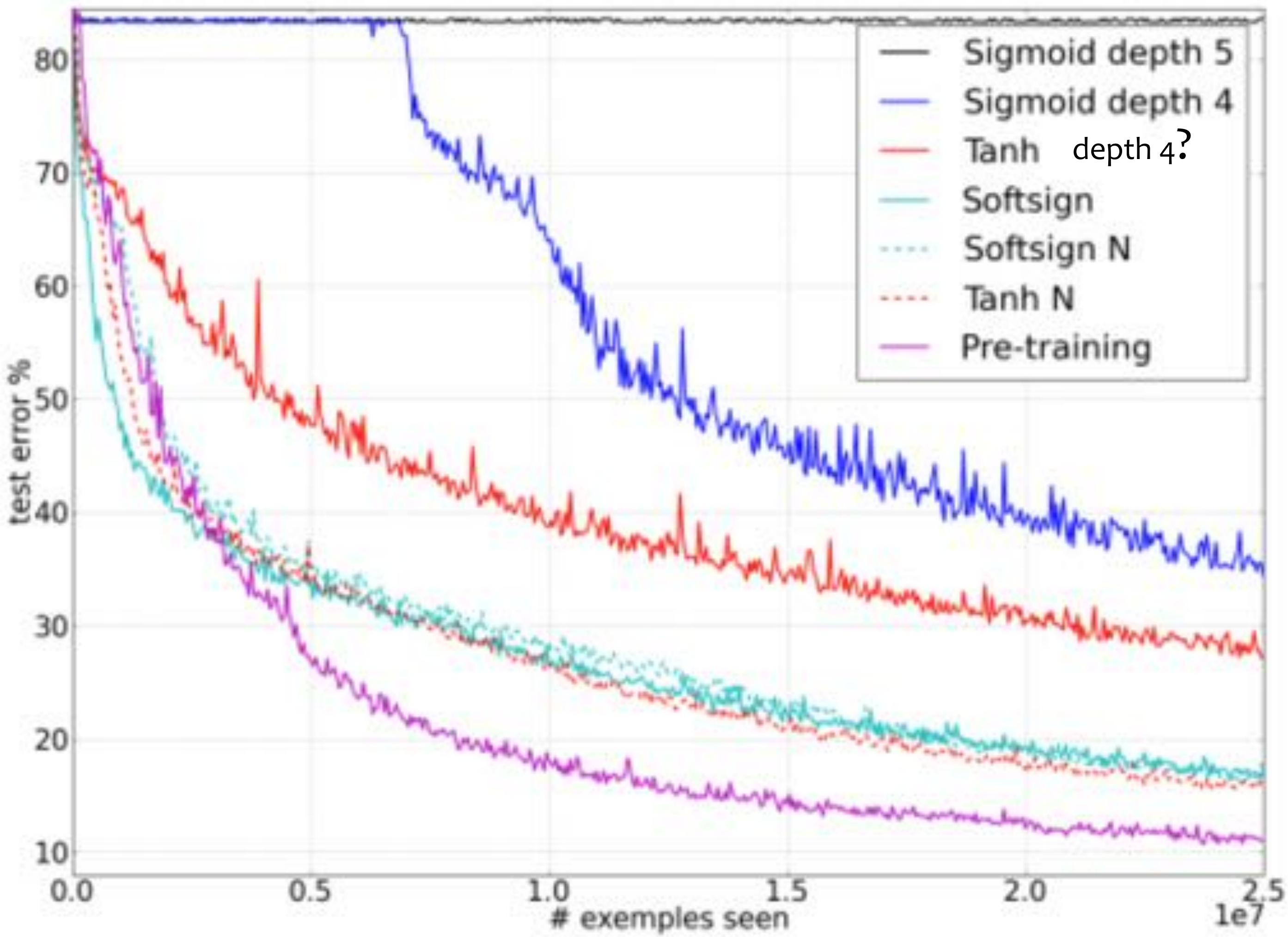
Like logistic function but shifted to range [-1, +1]



Slide from William Cohen

Understanding the difficulty of training deep feedforward neural networks

AI Stats 2010

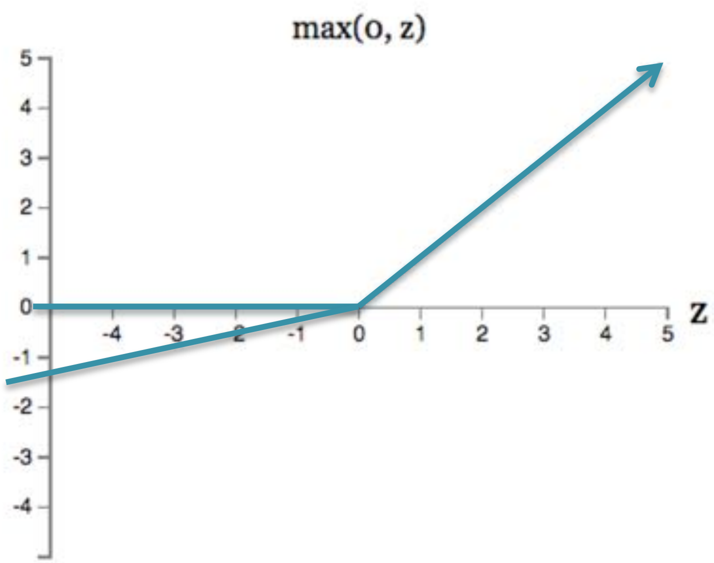


sigmoid
vs.
tanh

Figure from Glorot & Bentio (2010)

Activation Functions

- A new change: modifying the nonlinearity
  - reLU often used in vision tasks

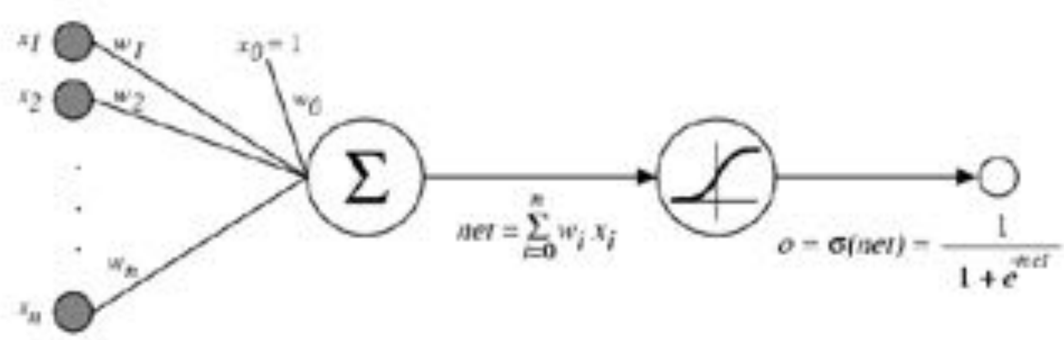


Alternate 2: rectified linear unit

Linear with a cutoff at zero

(Implementation: clip the gradient when you pass zero)

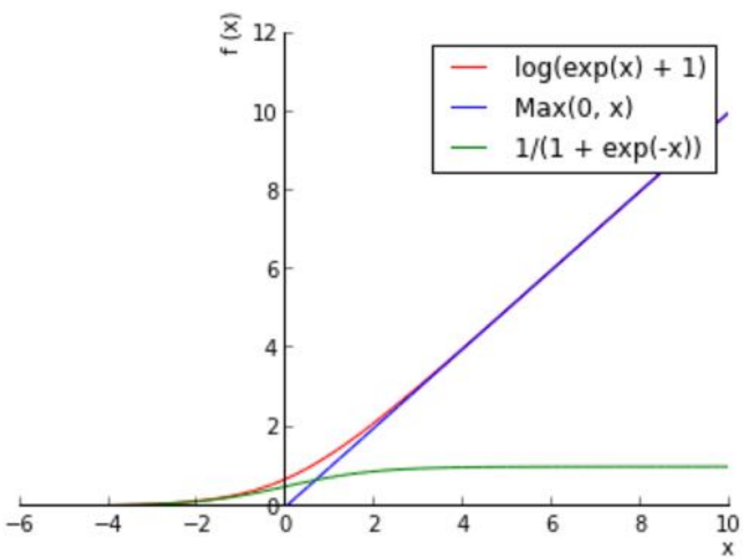
$$\max(0, w \cdot x + b).$$



Slide from William Cohen

Activation Functions

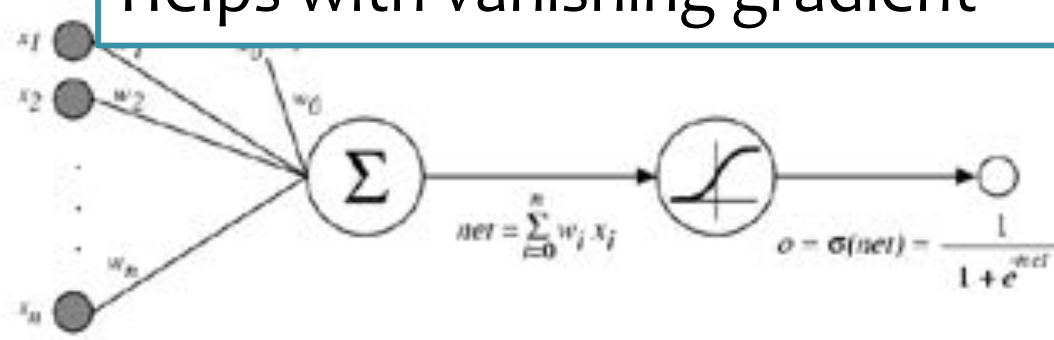
- A new change: modifying the nonlinearity
  - reLU often used in vision tasks



Alternate 2: rectified linear unit

Soft version: log(exp(x)+1)

Doesn’t saturate (at one end)
Sparsifies outputs
Helps with vanishing gradient

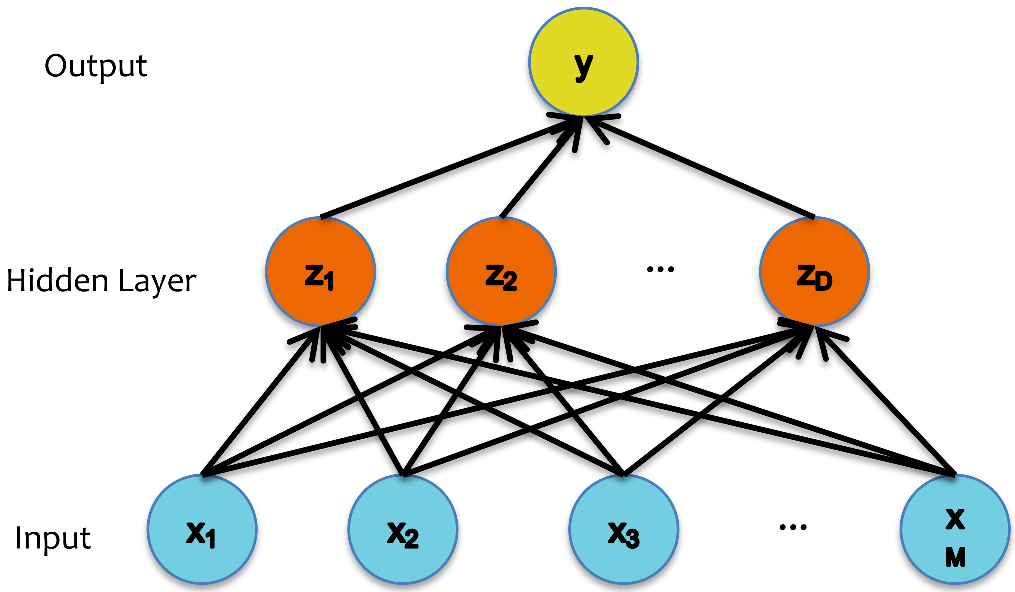


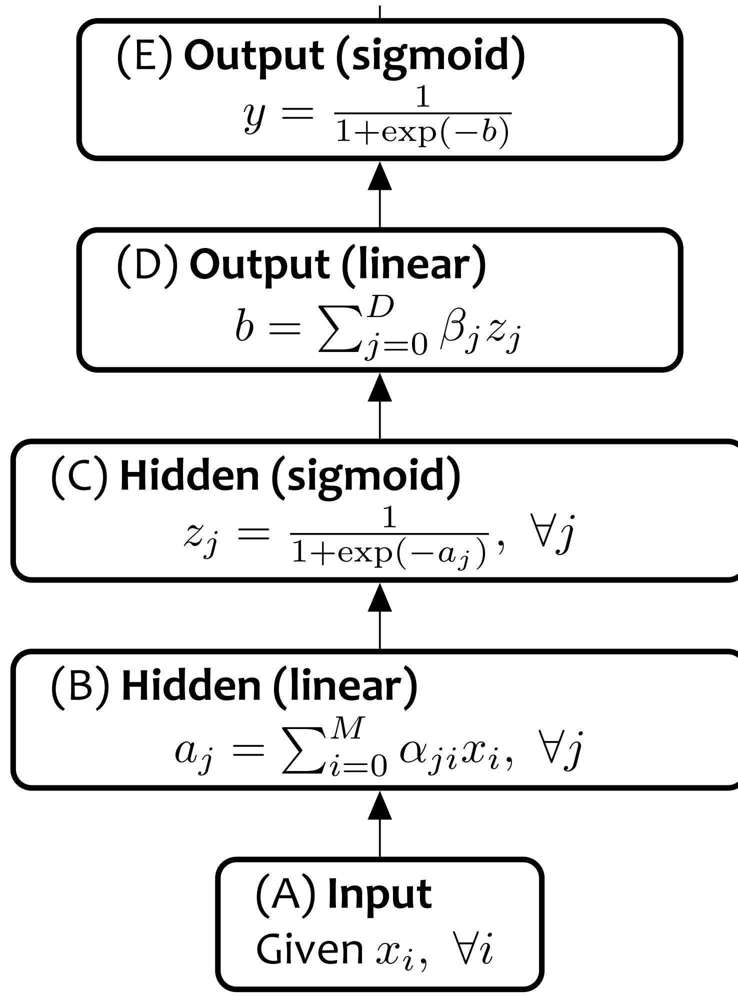
Slide from William Cohen

Decision Functions

Neural Network

Neural Network for Classification

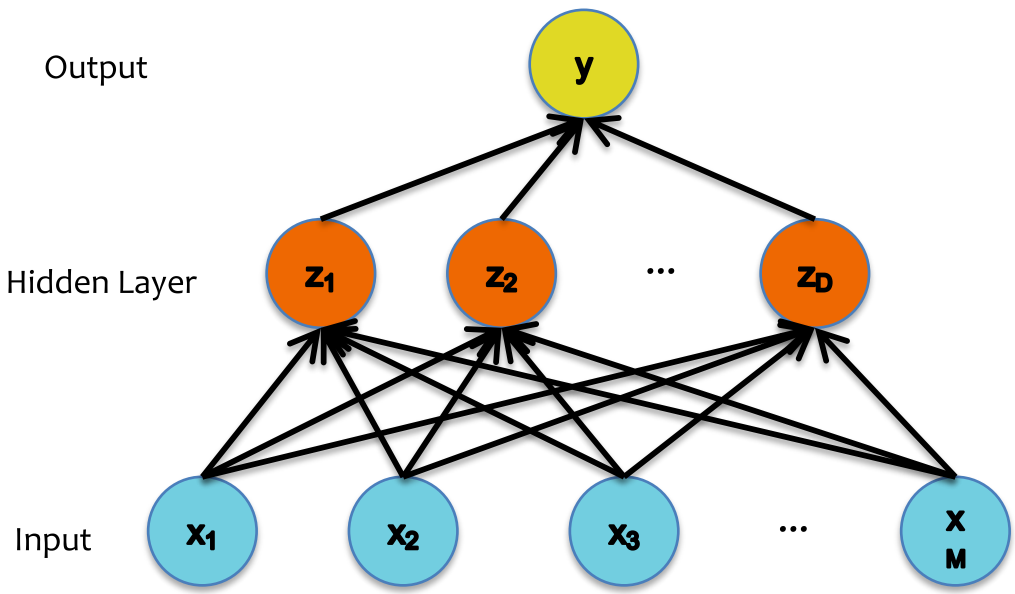


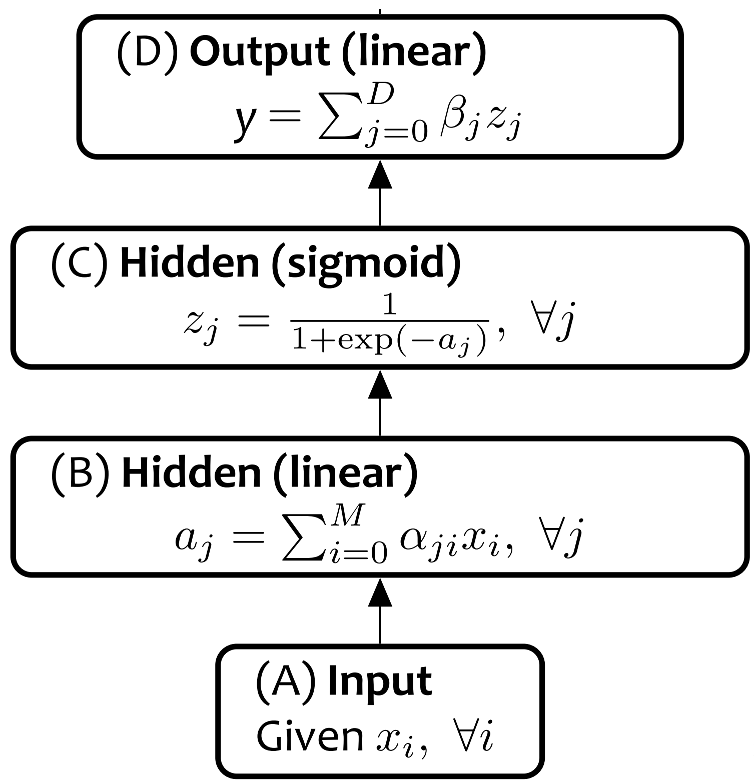


Decision Functions

Neural Network

Neural Network for Regression





Objective Functions for NNs

1. Quadratic Loss:
   - the same objective as Linear Regression
   - i.e. mean squared error
2. Cross-Entropy:
   - the same objective as Logistic Regression
   - i.e. negative log likelihood
   - This requires probabilities, so we add an additional “softmax” layer at the end of our network

	Forward	Backward
Quadratic	$J = \frac{1}{2}(y - y^*)^2$	$\frac{dJ}{dy} = y - y^*$
Cross Entropy	$J = y^* \log(y) + (1 - y^*) \log(1 - y)$	$\frac{dJ}{dy} = y^* \frac{1}{y} + (1 - y^*) \frac{1}{y - 1}$

Objective Functions for NNs

Cross-entropy vs. Quadratic loss

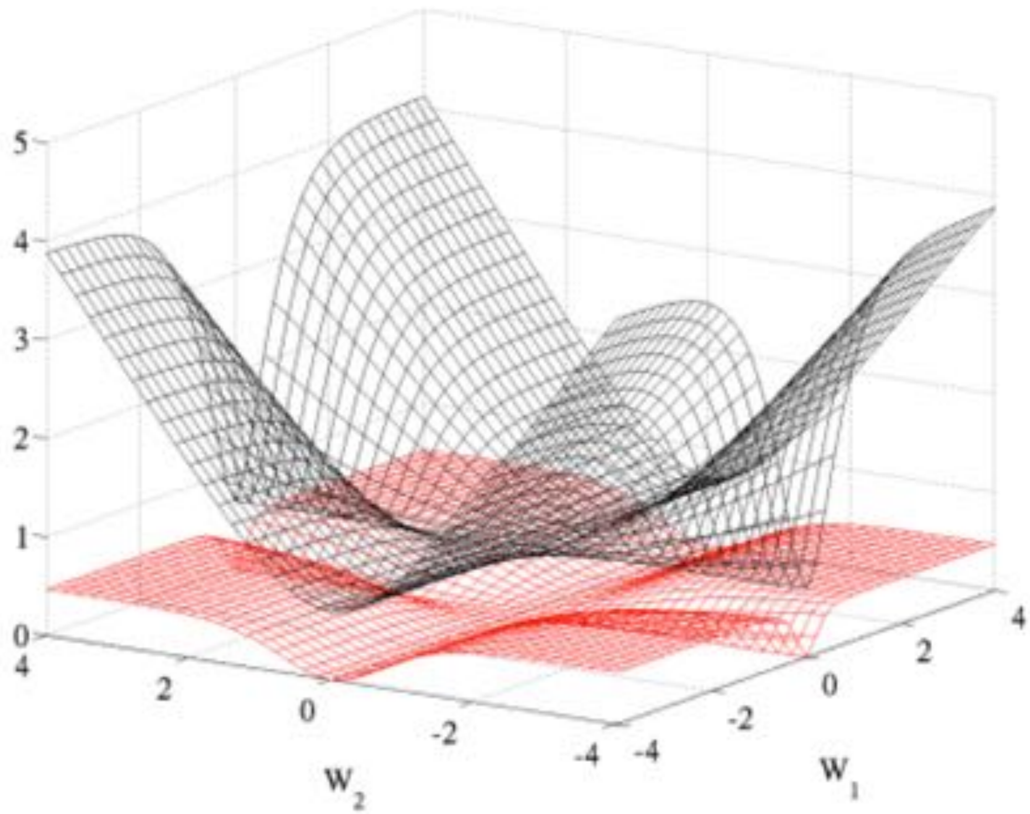
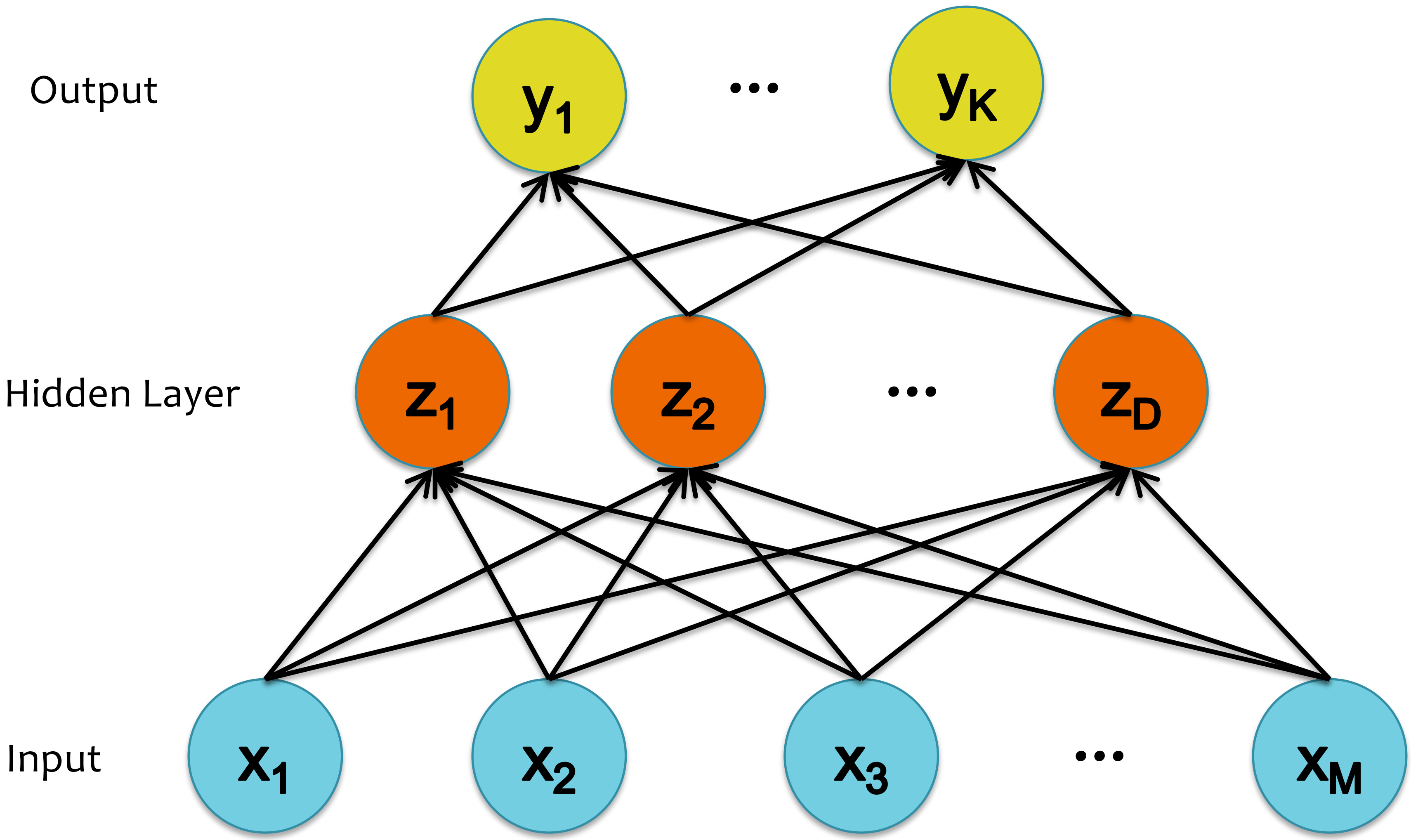


Figure 5: *Cross entropy (black, surface on top) and quadratic (red, bottom surface) cost as a function of two weights (one at each layer) of a network with two layers, W_1 respectively on the first layer and W_2 on the second, output layer.*

Figure from Glorot & Bentio (2010)

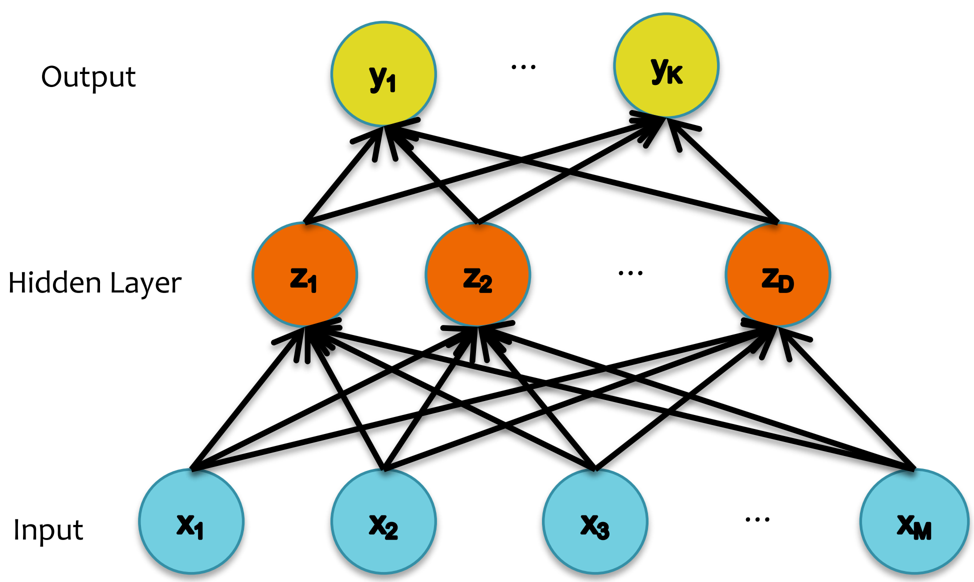
Multi-Class Output

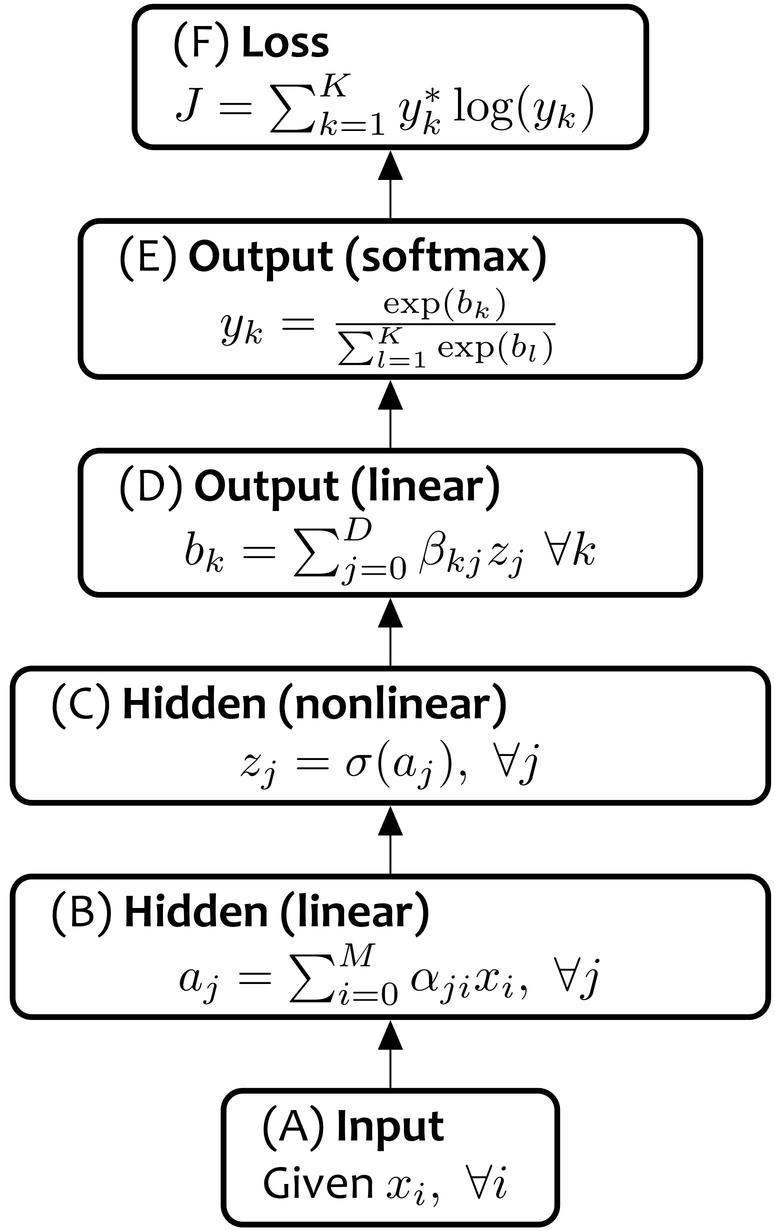


Multi-Class Output

Softmax:

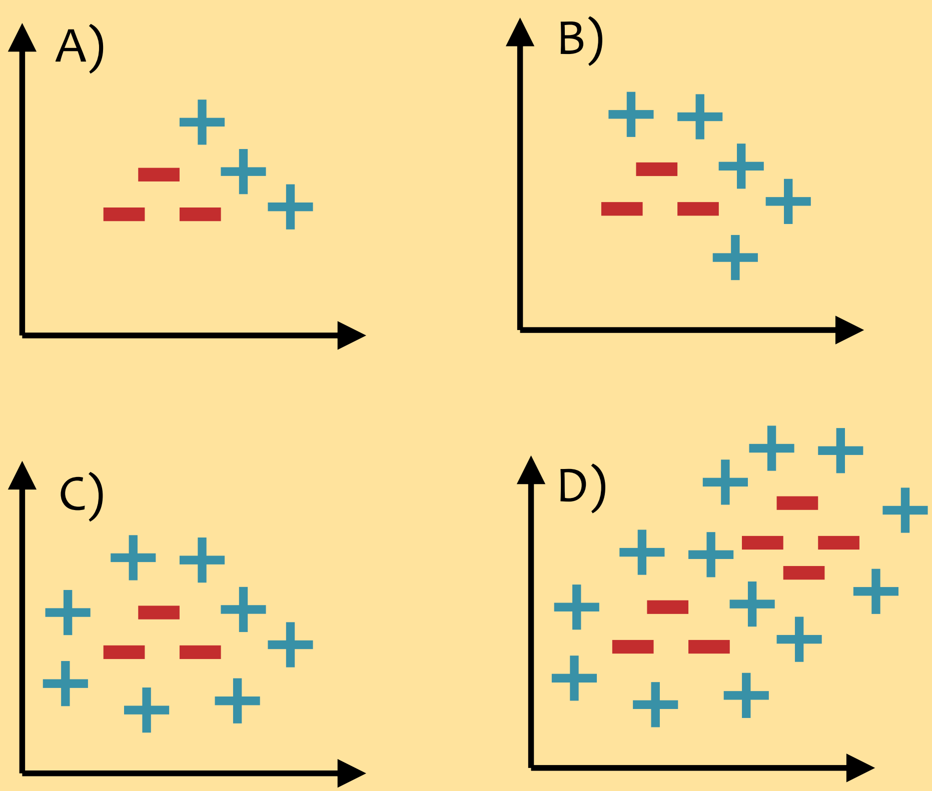
$$y_k = \frac{\exp(b_k)}{\sum_{l=1}^{K} \exp(b_l)}$$



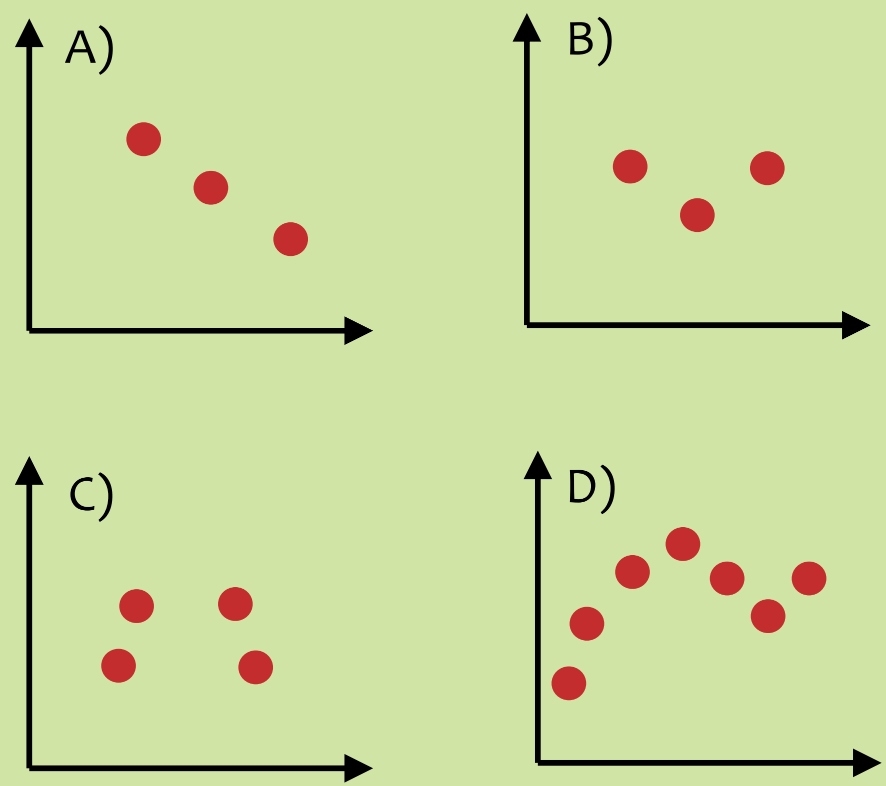


Neural Network Errors

Question A: On which of the datasets below could a one-hidden layer neural network achieve zero *classification* error? **Select all that apply.**

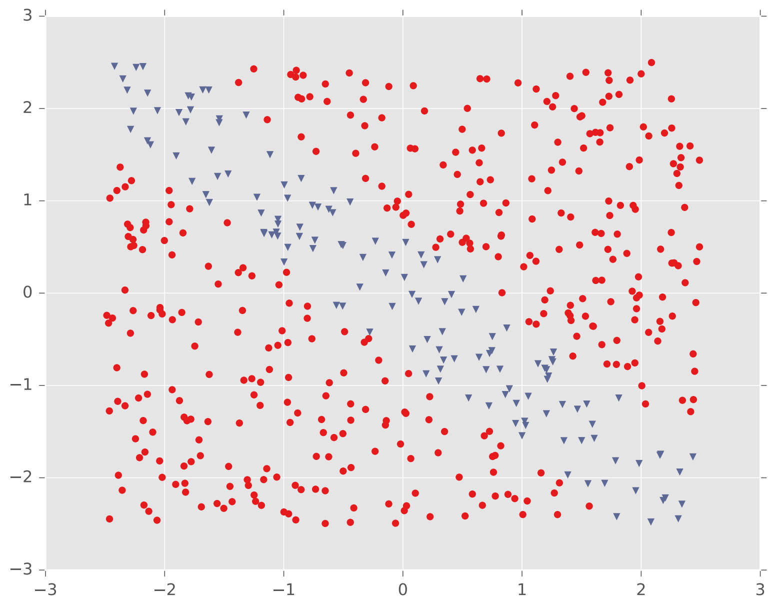


Question B: On which of the datasets below could a one-hidden layer neural network for *regression* achieve *nearly* zero MSE? **Select all that apply.**

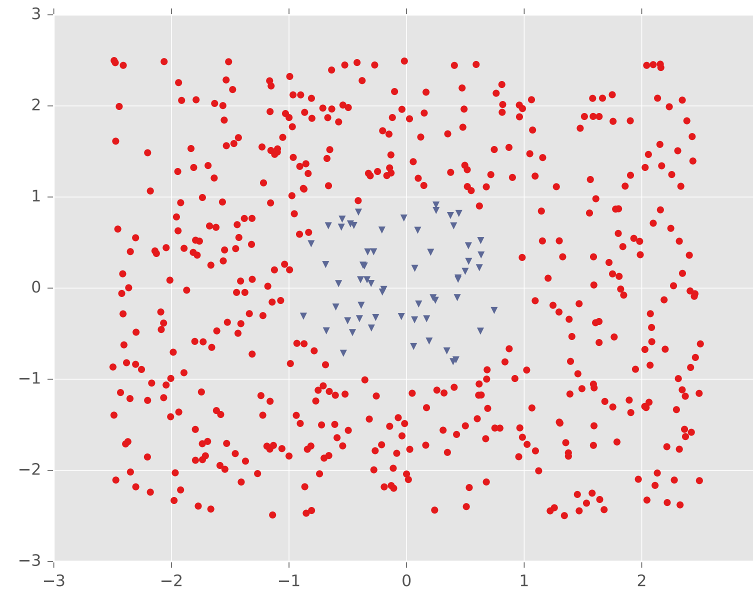


DECISION BOUNDARY EXAMPLES

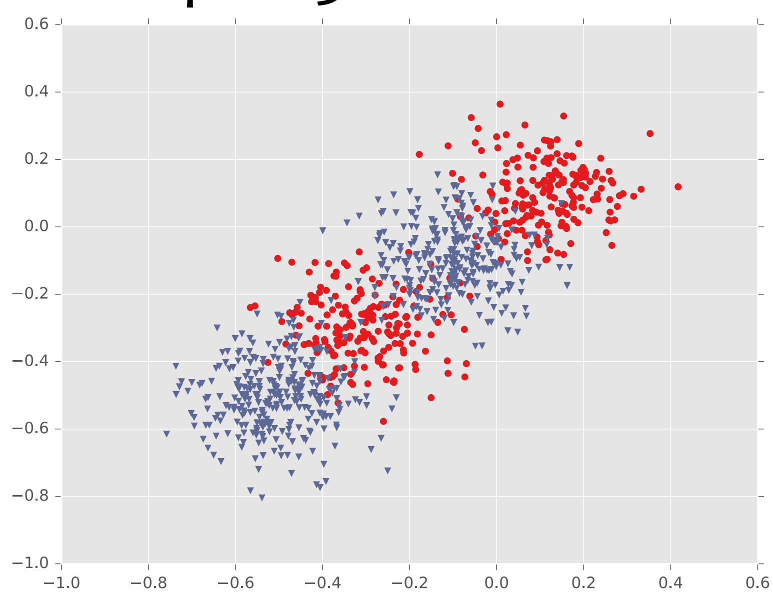
Example #1: Diagonal Band



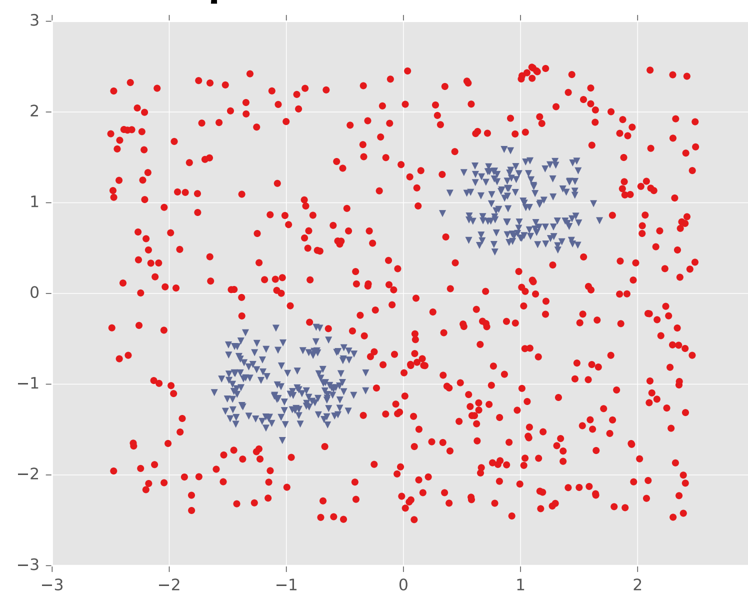
Example #2: One Pocket



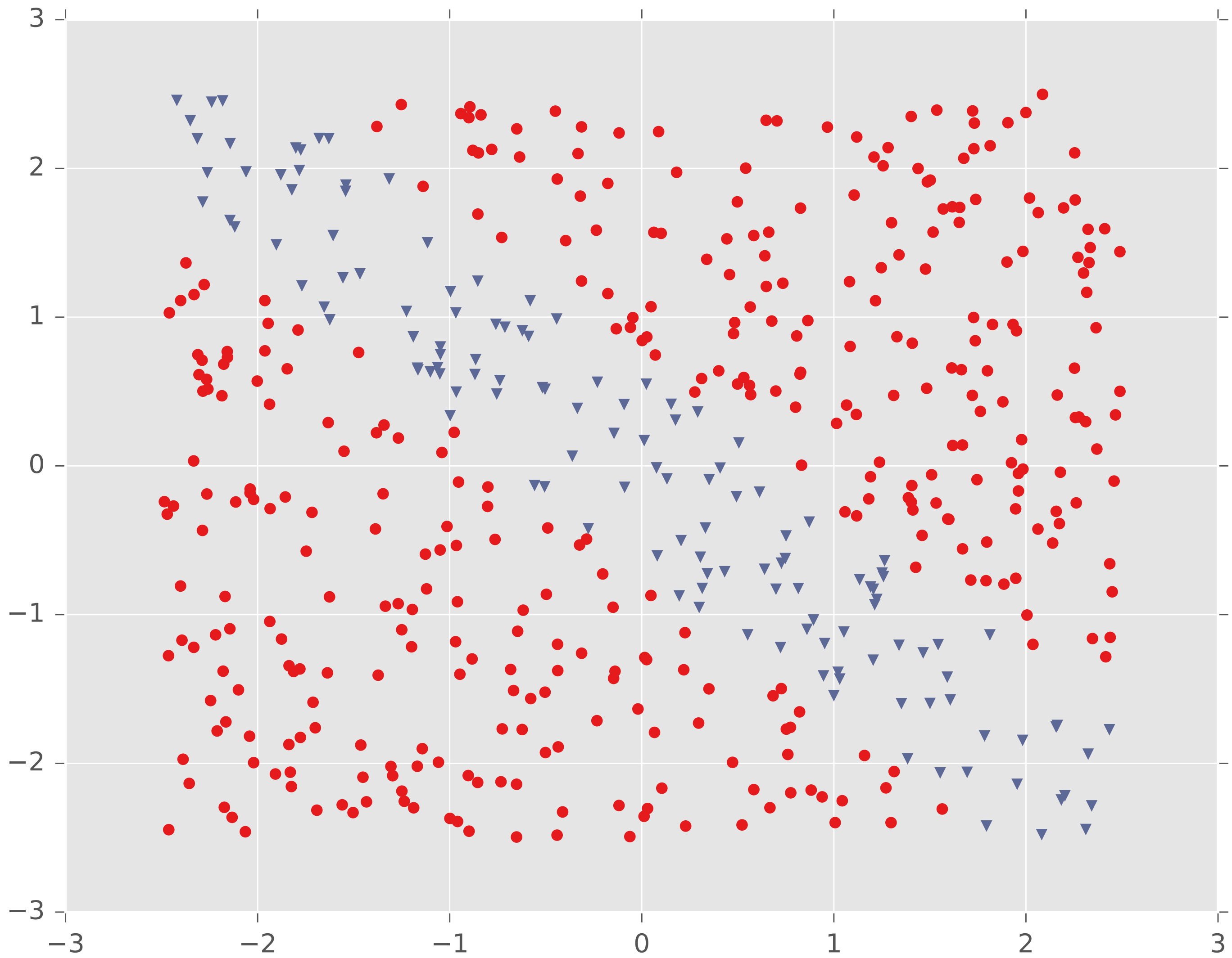
Example #3: Four Gaussians



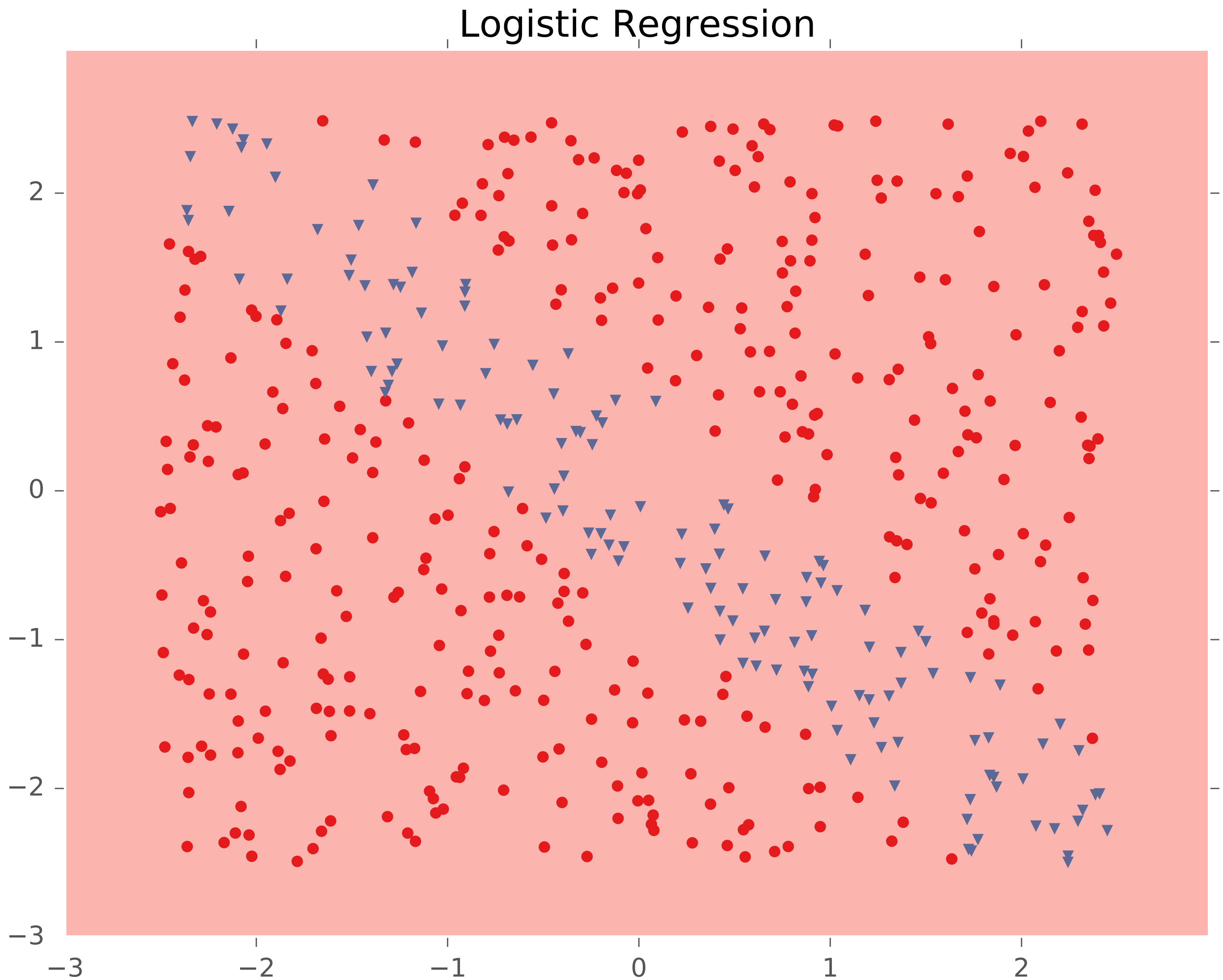
Example #4: Two Pockets



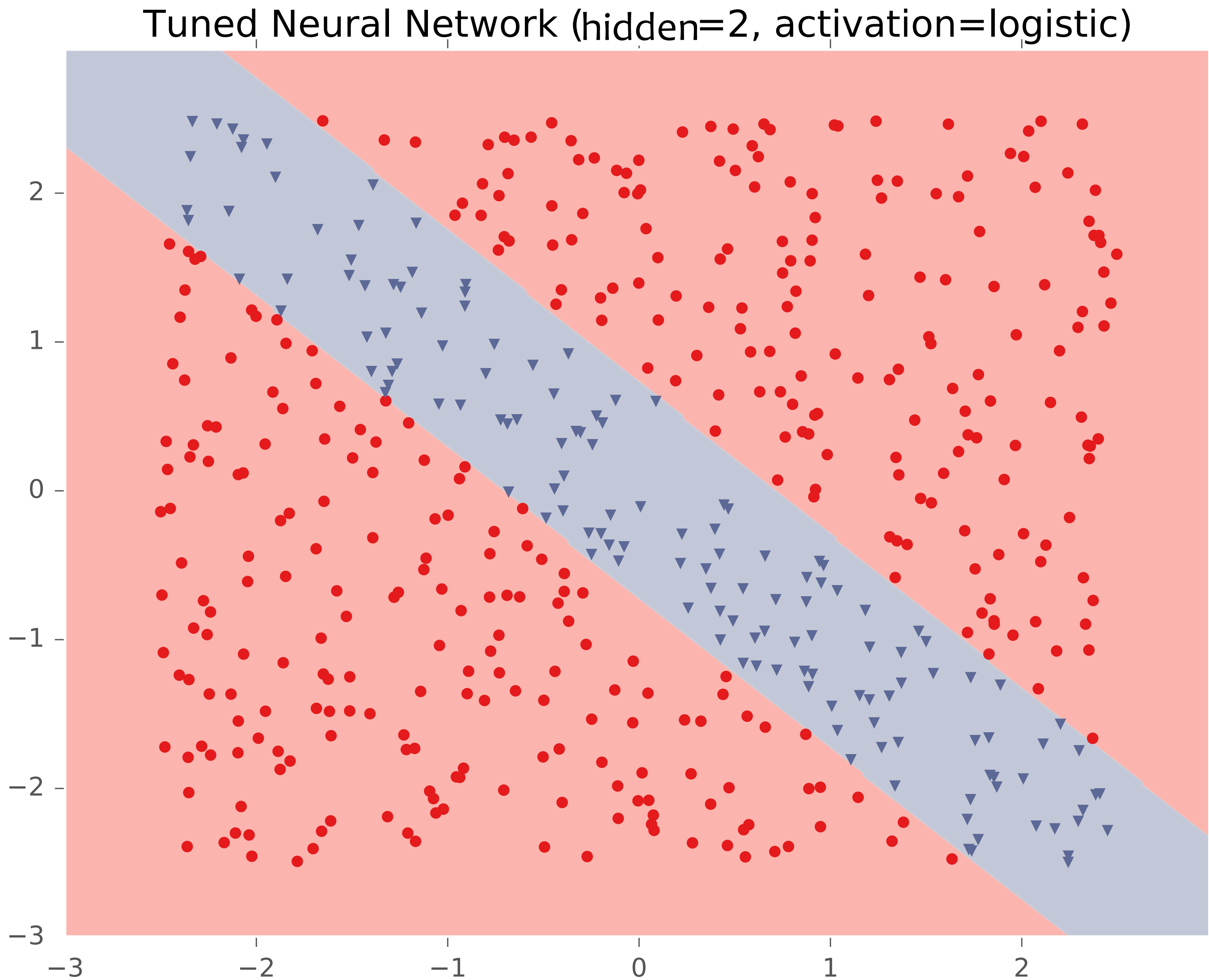
Example #1: Diagonal Band



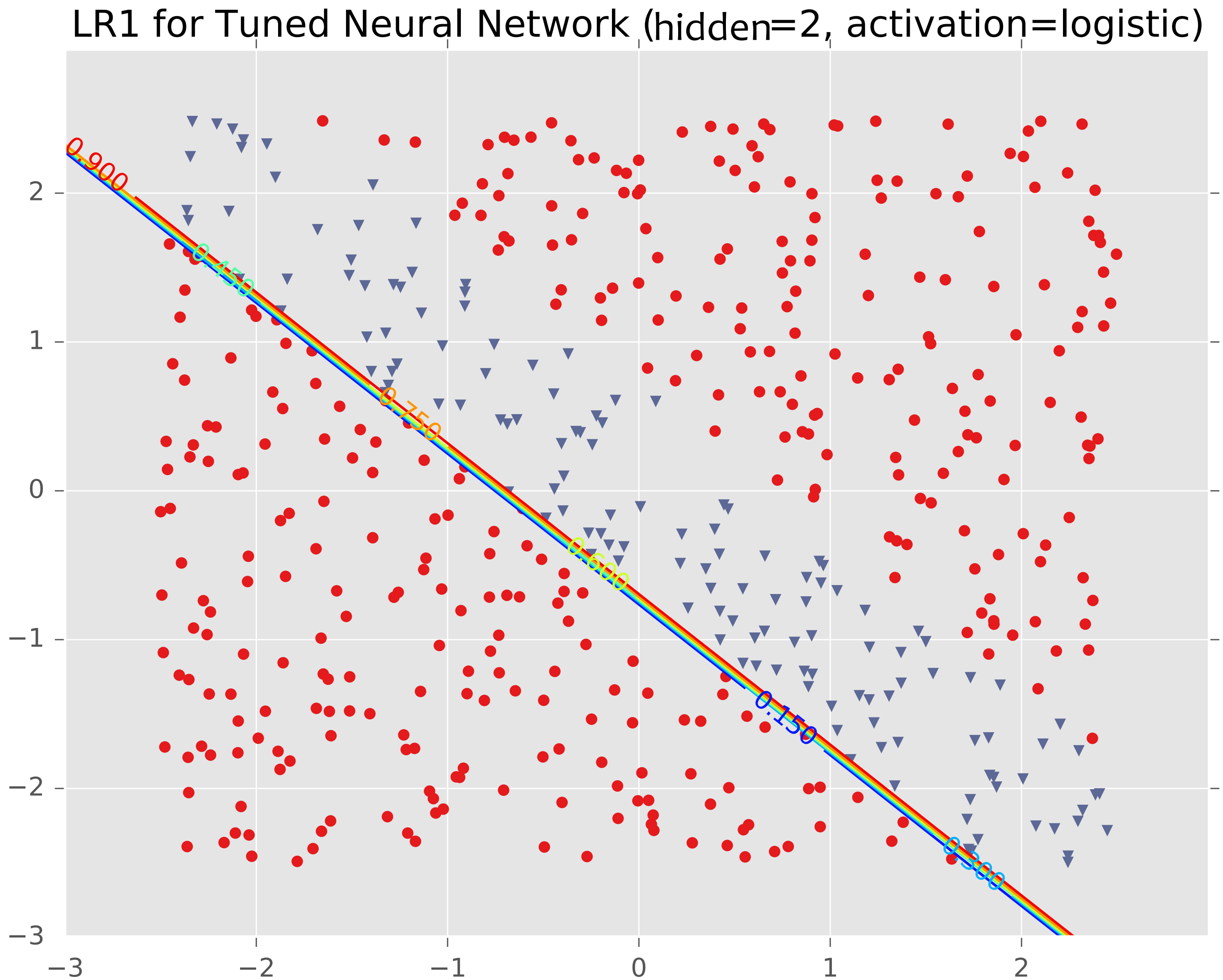
Example #1: Diagonal Band



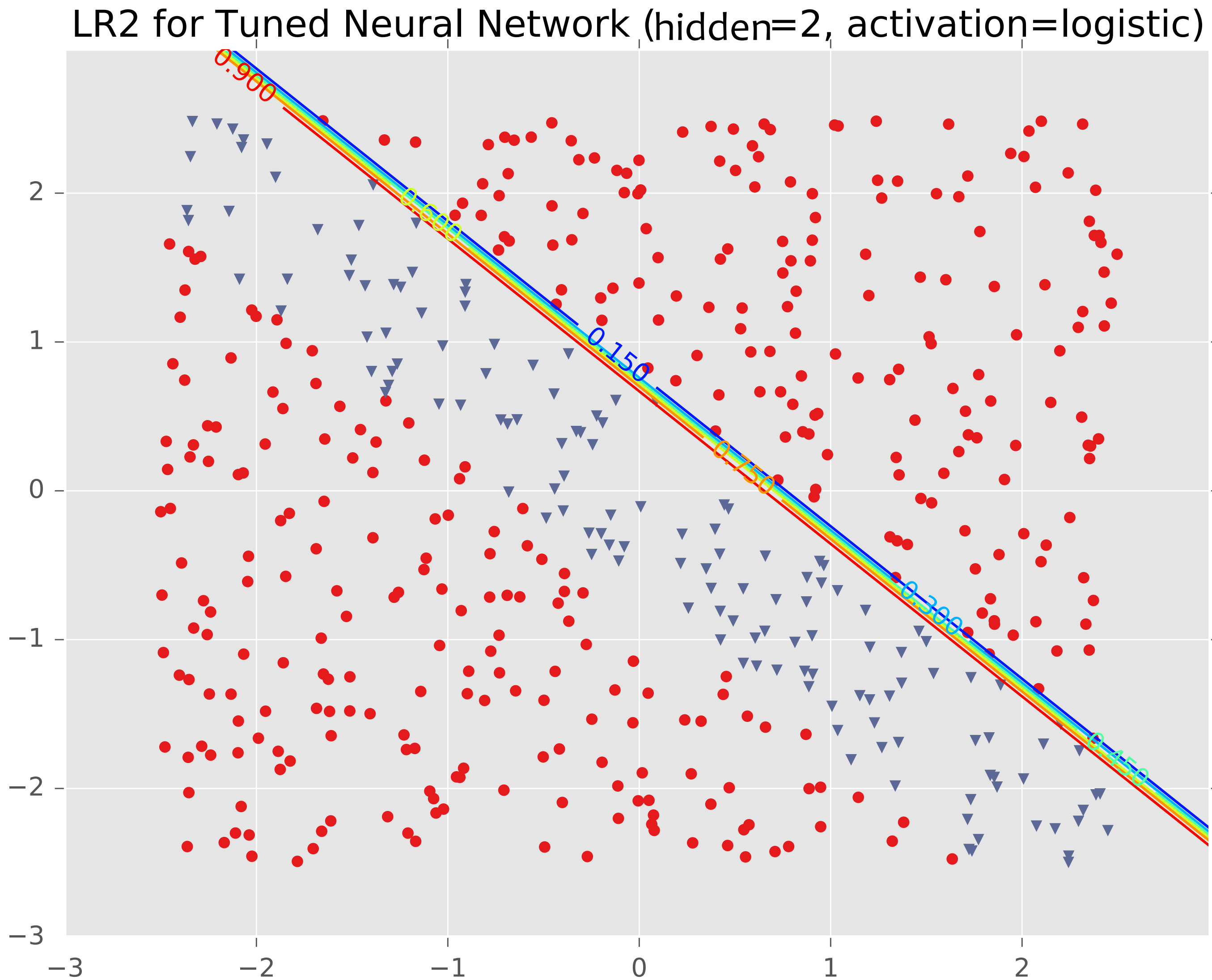
Example #1: Diagonal Band



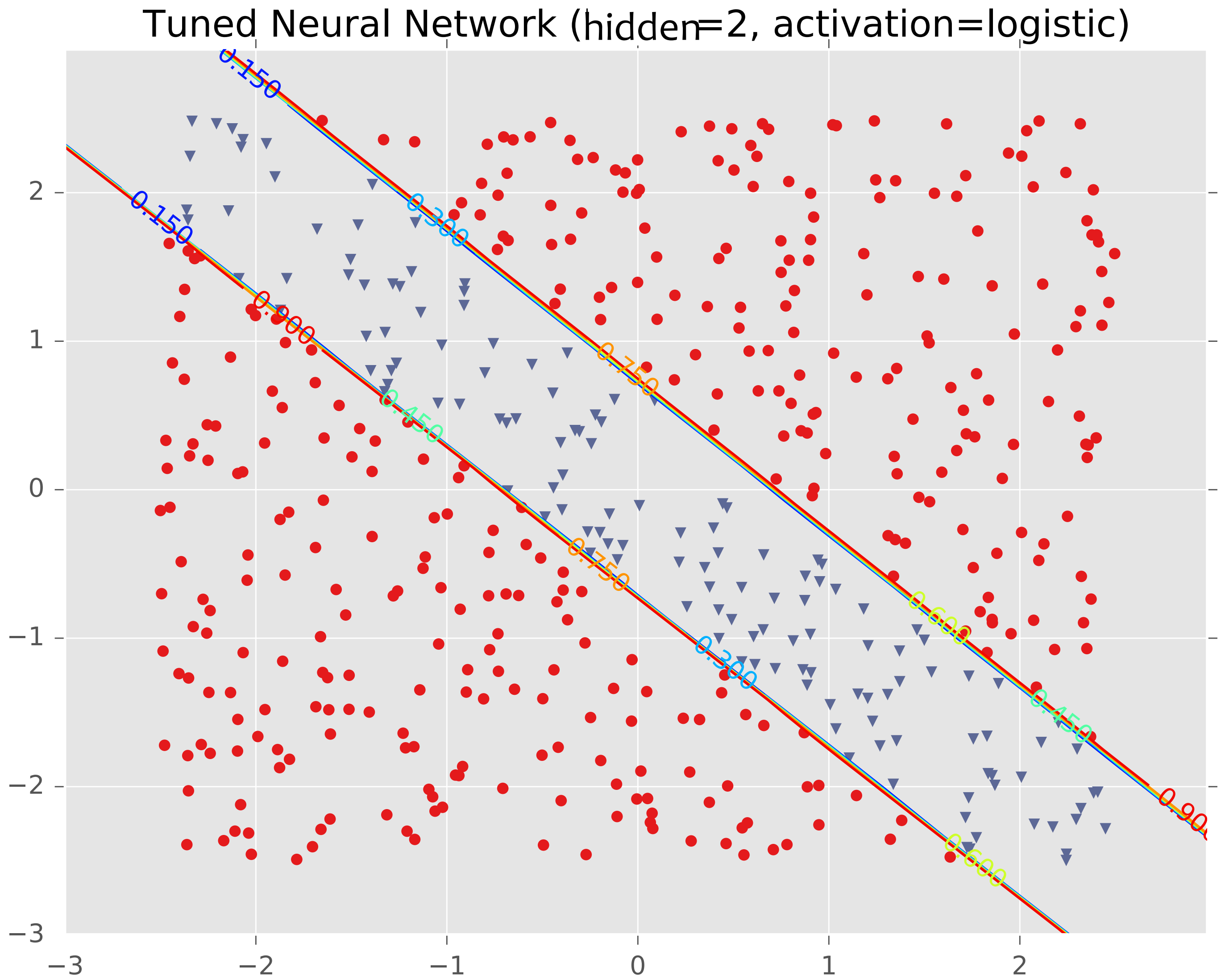
Example #1: Diagonal Band



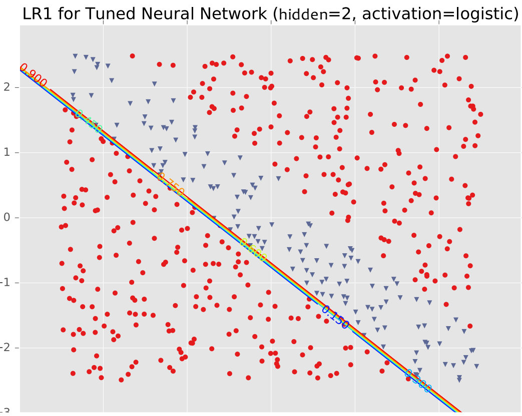
Example #1: Diagonal Band

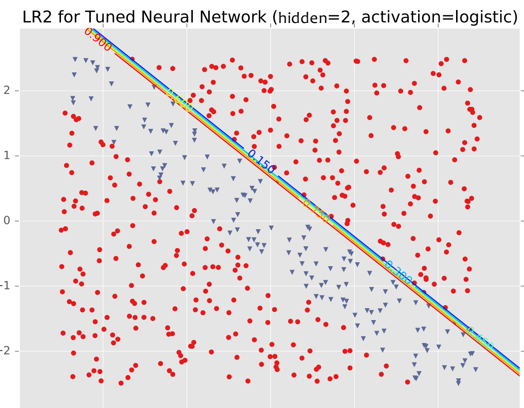


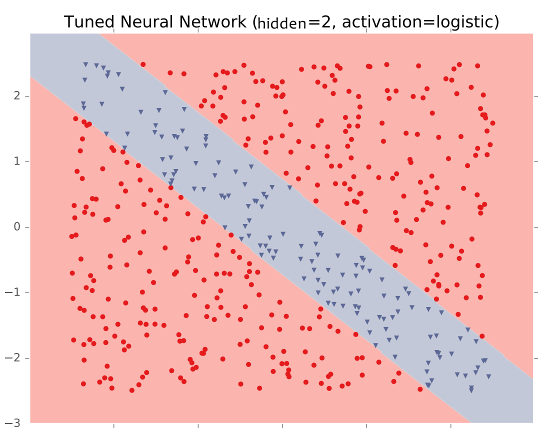
Example #1: Diagonal Band

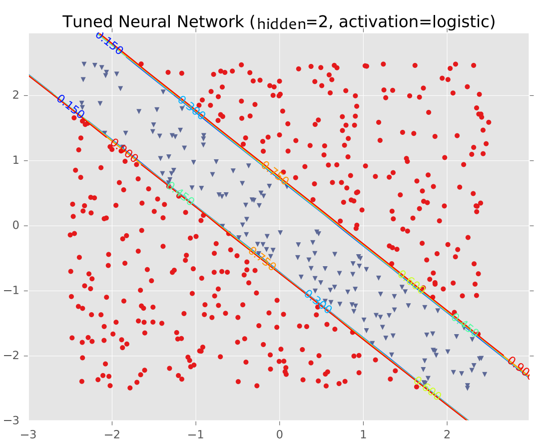


Example #1: Diagonal Band

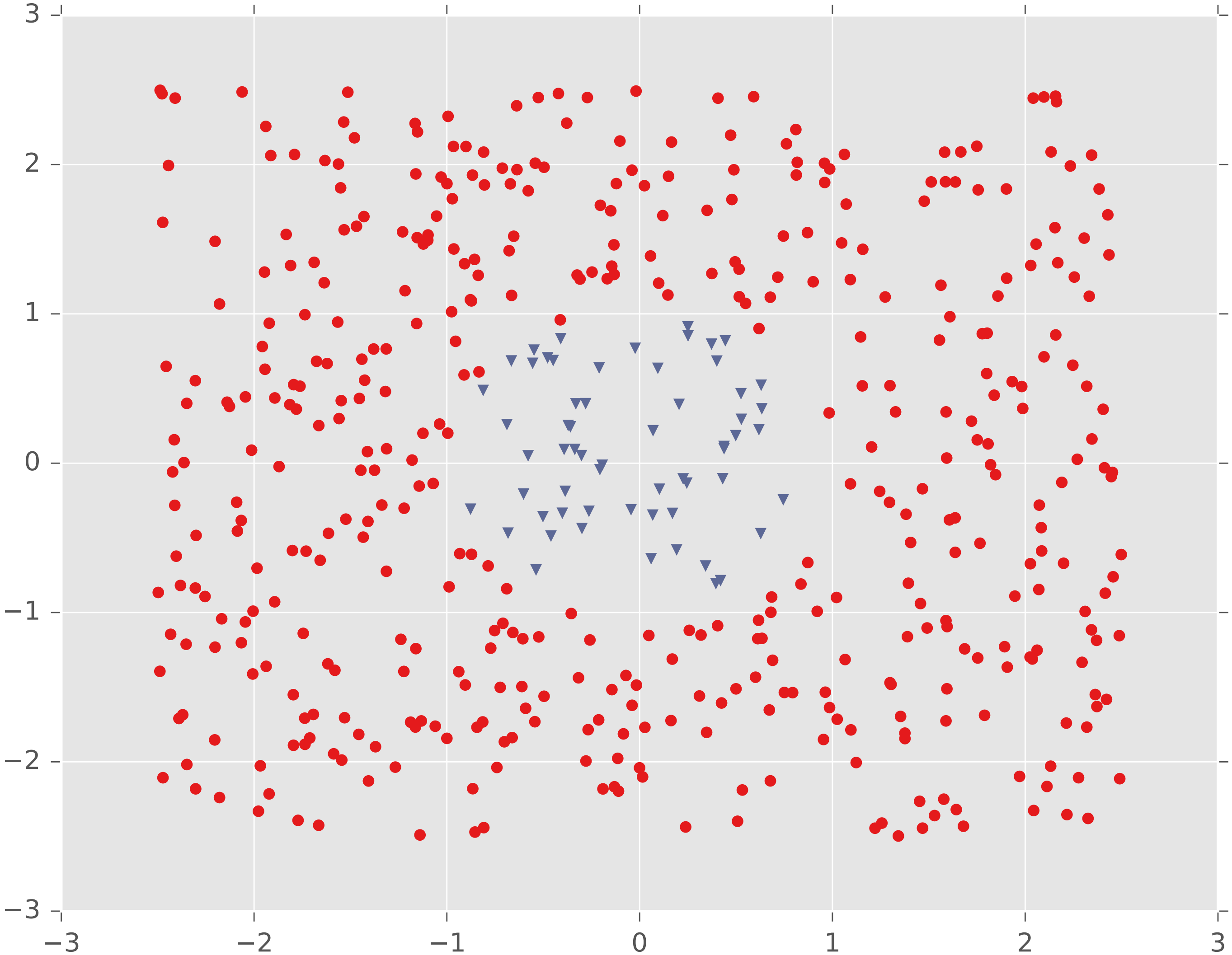






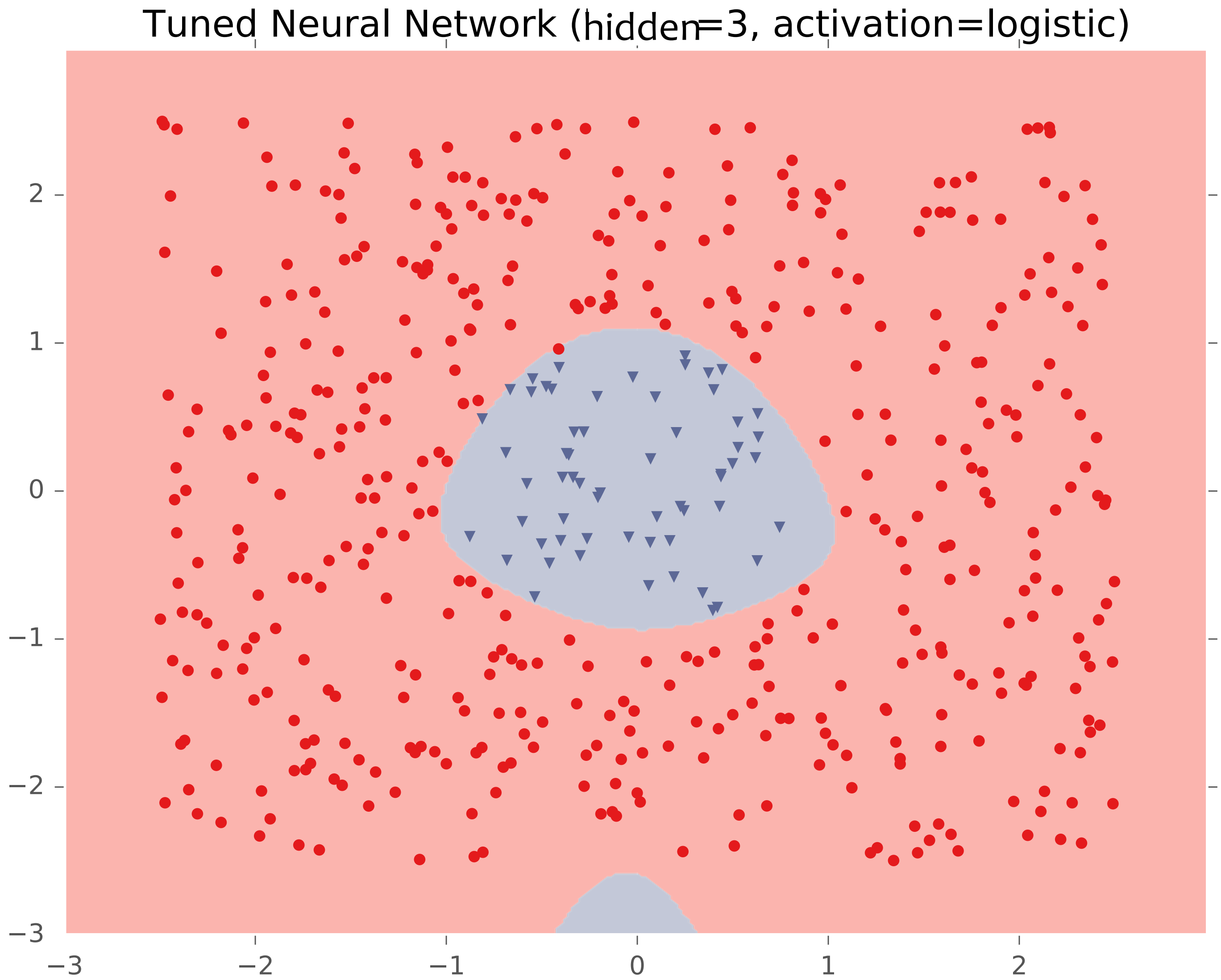


Example #2: One Pocket

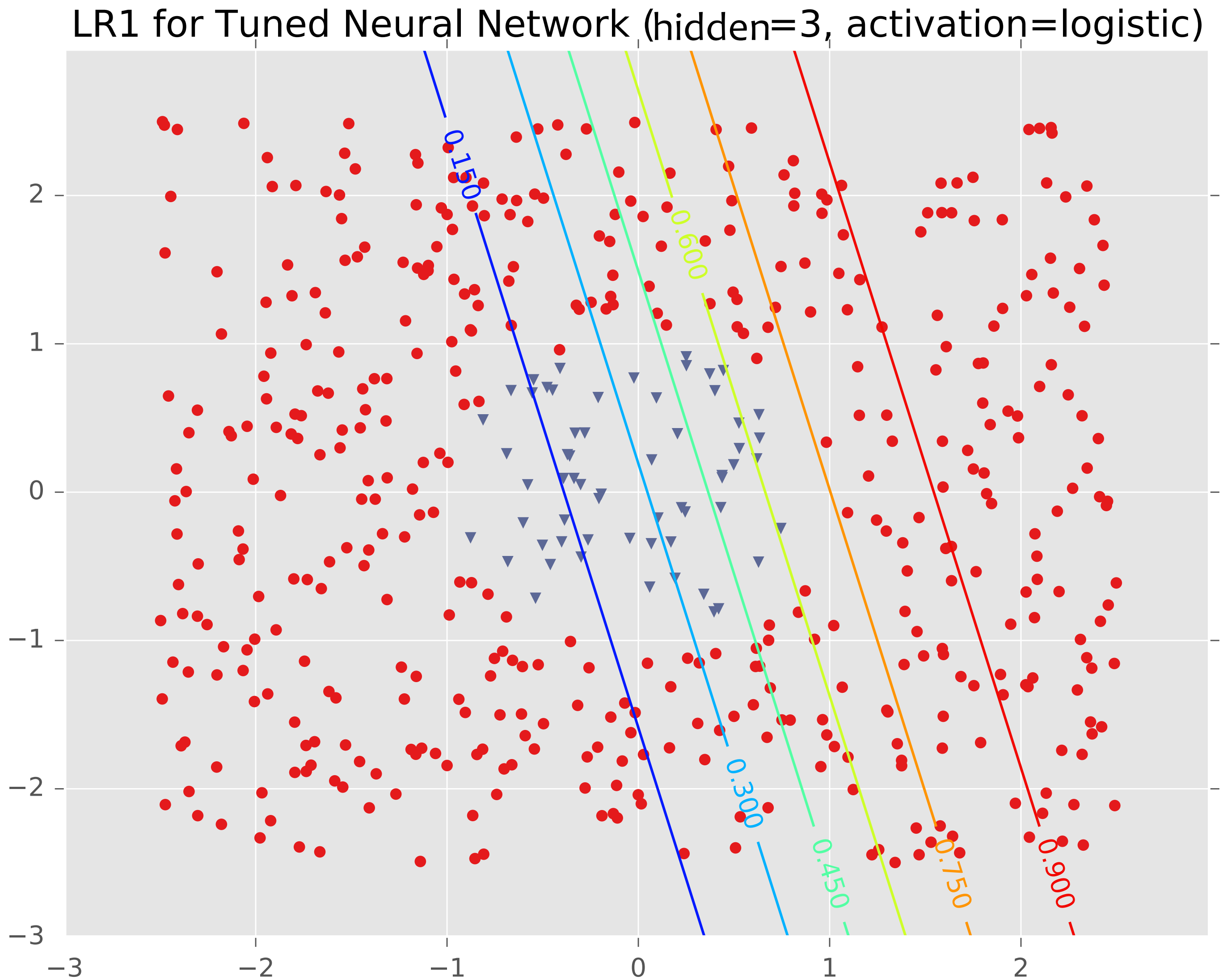


Example #2: One Pocket

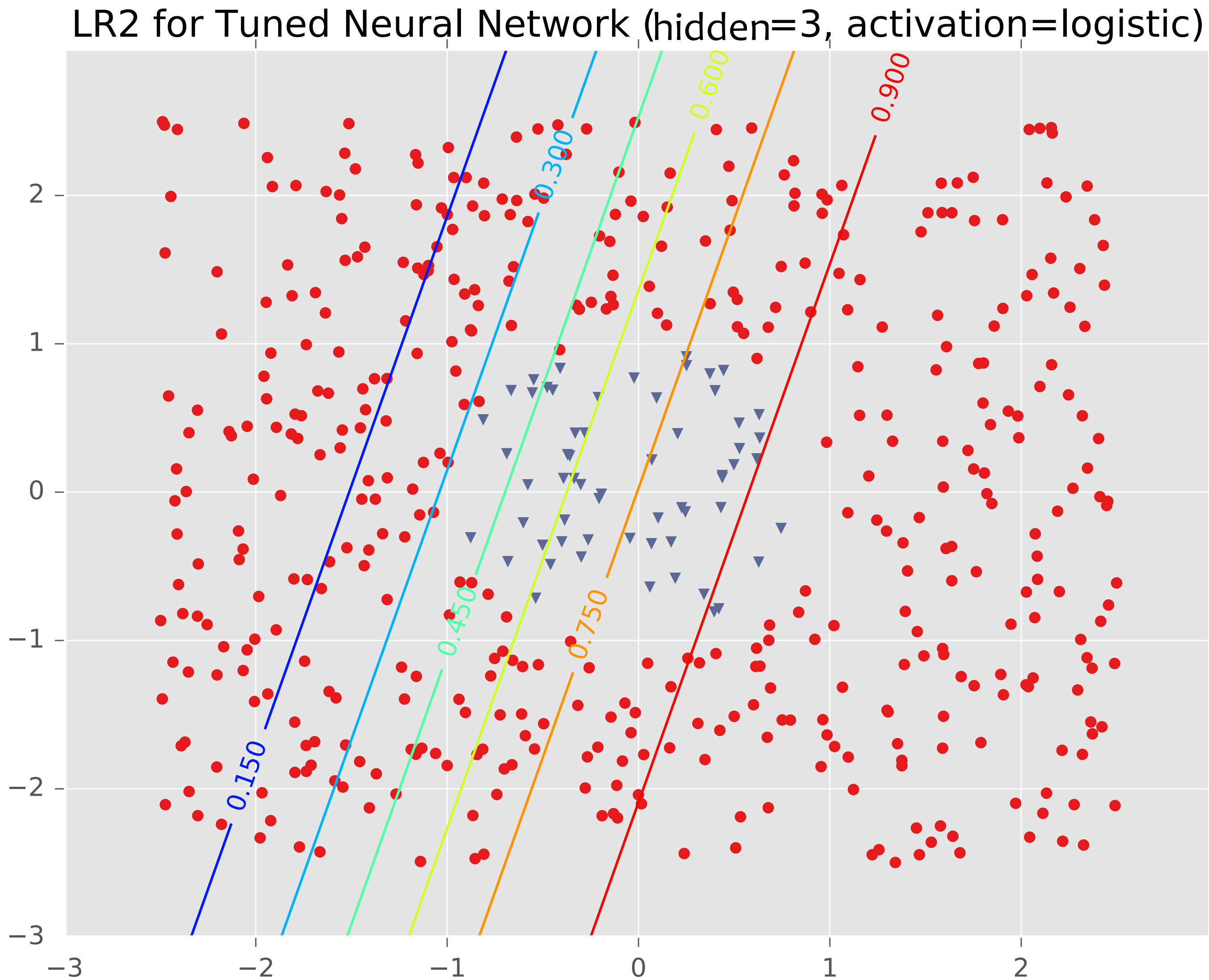
Example #2: One Pocket



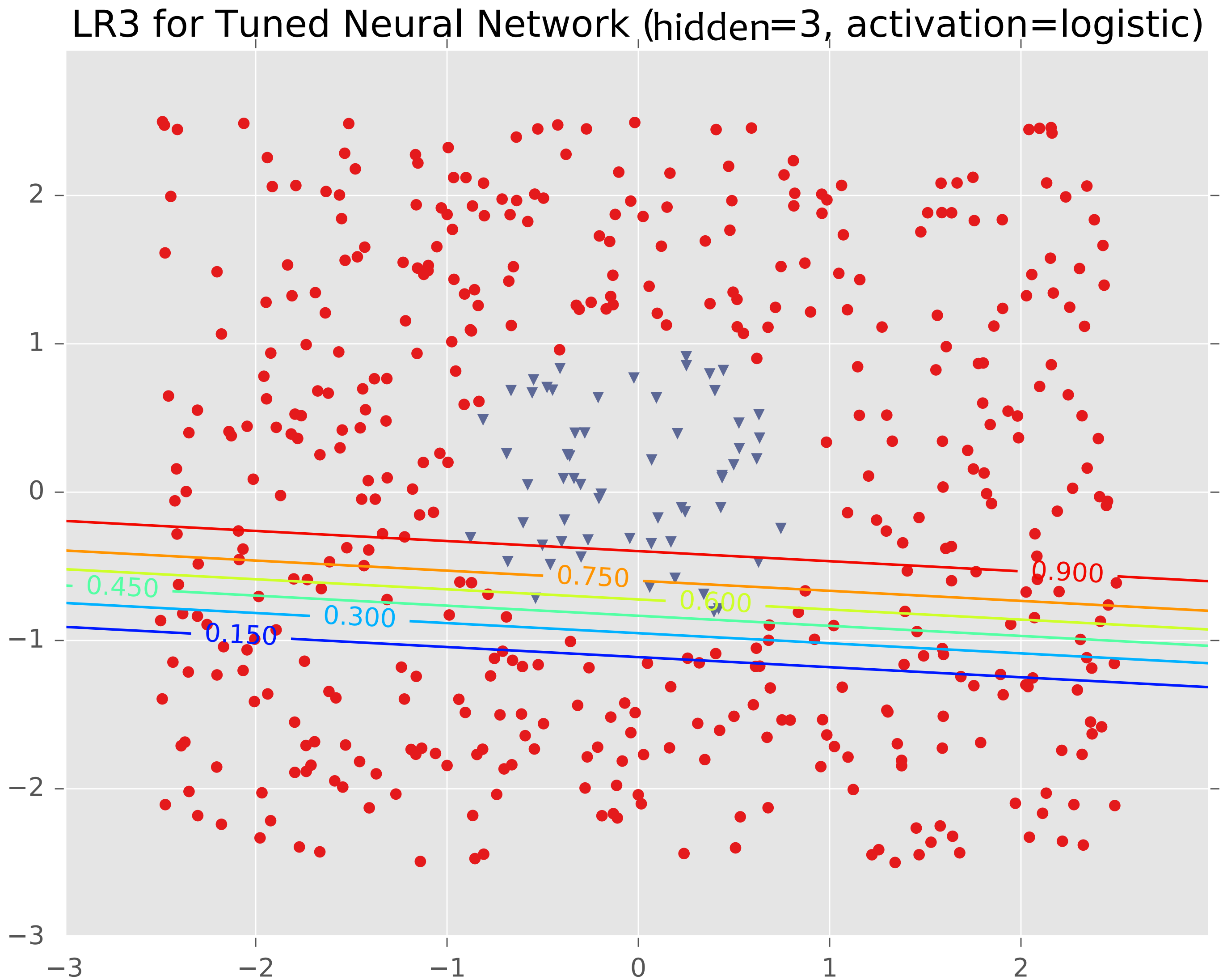
Example #2: One Pocket



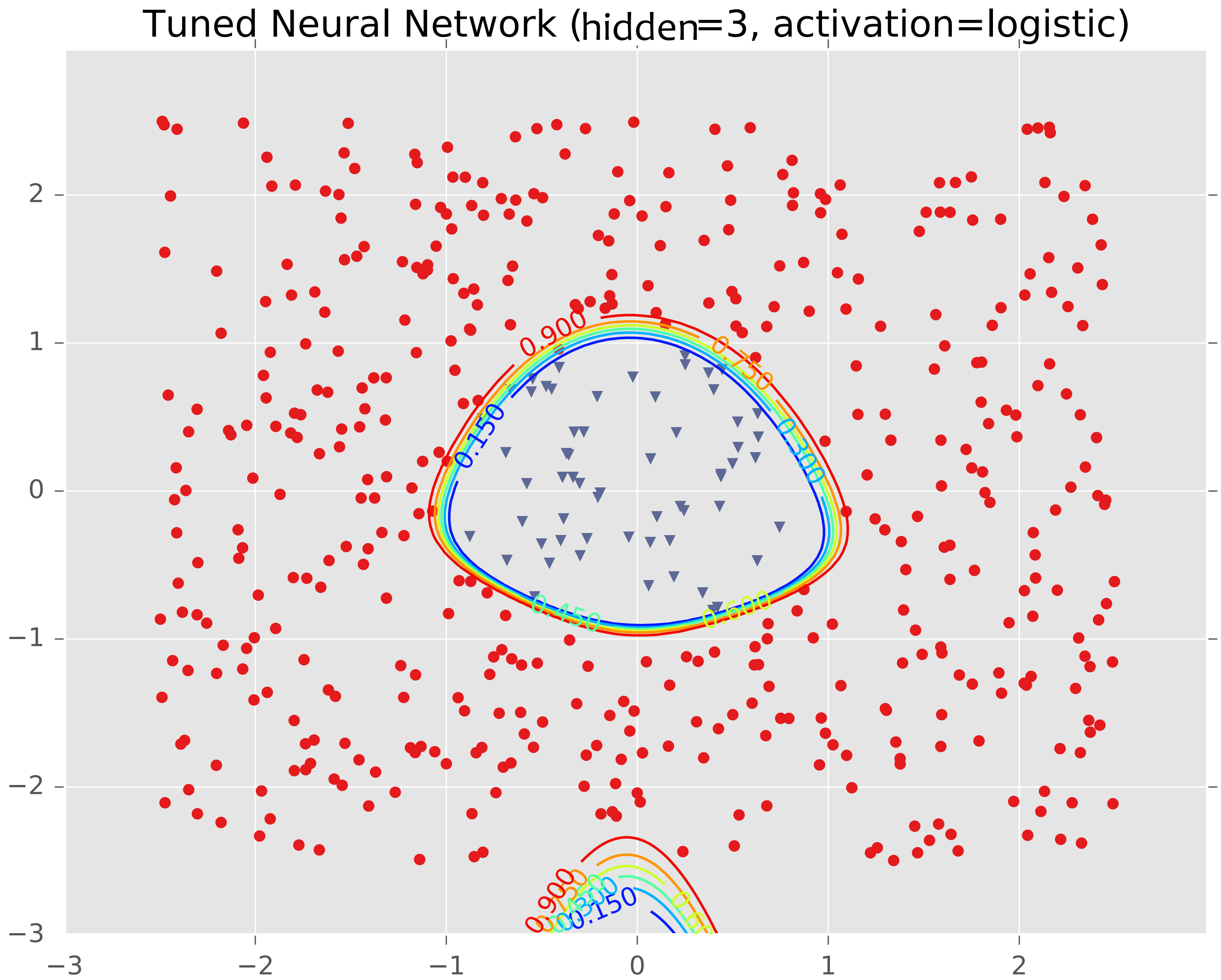
Example #2: One Pocket



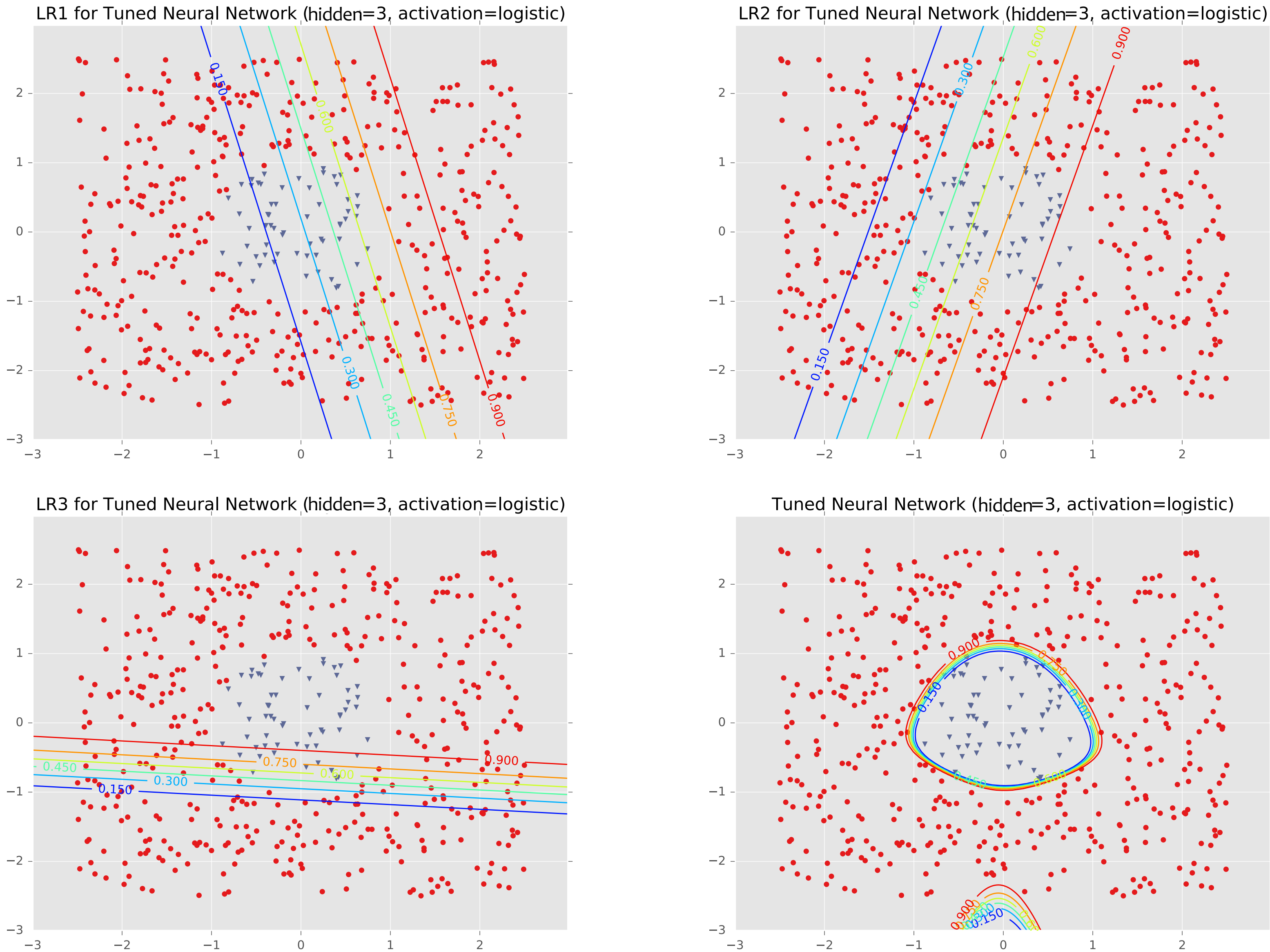
Example #2: One Pocket

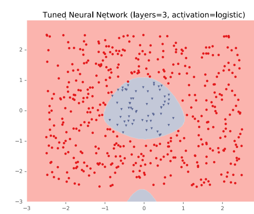


Example #2: One Pocket

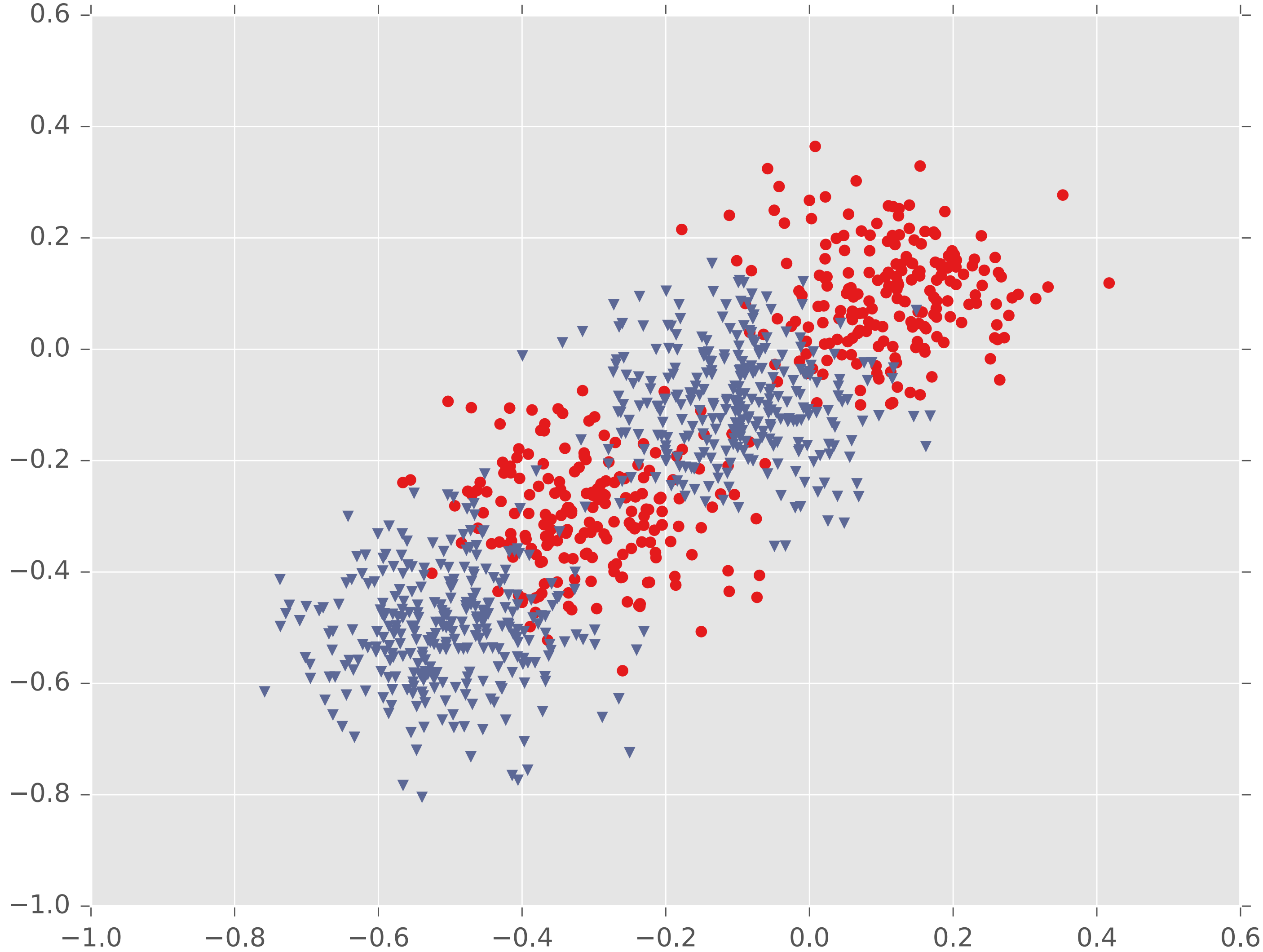


Example #2: One Pocket

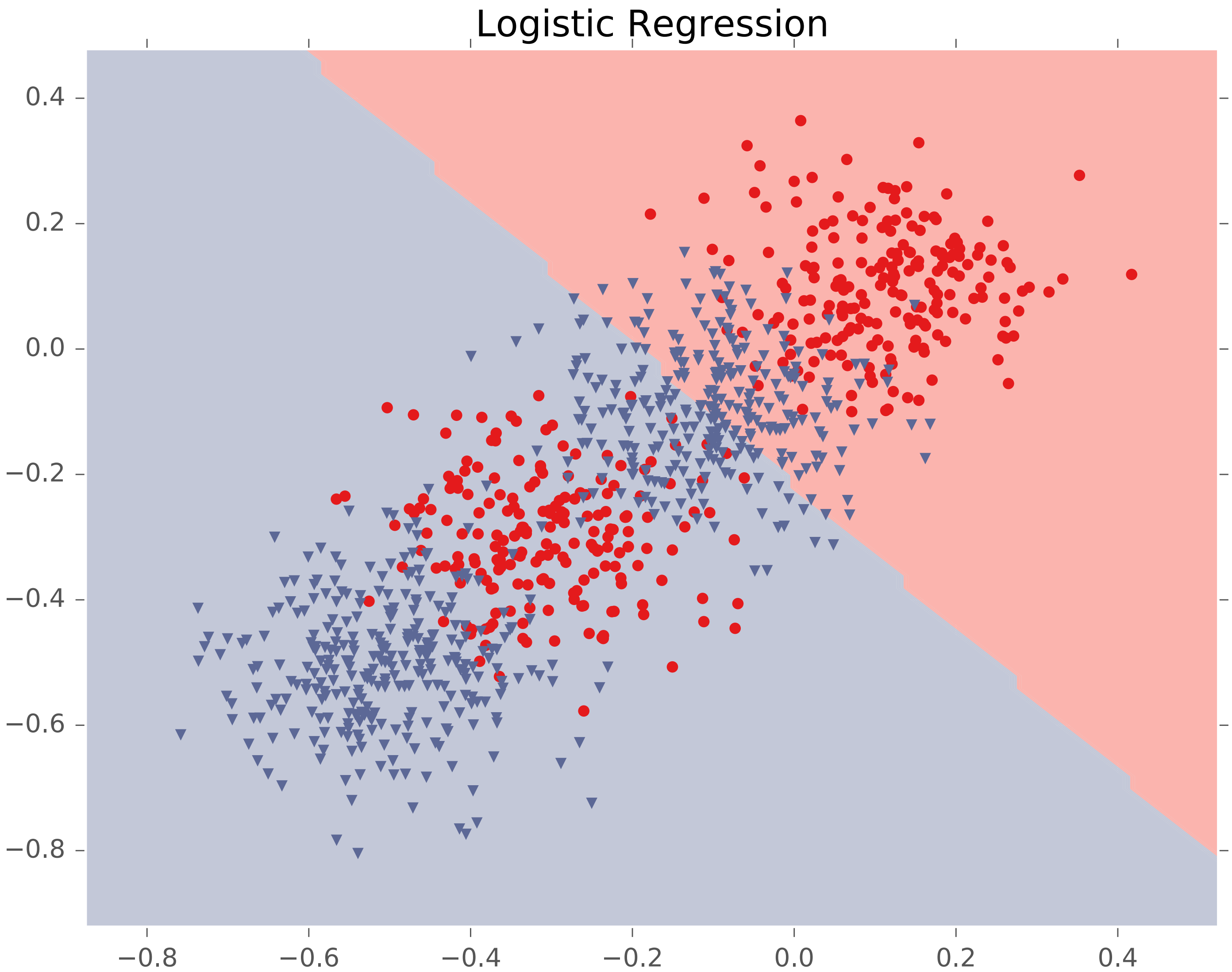




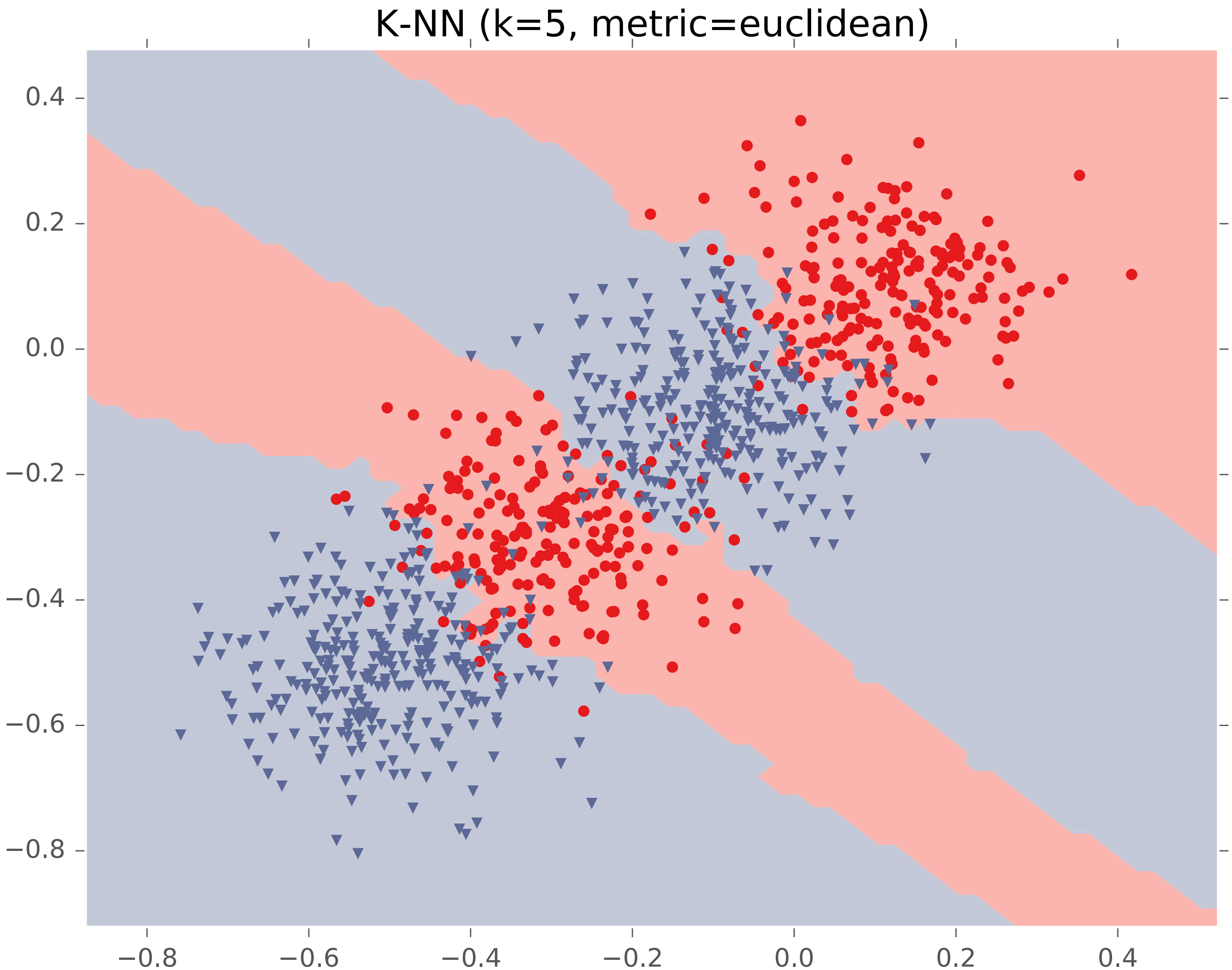
Example #3: Four Gaussians



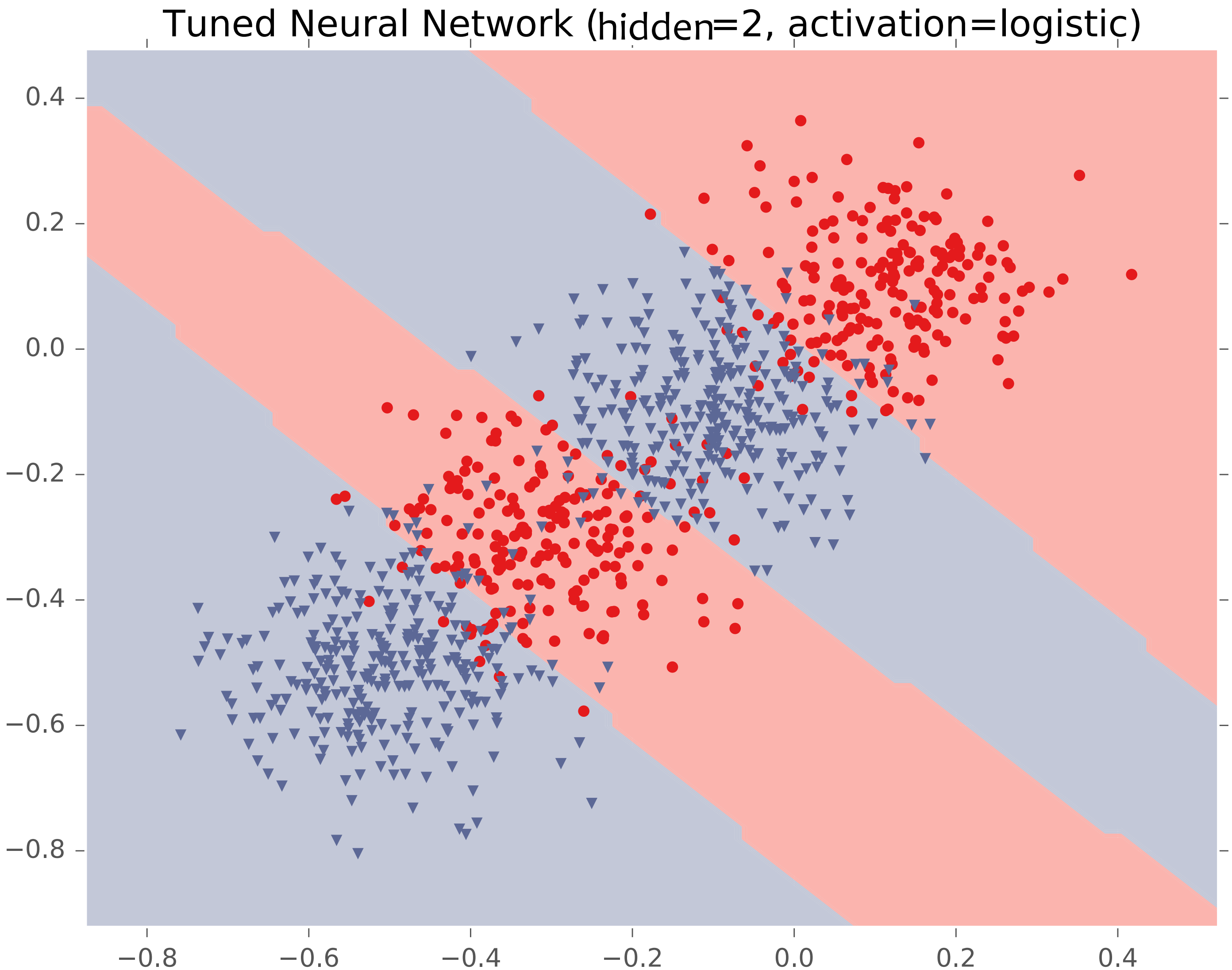
Example #3: Four Gaussians



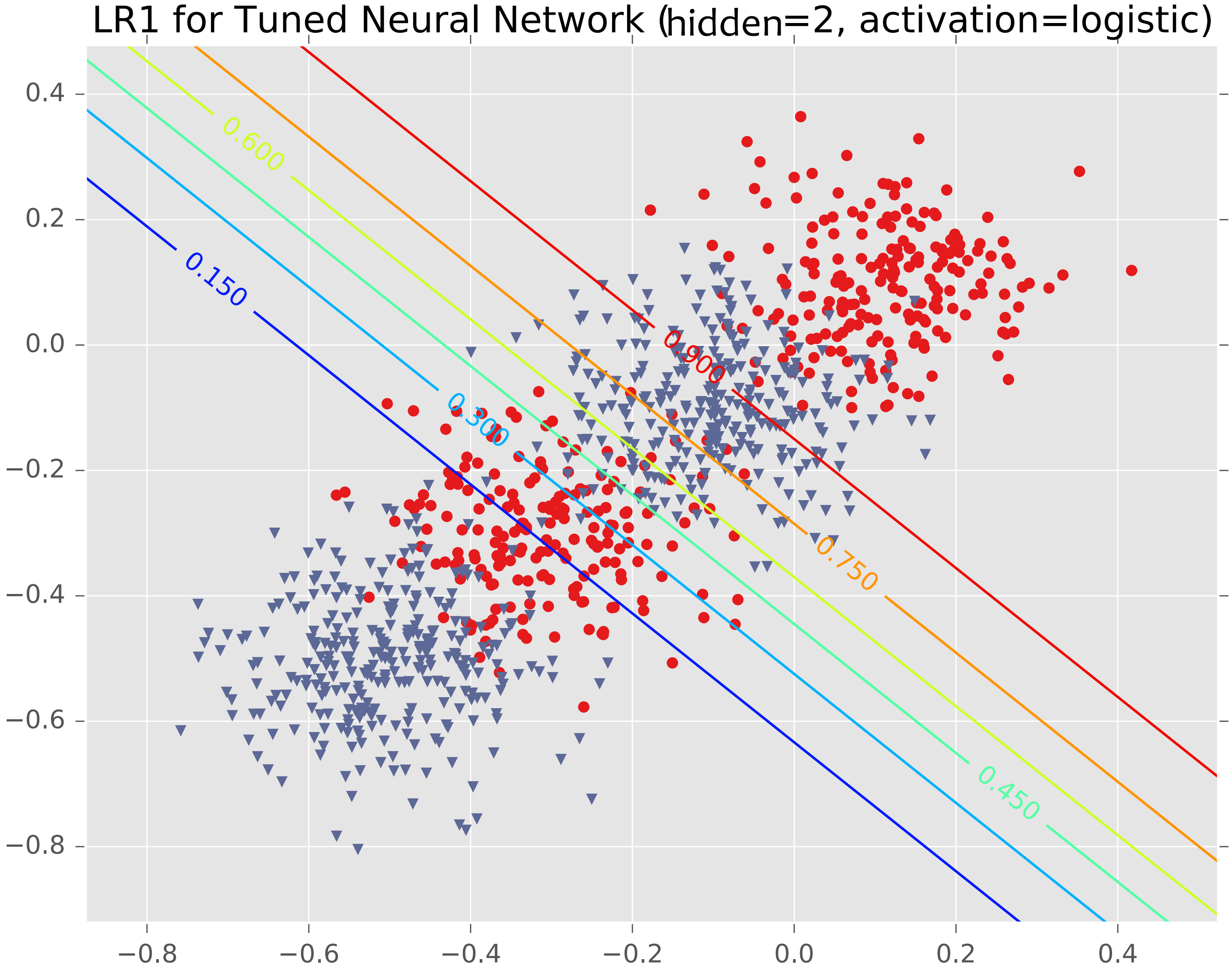
Example #3: Four Gaussians



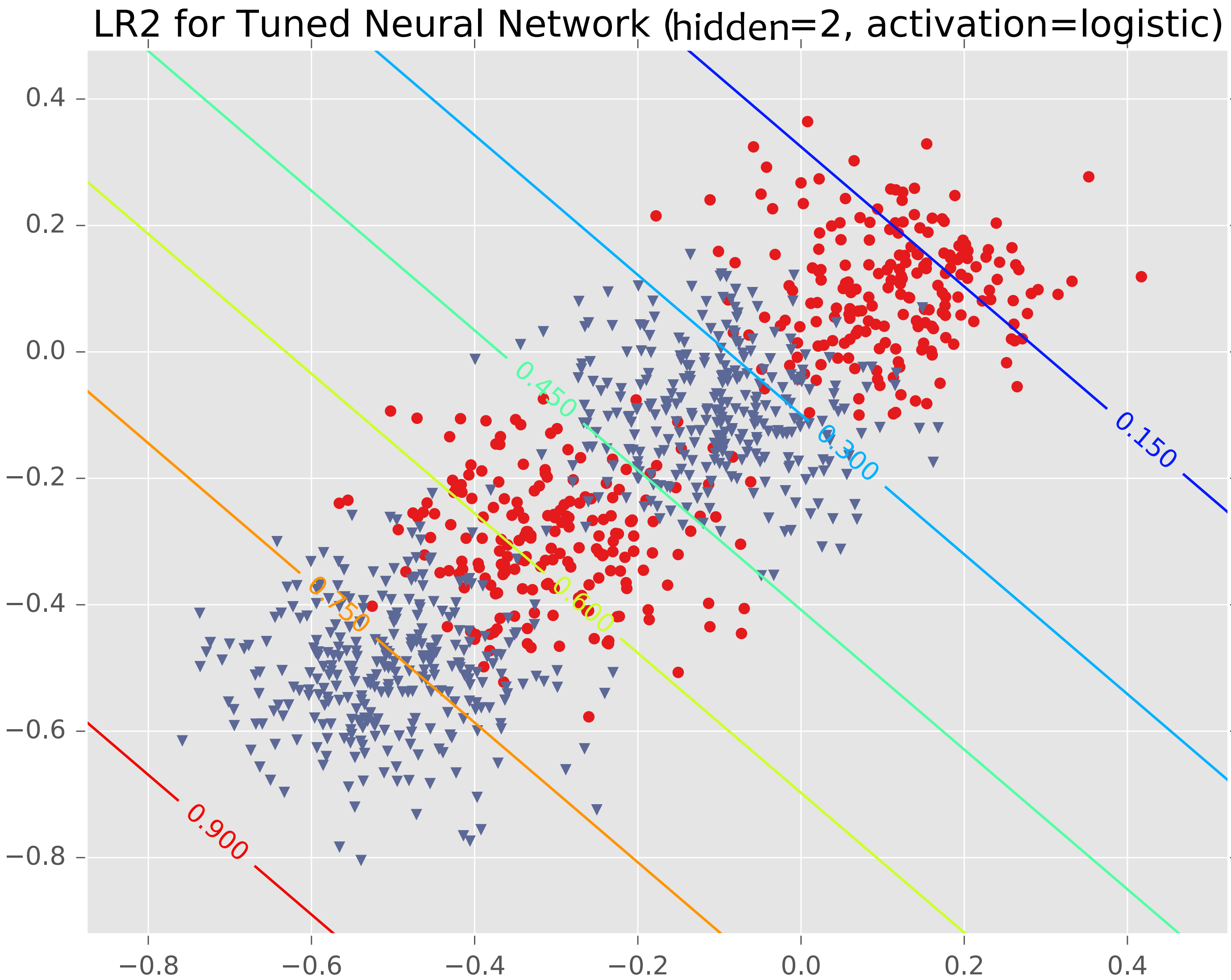
Example #3: Four Gaussians



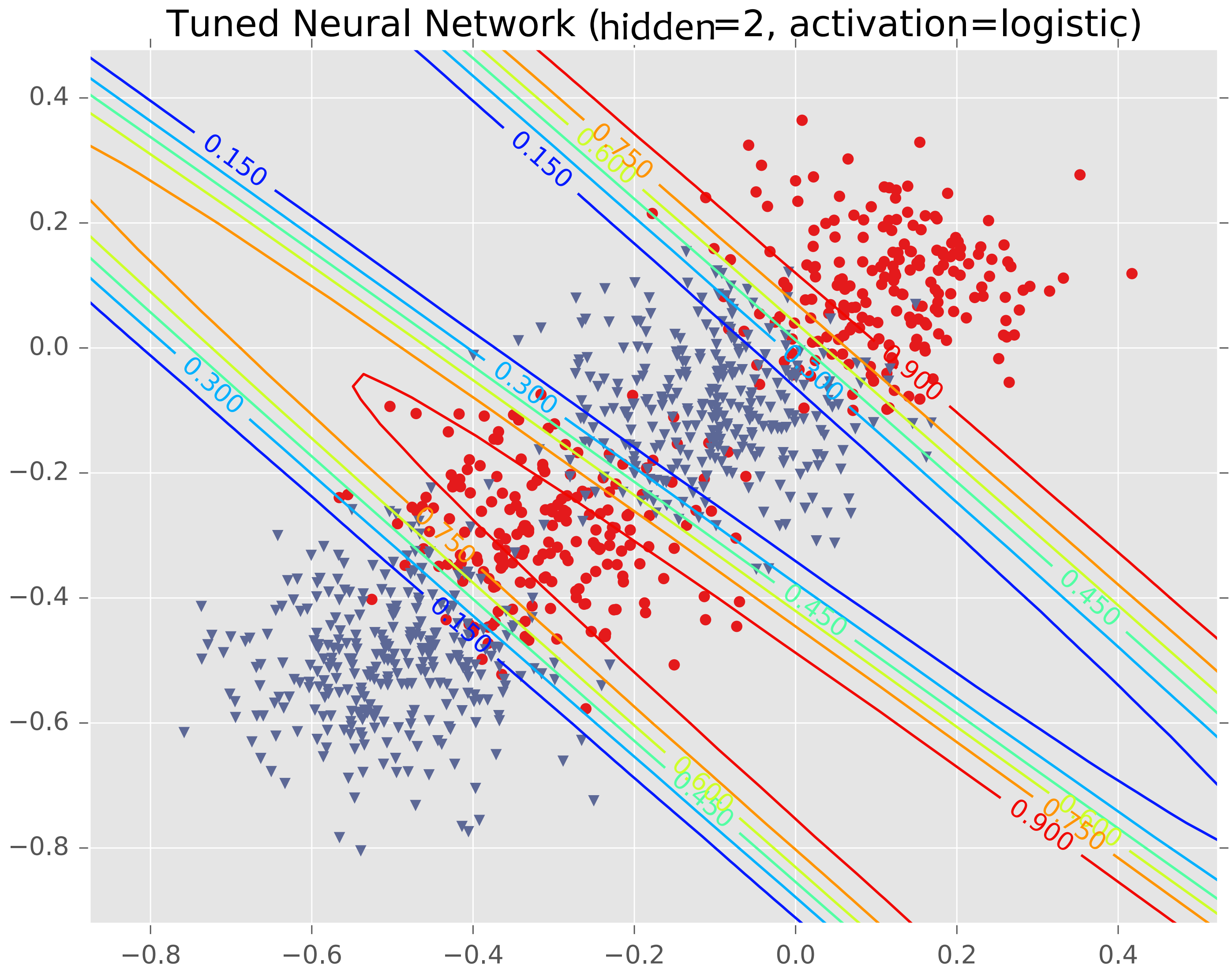
Example #3: Four Gaussians



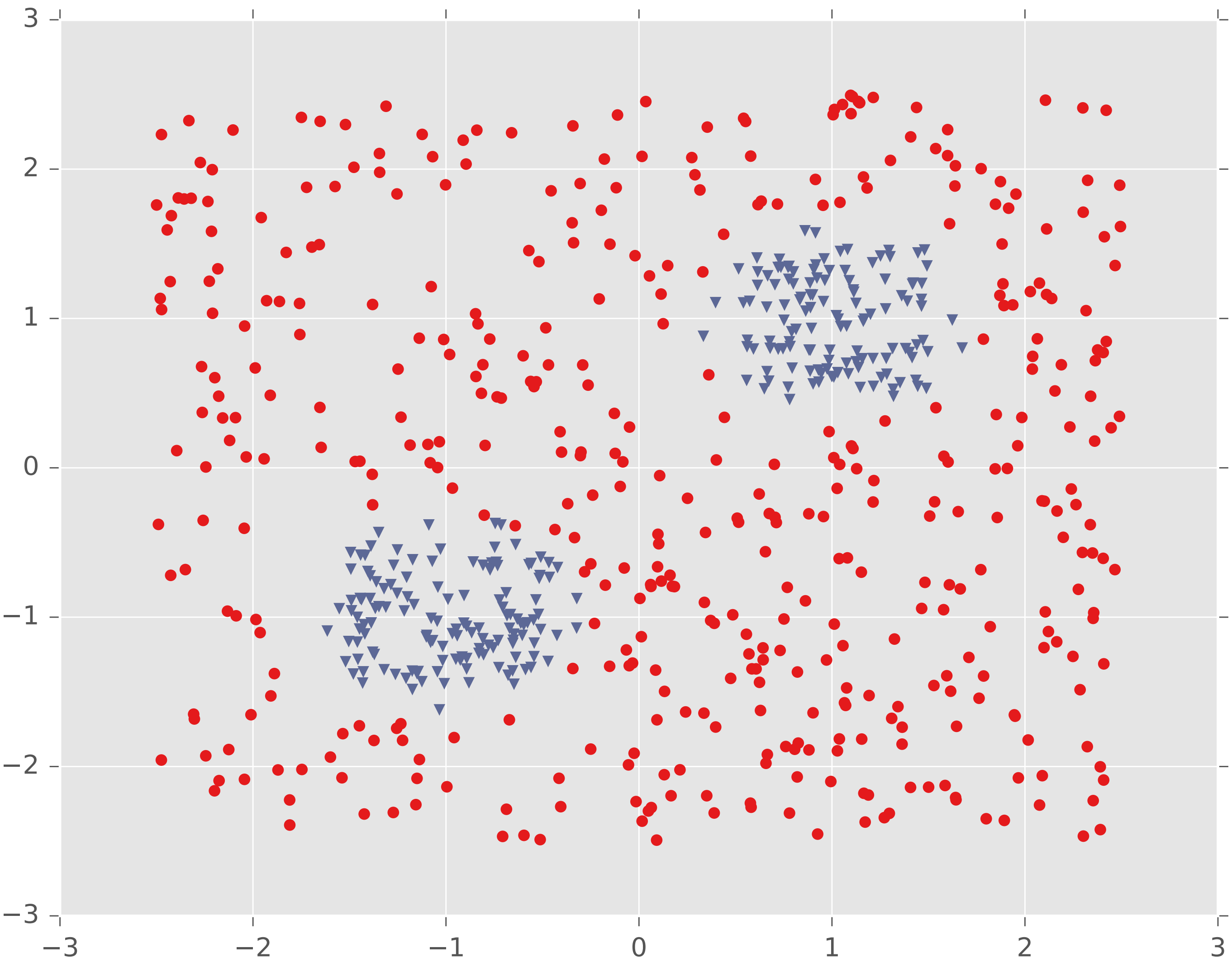
Example #3: Four Gaussians



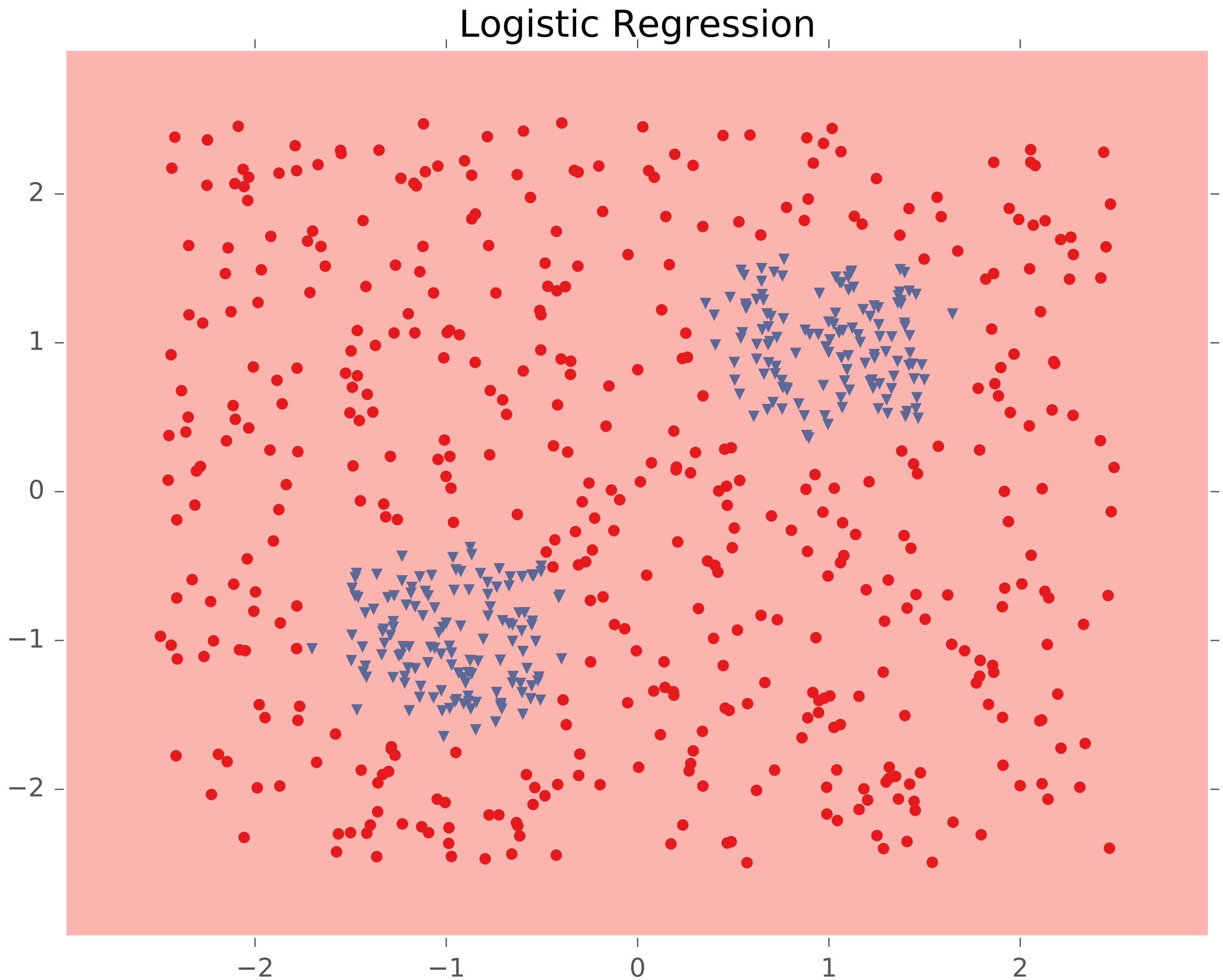
Example #3: Four Gaussians



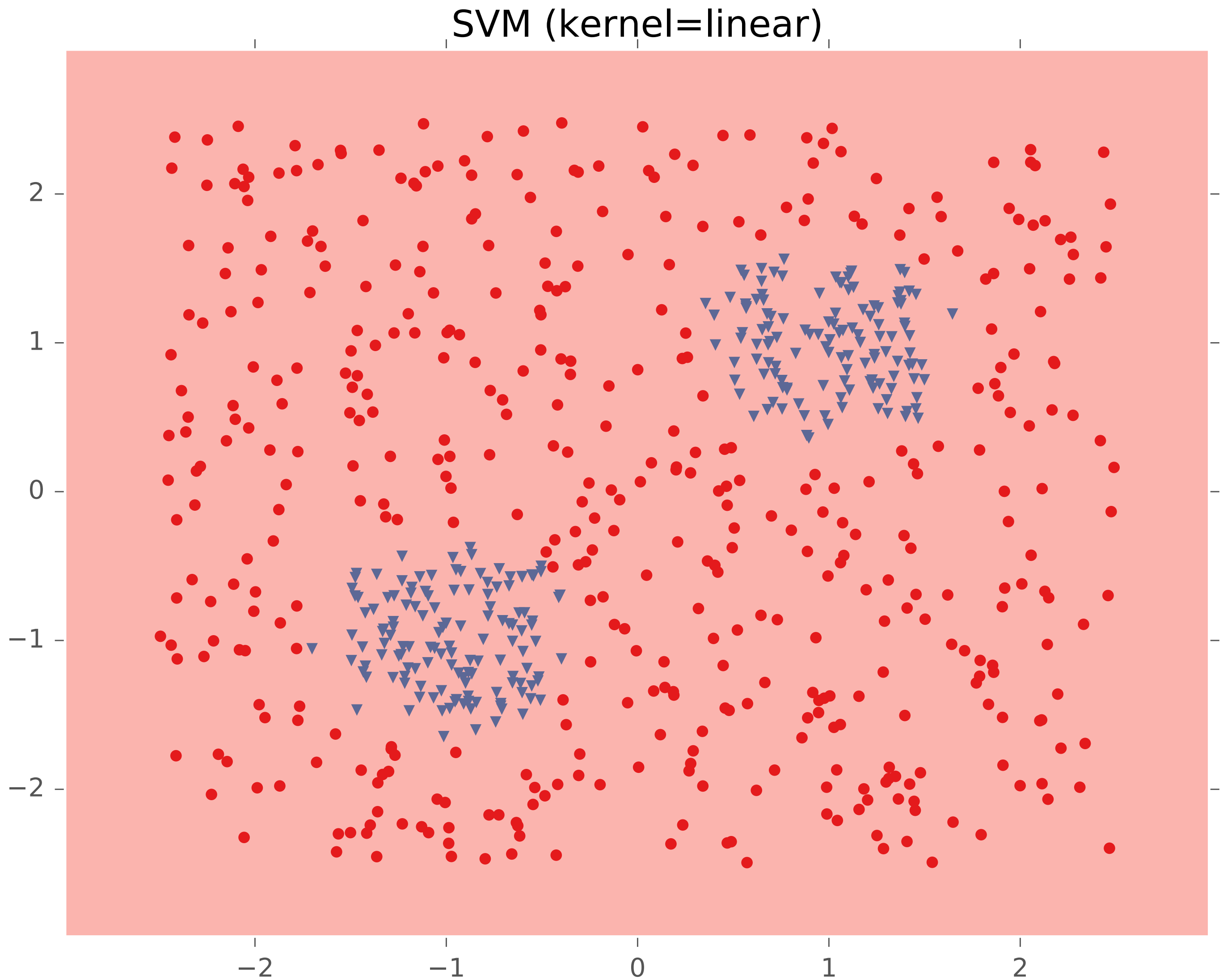
Example #4: Two Pockets



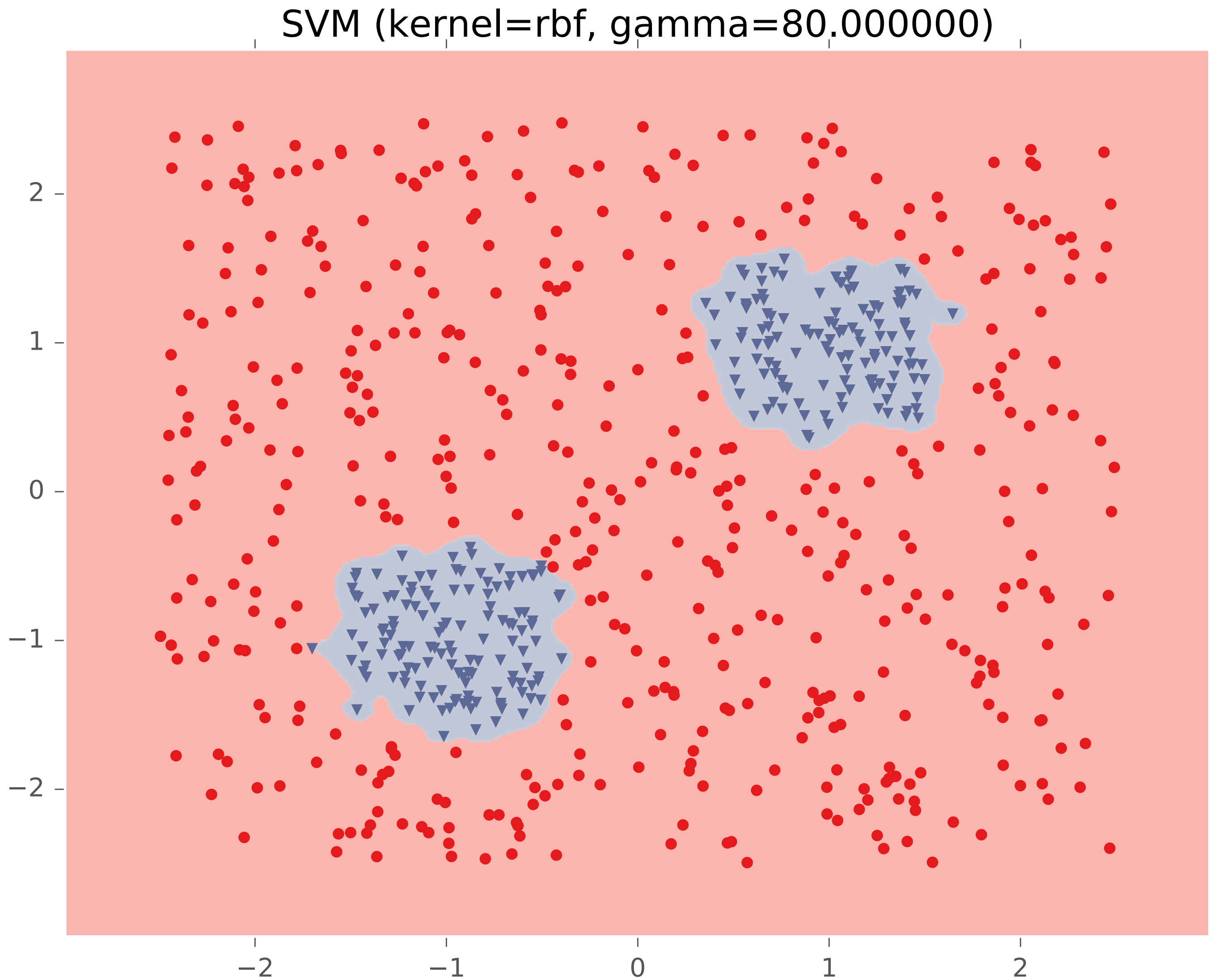
Example #4: Two Pockets



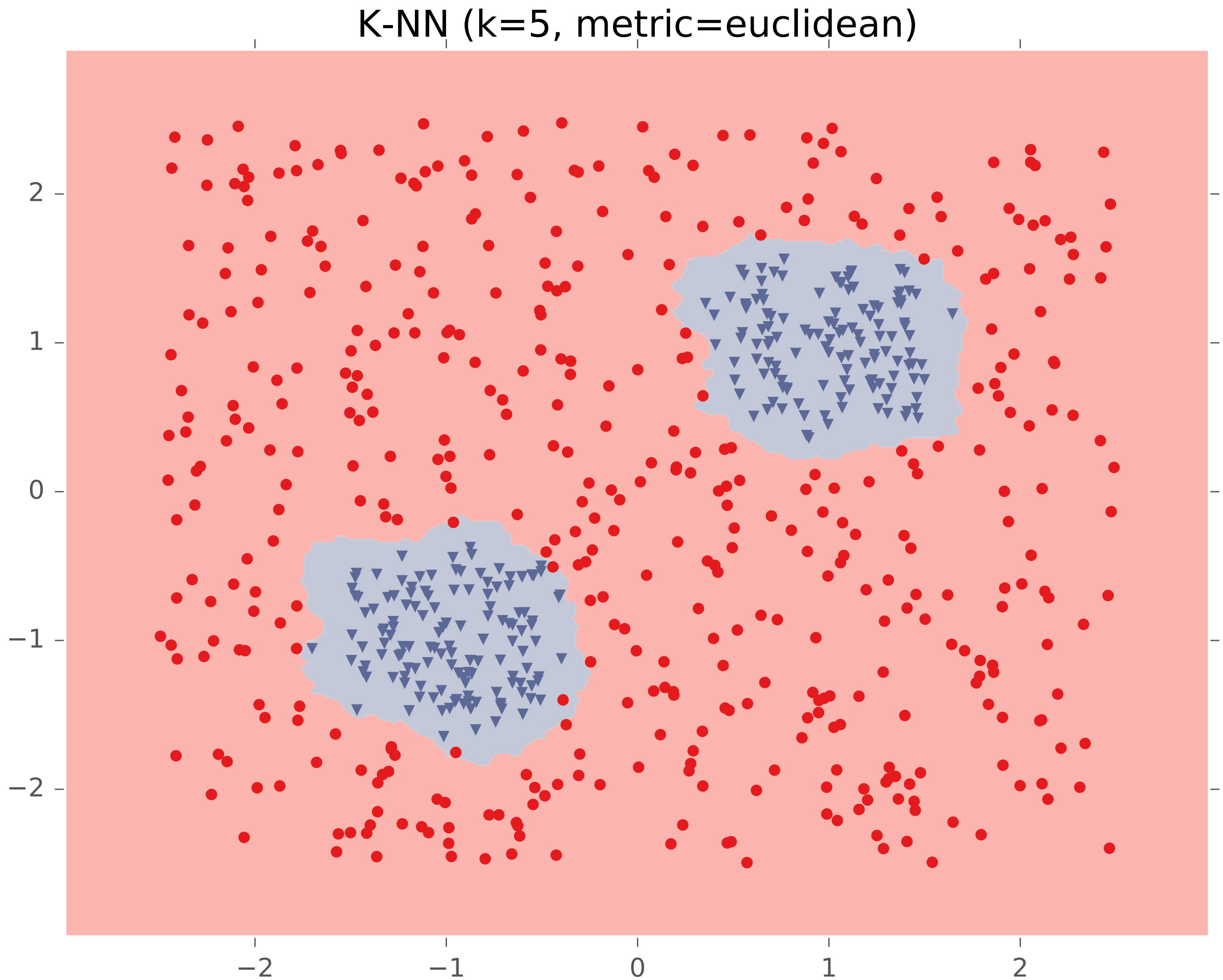
Example #4: Two Pockets



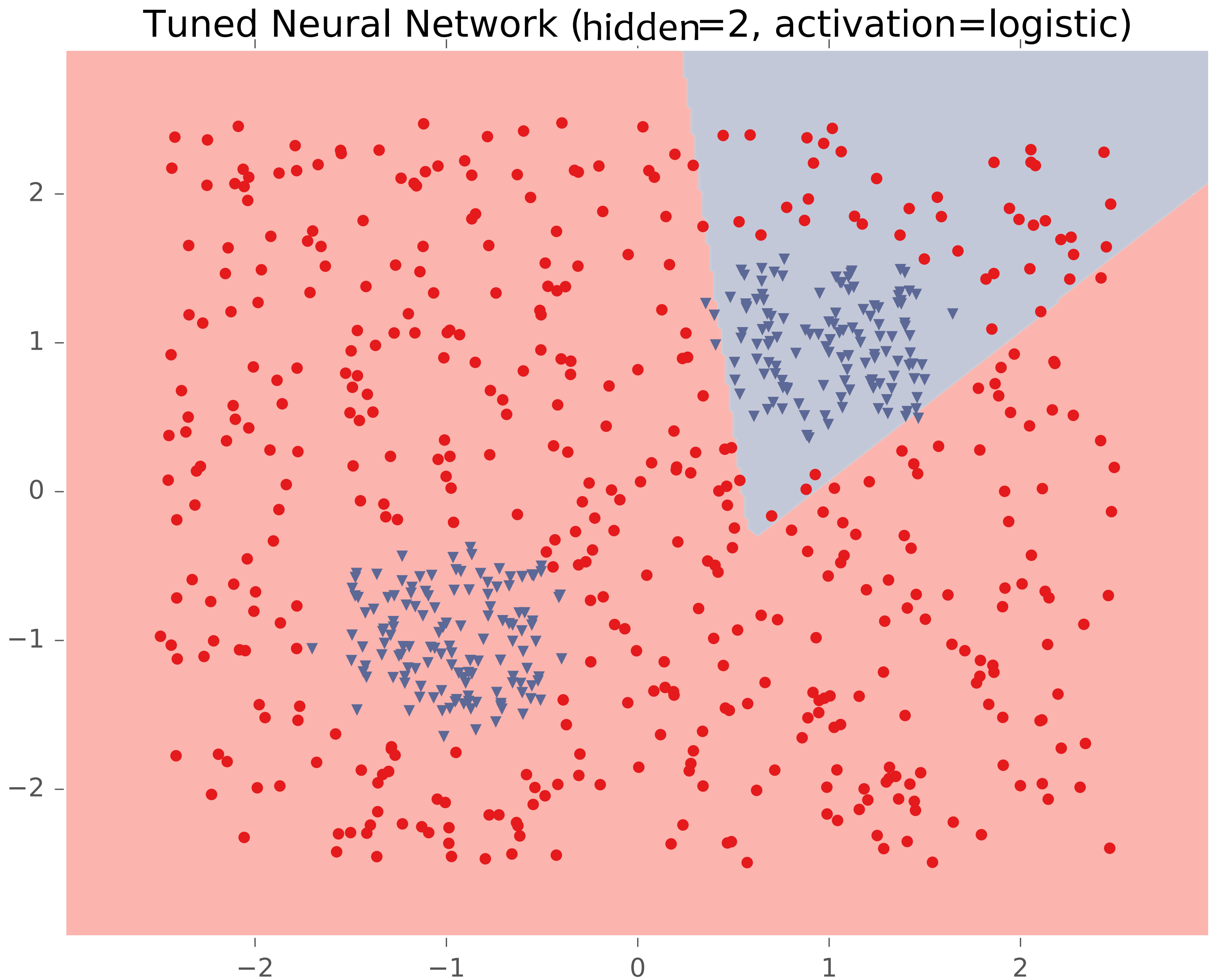
Example #4: Two Pockets



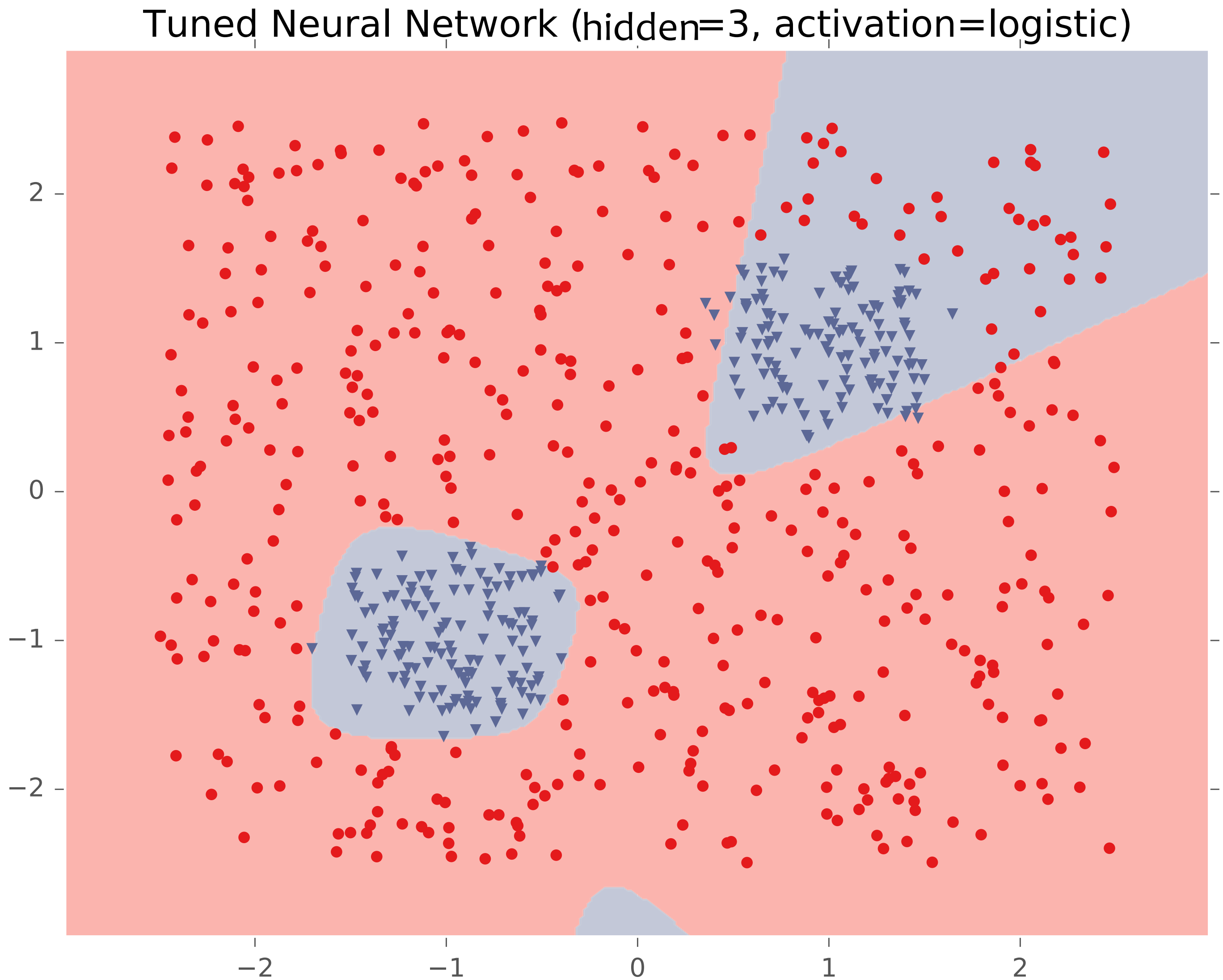
Example #4: Two Pockets



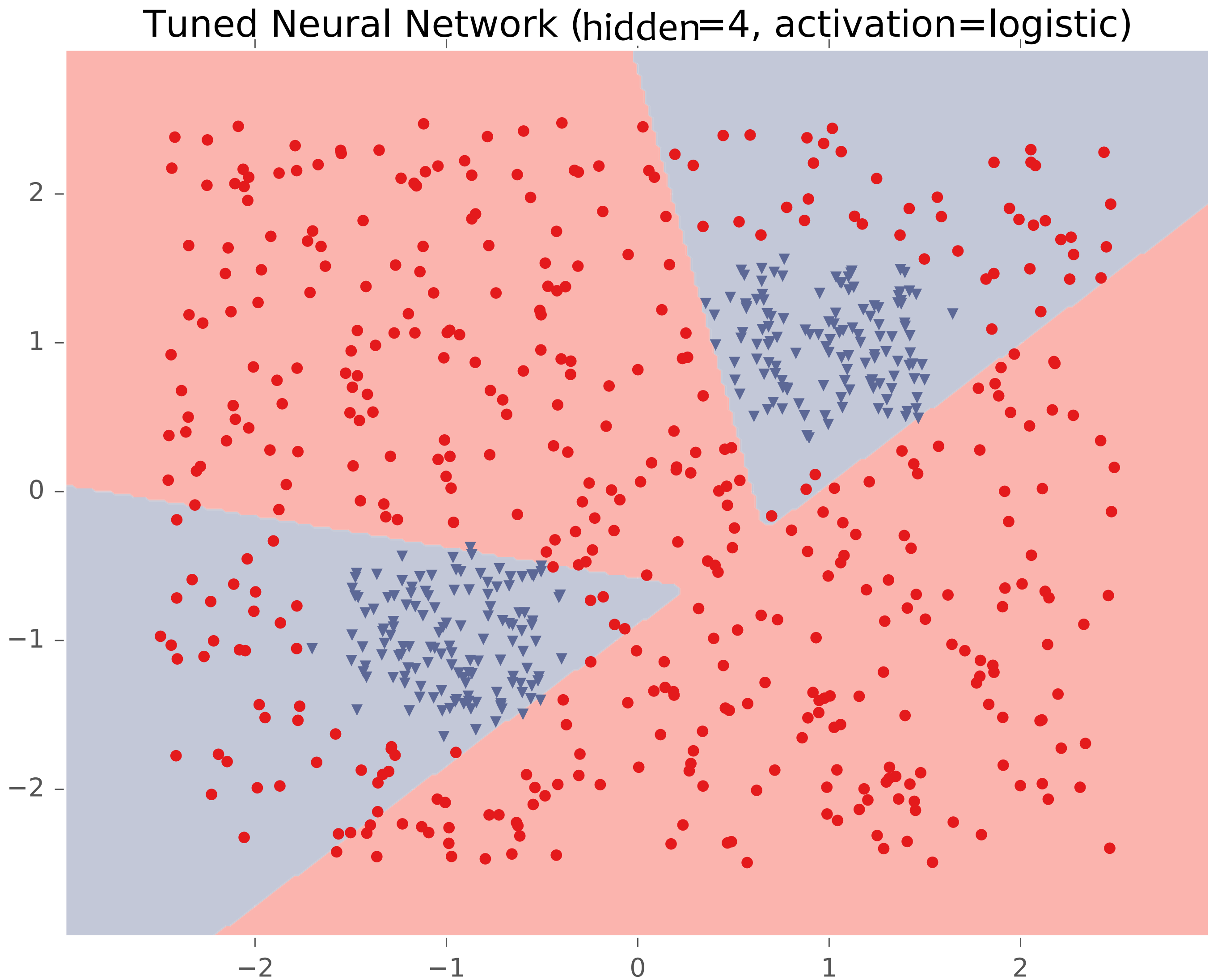
Example #4: Two Pockets



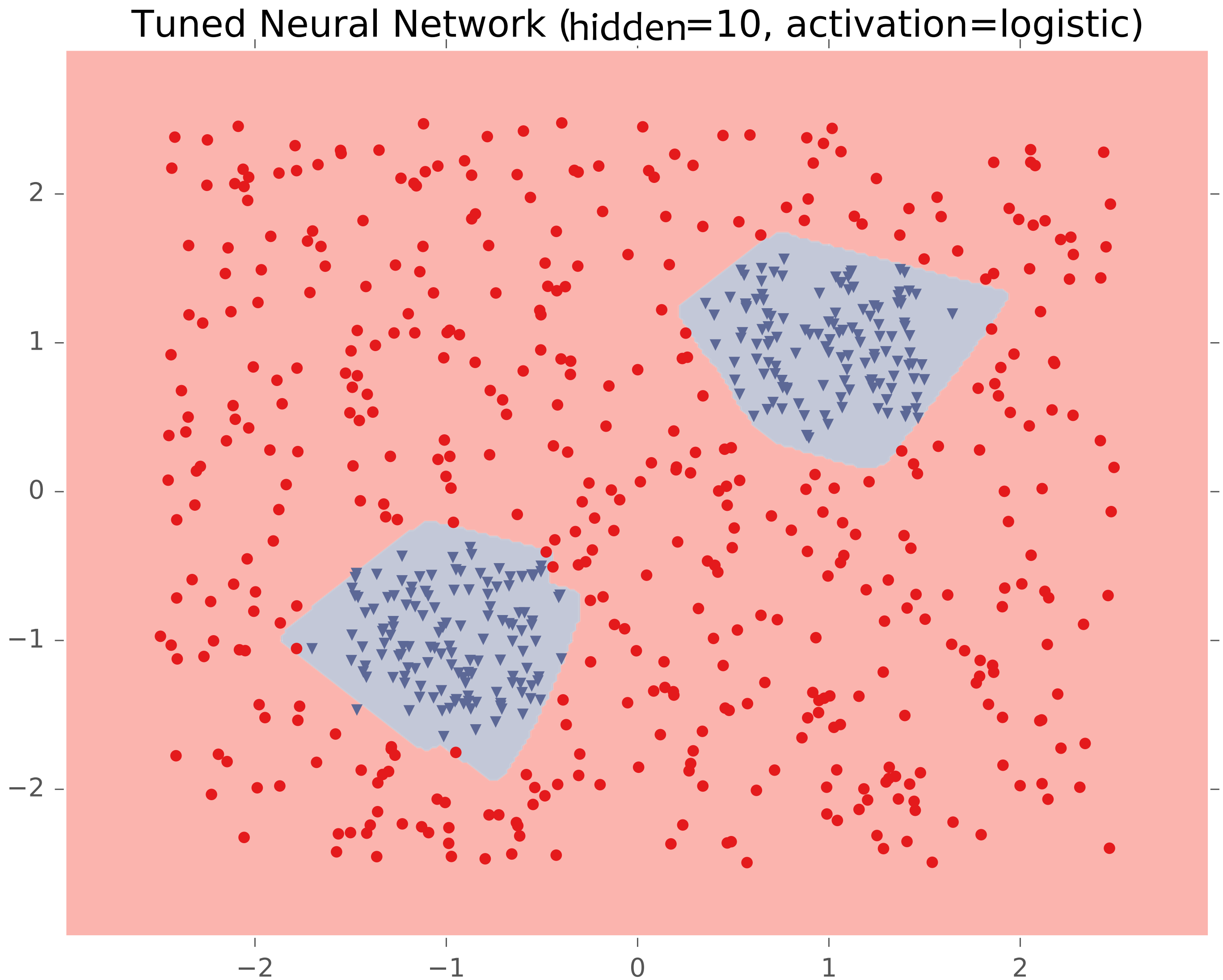
Example #4: Two Pockets



Example #4: Two Pockets



Example #4: Two Pockets



Neural Networks Objectives

You should be able to…

- Explain the biological motivations for a neural network
- Combine simpler models (e.g. linear regression, binary logistic regression, multinomial logistic regression) as components to build up feed-forward neural network architectures
- Explain the reasons why a neural network can model nonlinear decision boundaries for classification
- Compare and contrast feature engineering with learning features
- Identify (some of) the options available when designing the architecture of a neural network
- Implement a feed-forward neural network