



# 10-601 Introduction to Machine Learning

Machine Learning Department  
School of Computer Science  
Carnegie Mellon University

# Perceptron

Matt Gormley  
Lecture 6  
Sep. 13, 2019

# Q&A

**Q:** How do we define a distance function when the features are categorical (e.g. weather takes values {sunny, rainy, overcast})?

**A:** Step 1: Convert from categorical attributes to numeric features (e.g. binary)

Step 2: Select an appropriate distance function (e.g. Hamming distance)

# Q&A

**Q:** We pick the best hyperparameters by learning on the training data and evaluating error on the validation error. For our final model, should we then learn from training + validation?

**A:** Yes.

Let's assume that {train-original} is the original training data, and {test} is the provided test dataset.

1. Split {train-original} into {train-subset} and {validation}.
2. Pick the hyperparameters that when training on {train-subset} give the lowest error on {validation}. Call these hyperparameters {best-hyper}.
3. Retrain a new model using {best-hyper} on {train-original} = {train-subset}  $\cup$  {validation}.
4. Report test error by evaluating on {test}.

Alternatively, you could replace Step 1/2 with the following:

Pick the hyperparameters that give the lowest cross-validation error on {train-original}. Call these hyperparameters {best-hyper}.

# Reminders

- **Homework 2: Decision Trees**
  - Out: Wed, Sep. 04
  - Due: Wed, Sep. 18 at 11:59pm
- **Homework 3: KNN, Perceptron, Lin.Reg.**
  - Out: Wed, Sep. 18
  - Due: Wed, Sep. 25 at 11:59pm
- **Today's In-Class Poll**
  - <http://p6.mlcourse.org>

# Course Staff



Longxiang Zhang



Max



Yue Yin



Ayushi Sood



Eric Chen



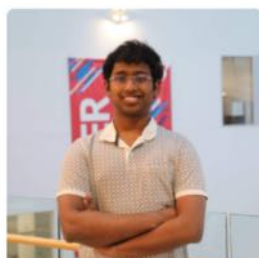
Bharath Prabhu



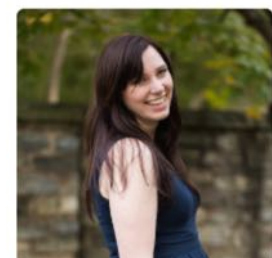
Manini Amin



Yang Yu



PSankalp Patro



Brynn Edmunds



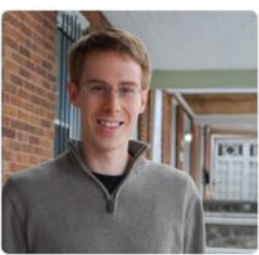
ANUPMA SHARAN



David Xu



Yuwei Qiu



Matt Gormley



Roni Rosenfeld



Lisa Hou



Yujia Chen



Kelly Shi



# Course Staff



Longxiang Zhang



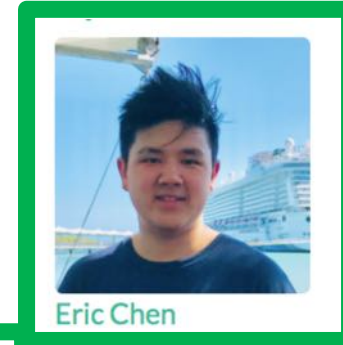
Max



Yue Yin



Avushi Seed



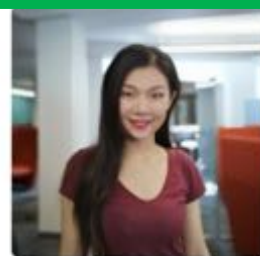
Eric Chen



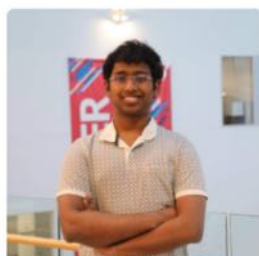
Bharath Prabhu



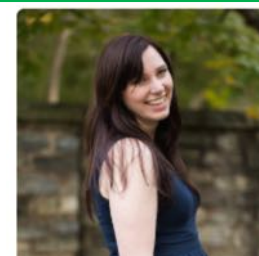
Manini Amin



Yang Yu



PSankalp Patro



Brynn Edmunds



ANUPMA SHARAN



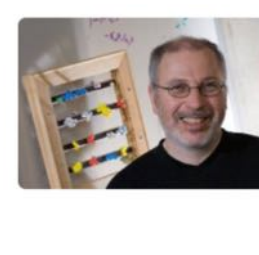
David Xu



Yuwei Oiu



Matt Gormley



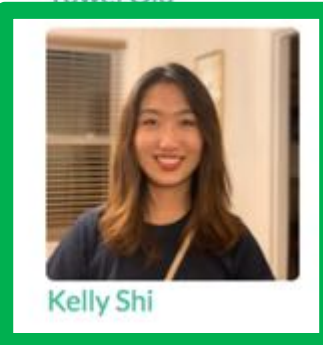
Roni Rosenfeld



Lisa Hou



Yujia Chen



Kelly Shi

Team A

# Course Staff



Longxiang Zhang



Max



Yue Yin



Ayushi Sood



Eric Chen



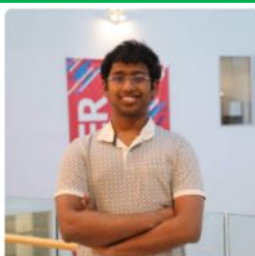
Bharath Prabhu



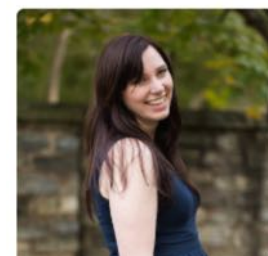
Manini Amin



Yang Yu



PSankalp Patro



Brynn Edmunds



ANUPMA SHARAN



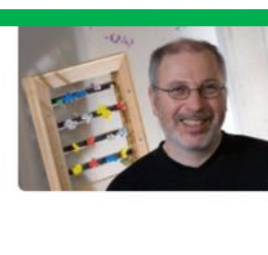
David Xu



Yuwei Qiu



Matt Gormley



Roni Rosenfeld



Lisa Hou



Yujia Chen



Kelly Shi

Team B



# Course Staff



Max



Yue Yin



Ayushi Sood



Eric Chen



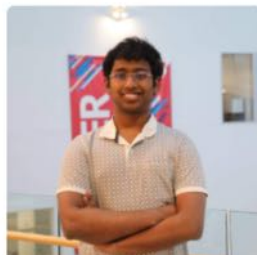
Bharath Prabhu



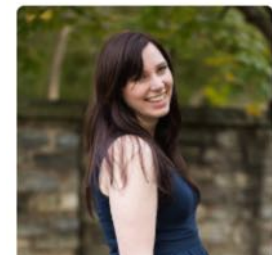
Manini Amin



Yang Yu



PSankalp Patro



Brynn Edmunds



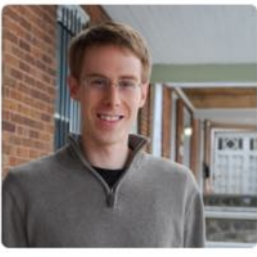
ANUPMA SHARAN



David Xu



Yuwei Qiu



Matt Gormley



Roni Rosenfeld



Lisa Hou



Yujia Chen



Kelly Shi

Team C



# Course Staff



Longxiang Zhang



Max



Yue Yin



Ayushi Sood



Eric Chen



Bharath Prabhu



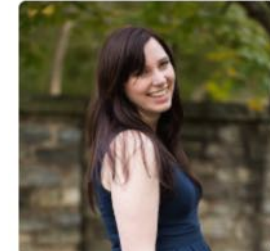
Minsik An



Yang Yu



PSankalp Patro



Brynn Edmunds



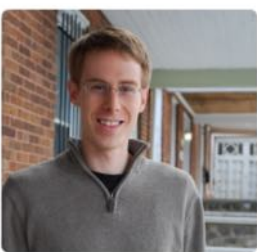
ANUPMA SHARAN



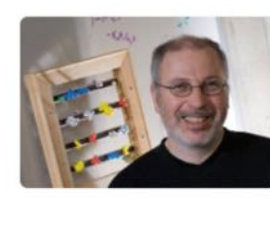
David Xu



Yuwei Qiu



Matt Gormley



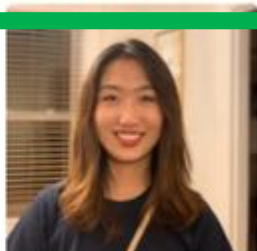
Roni Rosenfeld



Lisa Hou



Yujia Chen



Kelly Shi

Team D

# Course Staff



Longxiang Zhang



Max



Yue Yin



Ayushi Sood



Eric Chen



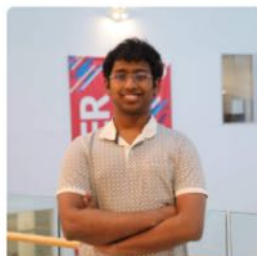
Bharath Prabhu



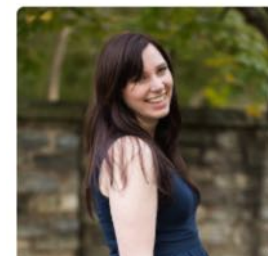
Manini Amin



Yang Yu



PSankalp Patro



Brynn Edmunds



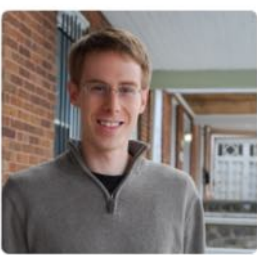
ANUPMA SHARAN



David Xu



Yuwei Qiu



Matt Gormley



Roni Rosenfeld



Lisa Hou



Yujia Chen



Kelly Shi

# **THE PERCEPTRON ALGORITHM**

# Perceptron: History

Imagine you are trying to build a new machine learning technique... your name is Frank Rosenblatt... and the year is 1957

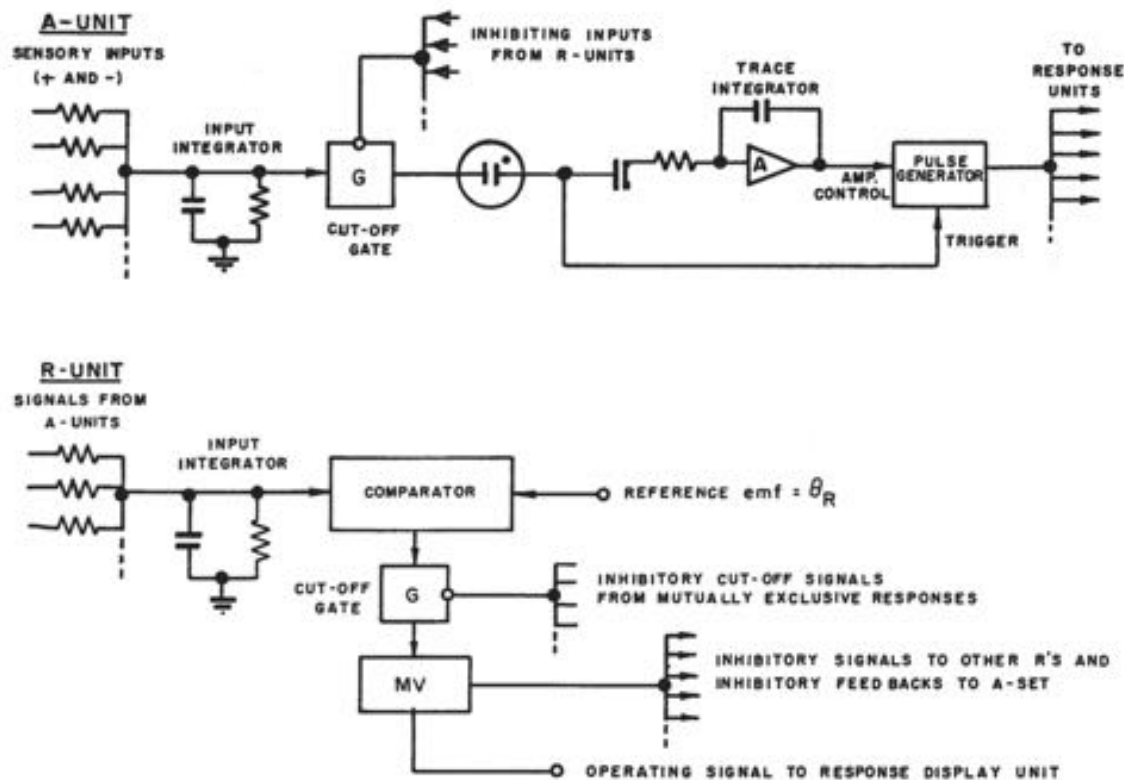


FIGURE 5  
DESIGN OF TYPICAL UNITS



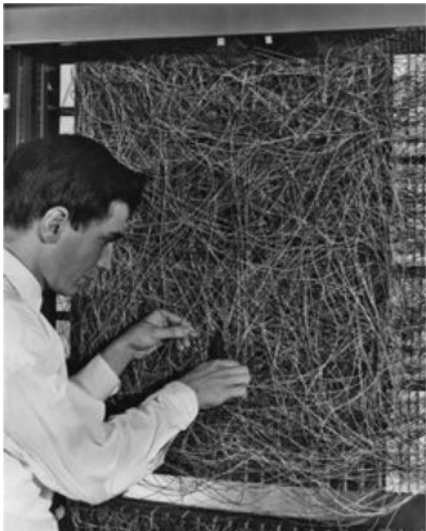
# Perceptron: History

Imagine you are trying to build a new machine learning technique... your name is Frank Rosenblatt... and the year is 1957



*The New Yorker*, December 6, 1958 P. 44

Talk story about the perceptron, a new electronic brain which hasn't been built, but which has been successfully simulated on the I.B.M. 704. Talk with Dr. Frank Rosenblatt, of the Cornell Aeronautical Laboratory, who is one of the two men who developed the prodigy; the other man is Dr. Marshall C. Yovits, of the Office of Naval Research, in Washington. Dr. Rosenblatt defined the perceptron as the first non-biological object which will achieve an organization o its external environment in a meaningful way. It interacts with its environment, forming concepts that have not been made ready for it by a human agent. If a triangle is held up, the perceptron's eye picks up the image & conveys it along a random succession of lines to the response units, where the image is registered. It can tell the difference betw. a cat and a dog, although it wouldn't be able to tell whether the dog was to theleft or right of the cat. Right now it is of no practical use, Dr. Rosenblatt conceded, but he said that one day it might be useful to send one into outer space to take in impressions for us.



# Linear Models for Classification

Key idea: Try to learn this hyperplane directly

Looking ahead:

- We'll see a number of commonly used Linear Classifiers
- These include:
  - Perceptron
  - Logistic Regression
  - Naïve Bayes (under certain conditions)
  - Support Vector Machines

Directly modeling the hyperplane would use a decision function:

$$h(\mathbf{x}) = \text{sign}(\boldsymbol{\theta}^T \mathbf{x})$$

for:

$$y \in \{-1, +1\}$$

# Geometry

## In-Class Exercise

Draw a picture of the region corresponding to:

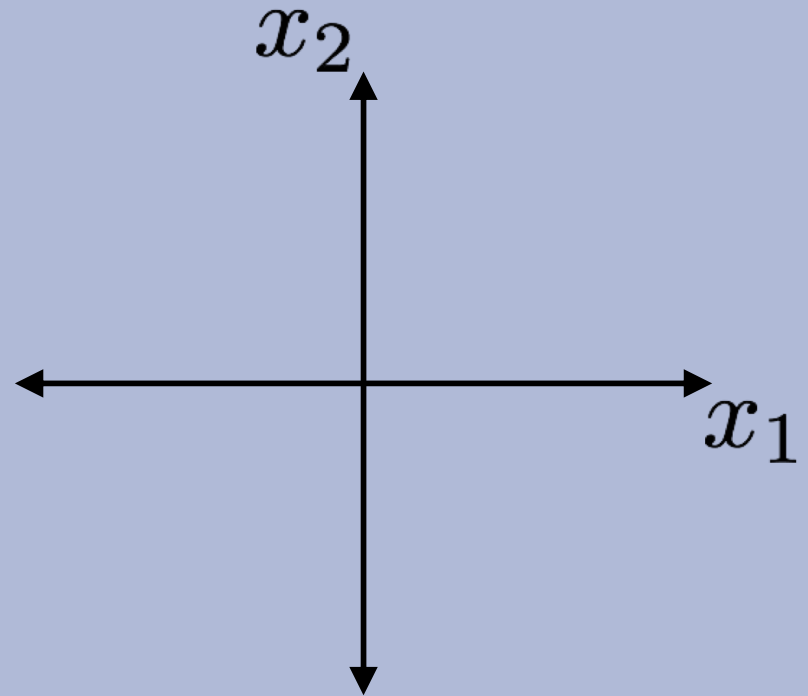
$$w_1x_1 + w_2x_2 + b > 0$$

$$\text{where } w_1 = 2, w_2 = 3, b = 6$$

Draw the vector

$$\mathbf{w} = [w_1, w_2]$$

## Answer Here:



# Visualizing Dot-Products

## *Chalkboard:*

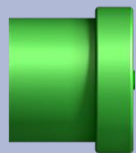
- vector in 2D
- line in 2D
- adding a bias term
- definition of orthogonality
- vector projection
- hyperplane definition
- half-space definitions



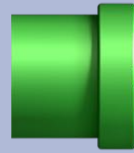
# Vector Projection

## Question:

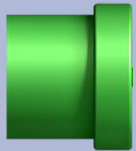
Which of the following is the projection of a vector  $\mathbf{a}$  onto a vector  $\mathbf{b}$ ?



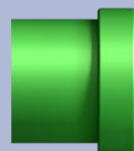
A.  $\frac{\mathbf{a}^T \mathbf{b}}{\mathbf{b}} \mathbf{a}$



D.  $\frac{(\mathbf{a} \cdot \mathbf{b})}{\|\mathbf{b}\|_2} \mathbf{b}$



B.  $\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a}^T \mathbf{b}}$



E.  $\frac{(\mathbf{a}^T \mathbf{b})}{\|\mathbf{b}\|_2^2} \mathbf{b}$



C.  $\frac{(\mathbf{a}^T \mathbf{b})}{\|\mathbf{b}\|_2} \mathbf{b}$



F.  $\frac{(\mathbf{a}^T \mathbf{b})^2}{\|\mathbf{b}\|_2} \mathbf{b}$

# Linear Models for Classification

Key idea: Try to learn this hyperplane directly

Looking ahead:

- We'll see a number of commonly used Linear Classifiers
- These include:
  - Perceptron
  - Logistic Regression
  - Naïve Bayes (under certain conditions)
  - Support Vector Machines

Directly modeling the hyperplane would use a decision function:

$$h(\mathbf{x}) = \text{sign}(\boldsymbol{\theta}^T \mathbf{x})$$

for:

$$y \in \{-1, +1\}$$

# Online vs. Batch Learning

## **Batch Learning**

Learn from all the examples at once

## **Online Learning**

Gradually learn as each example is received

# Online Learning

## Examples

1. **Stock market** prediction (what will the value of Alphabet Inc. be tomorrow?)
2. **Email** classification (distribution of both spam and regular mail changes over time, but the target function stays fixed - last year's spam still looks like spam)
3. **Recommendation** systems. Examples: recommending movies; predicting whether a user will be interested in a new news article
4. **Ad placement** in a new market



# Online Learning

**For  $i = 1, 2, 3, \dots$ :**

- **Receive** an unlabeled instance  $\mathbf{x}^{(i)}$
- **Predict**  $y' = h_{\theta}(\mathbf{x}^{(i)})$
- **Receive** true label  $y^{(i)}$
- **Suffer loss** if a mistake was made,  $y' \neq y^{(i)}$
- **Update** parameters  $\theta$

**Goal:**

- **Minimize** the number of **mistakes**

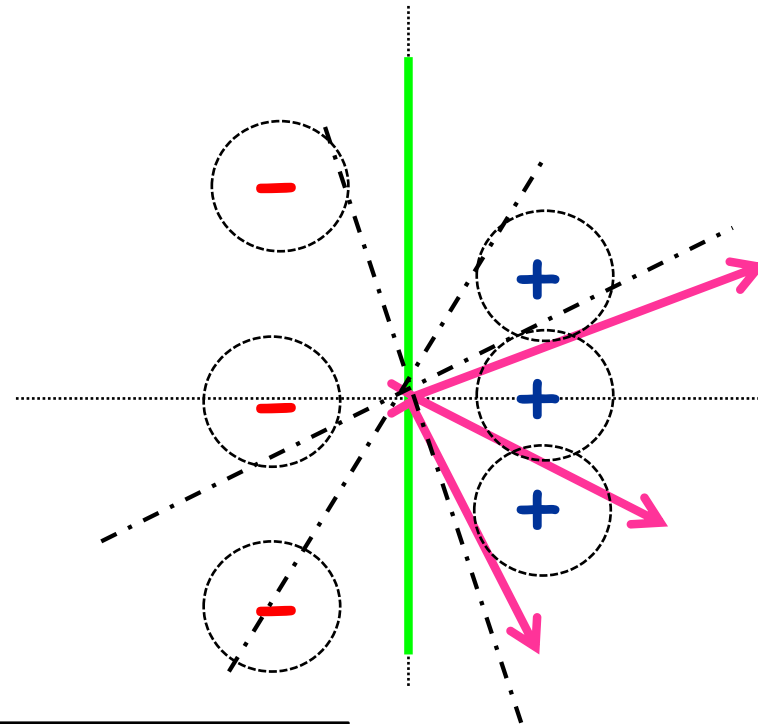
# Perceptron

## *Chalkboard:*

- (Online) Perceptron Algorithm
- Why do we need a bias term?
- Inductive Bias of Perceptron
- Limitations of Linear Models

# Perceptron Algorithm: Example

Example:  $(-1,2) -$  ✗  
 $(1,0) +$  ✓  
 $(1,1) +$  ✗  
 $(-1,0) -$  ✓  
 $(-1,-2) -$  ✗  
 $(1,-1) +$  ✓



## Perceptron Algorithm: (without the bias term)

- Set  $t=1$ , start with all-zeroes weight vector  $w_1$ .
- Given example  $x$ , predict positive iff  $w_t \cdot x \geq 0$ .
- On a mistake, update as follows:
  - Mistake on positive, update  $w_{t+1} \leftarrow w_t + x$
  - Mistake on negative, update  $w_{t+1} \leftarrow w_t - x$

$$w_1 = (0,0)$$

$$w_2 = w_1 - (-1,2) = (1,-2)$$

$$w_3 = w_2 + (1,1) = (2,-1)$$

$$w_4 = w_3 - (-1,-2) = (3,1)$$

# Background: Hyperplanes

*Notation Trick:* fold the bias  $b$  and the weights  $\mathbf{w}$  into a single vector  $\boldsymbol{\theta}$  by prepending a constant to  $\mathbf{x}$  and increasing dimensionality by one!

Hyperplane (Definition 1):

$$\mathcal{H} = \{\mathbf{x} : \mathbf{w}^T \mathbf{x} = b\}$$

Hyperplane (Definition 2):

$$\mathcal{H} = \{\mathbf{x} : \boldsymbol{\theta}^T \mathbf{x} = 0$$

$$\text{and } x_0 = 1\}$$

$$\boldsymbol{\theta} = [b, w_1, \dots, w_M]^T$$

Half-spaces:

$$\mathcal{H}^+ = \{\mathbf{x} : \boldsymbol{\theta}^T \mathbf{x} > 0 \text{ and } x_0 = 1\}$$

$$\mathcal{H}^- = \{\mathbf{x} : \boldsymbol{\theta}^T \mathbf{x} < 0 \text{ and } x_0 = 1\}$$

# (Online) Perceptron Algorithm

**Data:** Inputs are continuous vectors of length  $M$ . Outputs are discrete.

$$(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \dots$$

$$\text{where } \mathbf{x} \in \mathbb{R}^M \text{ and } y \in \{+1, -1\}$$

**Prediction:** Output determined by hyperplane.

$$\hat{y} = h_{\boldsymbol{\theta}}(\mathbf{x}) = \text{sign}(\boldsymbol{\theta}^T \mathbf{x})$$

$$\text{sign}(a) = \begin{cases} 1, & \text{if } a \geq 0 \\ -1, & \text{otherwise} \end{cases}$$

$$\text{Assume } \boldsymbol{\theta} = [b, w_1, \dots, w_M]^T \text{ and } x_0 = 1$$

**Learning:** Iterative procedure:

- **initialize** parameters to vector of all zeroes
- **while** not converged
  - **receive** next example  $(\mathbf{x}^{(i)}, y^{(i)})$
  - **predict**  $y' = h(\mathbf{x}^{(i)})$
  - **if** positive mistake: **add**  $\mathbf{x}^{(i)}$  to parameters
  - **if** negative mistake: **subtract**  $\mathbf{x}^{(i)}$  from parameters



# (Online) Perceptron Algorithm

**Data:** Inputs are continuous vectors of length  $M$ . Outputs are discrete.

$$(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \dots$$

where  $\mathbf{x} \in \mathbb{R}^M$  and  $y \in \{+1, -1\}$

**Prediction:** Output determined by

$$\hat{y} = h_{\theta}(\mathbf{x}) = \text{sign}(\boldsymbol{\theta}^T \mathbf{x})$$

Assume  $\boldsymbol{\theta} = [b, w_1, \dots, w_M]$

*Implementation Trick: same behavior as our “add on positive mistake and subtract on negative mistake” version, because  $y^{(i)}$  takes care of the sign*

**Learning:**

**Algorithm 1** Perceptron Learning Alg

```
1: procedure PERCEPTRON( $\mathcal{D} = \{(\mathbf{x}^{(i)}, y^{(i)})\}$ )
2:    $\boldsymbol{\theta} \leftarrow \mathbf{0}$ 
3:   for  $i \in \{1, 2, \dots\}$  do
4:      $\hat{y} \leftarrow \text{sign}(\boldsymbol{\theta}^T \mathbf{x}^{(i)})$ 
5:     if  $\hat{y} \neq y^{(i)}$  then
6:        $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + y^{(i)} \mathbf{x}^{(i)}$ 
7:   return  $\boldsymbol{\theta}$ 
```

▷ Initialize parameters  
▷ For each example  
▷ Predict  
▷ If mistake  
▷ Update parameters



# (Batch) Perceptron Algorithm

Learning for Perceptron also works if we have a fixed training dataset,  $D$ . We call this the “batch” setting in contrast to the “online” setting that we’ve discussed so far.

---

## Algorithm 1 Perceptron Learning Algorithm (Batch)

---

```
1: procedure PERCEPTRON( $\mathcal{D} = \{(\mathbf{x}^{(1)}, y^{(1)}), \dots, (\mathbf{x}^{(N)}, y^{(N)})\}$ )
2:    $\theta \leftarrow \mathbf{0}$  ▷ Initialize parameters
3:   while not converged do
4:     for  $i \in \{1, 2, \dots, N\}$  do ▷ For each example
5:        $\hat{y} \leftarrow \text{sign}(\theta^T \mathbf{x}^{(i)})$  ▷ Predict
6:       if  $\hat{y} \neq y^{(i)}$  then ▷ If mistake
7:          $\theta \leftarrow \theta + y^{(i)} \mathbf{x}^{(i)}$  ▷ Update parameters
8:   return  $\theta$ 
```

---

# (Batch) Perceptron Algorithm

Learning for Perceptron also works if we have a fixed training dataset,  $D$ . We call this the “batch” setting in contrast to the “online” setting that we’ve discussed so far.

## Discussion:

The Batch Perceptron Algorithm can be derived in two ways.

1. By extending the online Perceptron algorithm to the batch setting (as mentioned above)
2. By applying **Stochastic Gradient Descent (SGD)** to minimize a so-called **Hinge Loss** on a linear separator

# Extensions of Perceptron

- **Voted Perceptron**
  - generalizes better than (standard) perceptron
  - memory intensive (keeps around every weight vector seen during training, so each one can vote)
- **Averaged Perceptron**
  - empirically similar performance to voted perceptron
  - can be implemented in a memory efficient way (running averages are efficient)
- **Kernel Perceptron**
  - Choose a kernel  $K(x', x)$
  - Apply the **kernel trick** to Perceptron
  - Resulting algorithm is **still very simple**
- **Structured Perceptron**
  - Basic idea can also be applied when  $y$  ranges over an exponentially large set
  - Mistake bound **does not** depend on the size of that set

# Perceptron Exercises

## Question:

The parameter vector  $w$  learned by the Perceptron algorithm can be **written as a linear combination** of the feature vectors  $x^{(1)}, x^{(2)}, \dots, x^{(N)}$ .

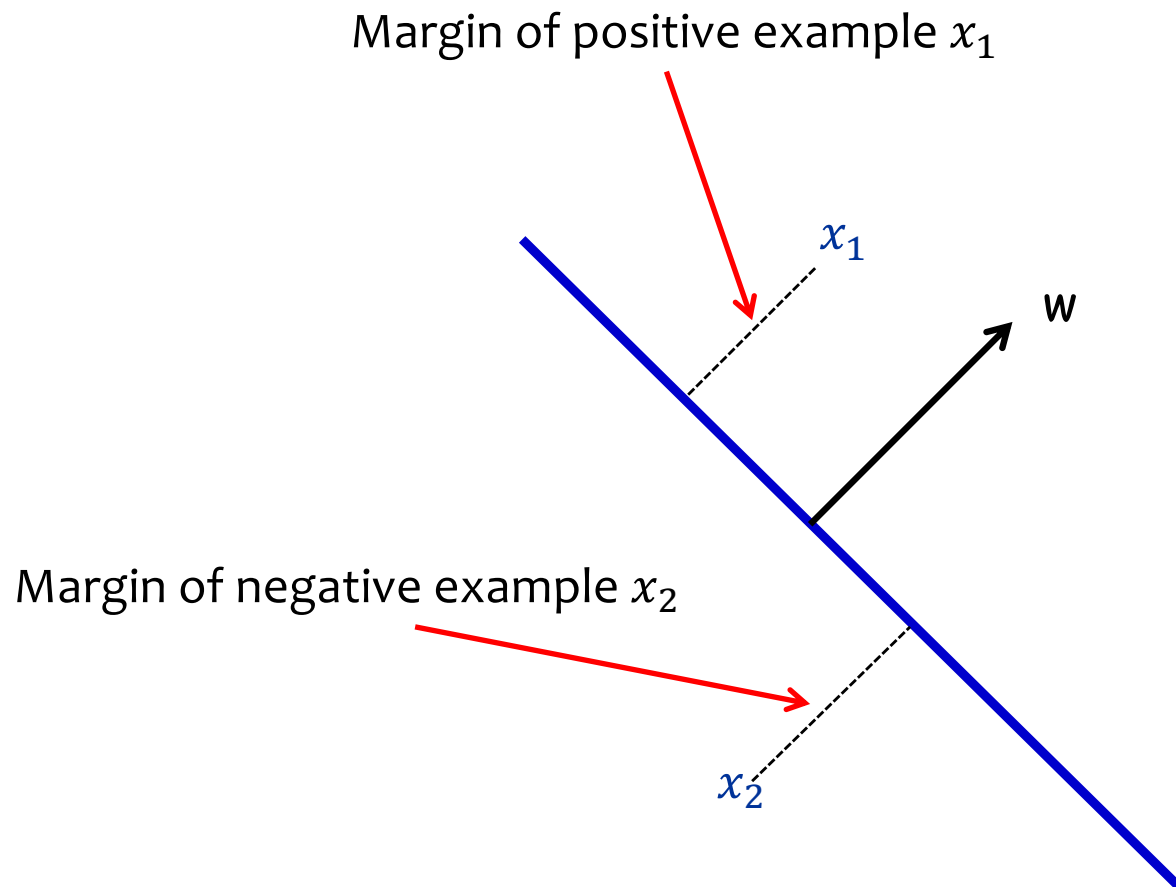
- A. True, if you replace “linear” with “polynomial” above
- B. True, for all datasets
- C. False, for all datasets
- D. True, but only for certain datasets
- E. False, but only for certain datasets



# **ANALYSIS OF PERCEPTRON**

# Geometric Margin

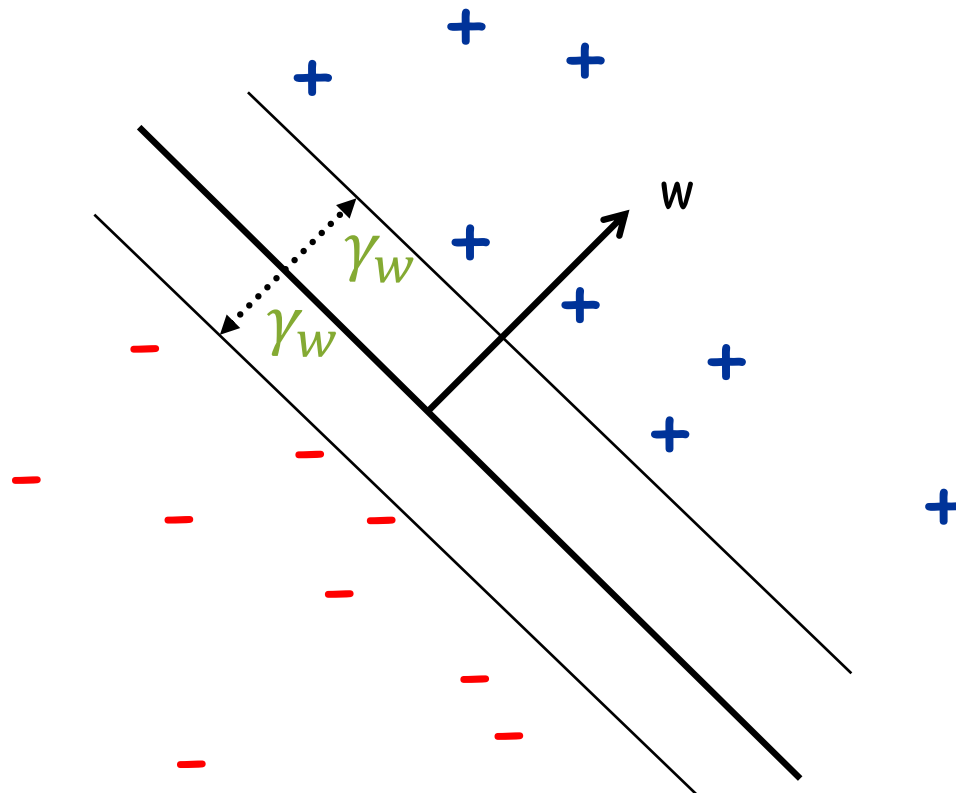
**Definition:** The **margin** of example  $x$  w.r.t. a linear sep.  $w$  is the distance from  $x$  to the plane  $w \cdot x = 0$  (or the negative if on wrong side)



# Geometric Margin

**Definition:** The **margin** of example  $x$  w.r.t. a linear sep.  $w$  is the distance from  $x$  to the plane  $w \cdot x = 0$  (or the negative if on wrong side)

**Definition:** The **margin**  $\gamma_w$  of a set of examples  $S$  wrt a linear separator  $w$  is the smallest margin over points  $x \in S$ .

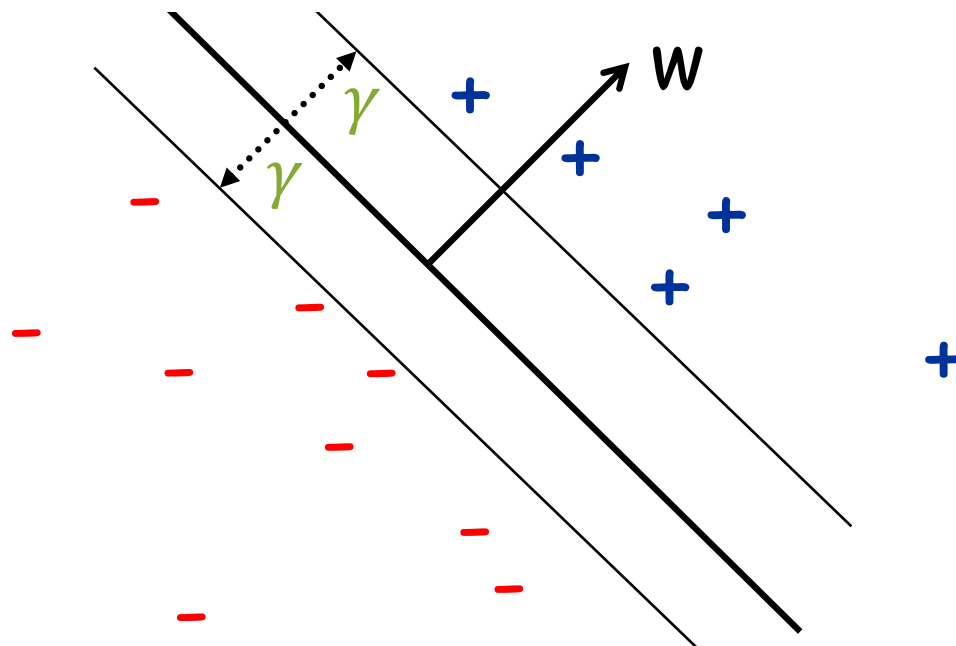


# Geometric Margin

**Definition:** The **margin** of example  $x$  w.r.t. a linear sep.  $w$  is the distance from  $x$  to the plane  $w \cdot x = 0$  (or the negative if on wrong side)

**Definition:** The **margin**  $\gamma_w$  of a set of examples  $S$  wrt a linear separator  $w$  is the smallest margin over points  $x \in S$ .

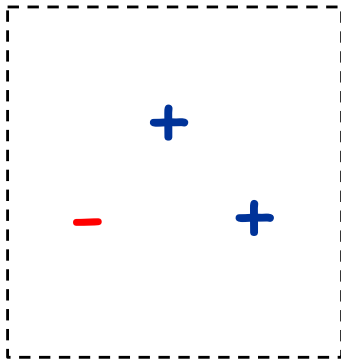
**Definition:** The **margin**  $\gamma$  of a set of examples  $S$  is the **maximum**  $\gamma_w$  over all linear separators  $w$ .



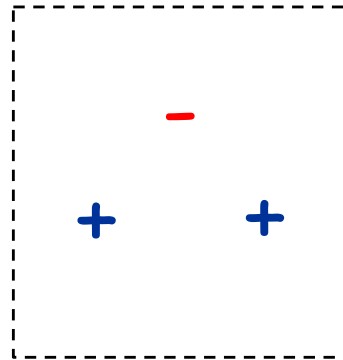
# Linear Separability

**Def:** For a **binary classification** problem, a set of examples  $S$  is **linearly separable** if there exists a linear decision boundary that can separate the points

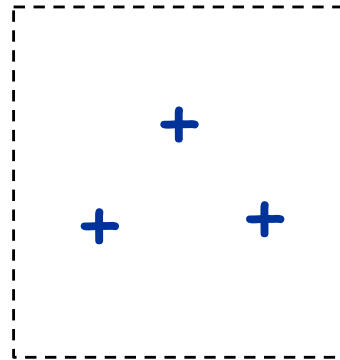
Case 1:



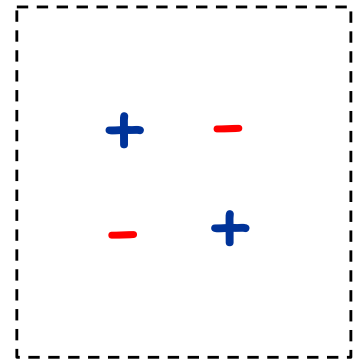
Case 2:



Case 3:



Case 4:



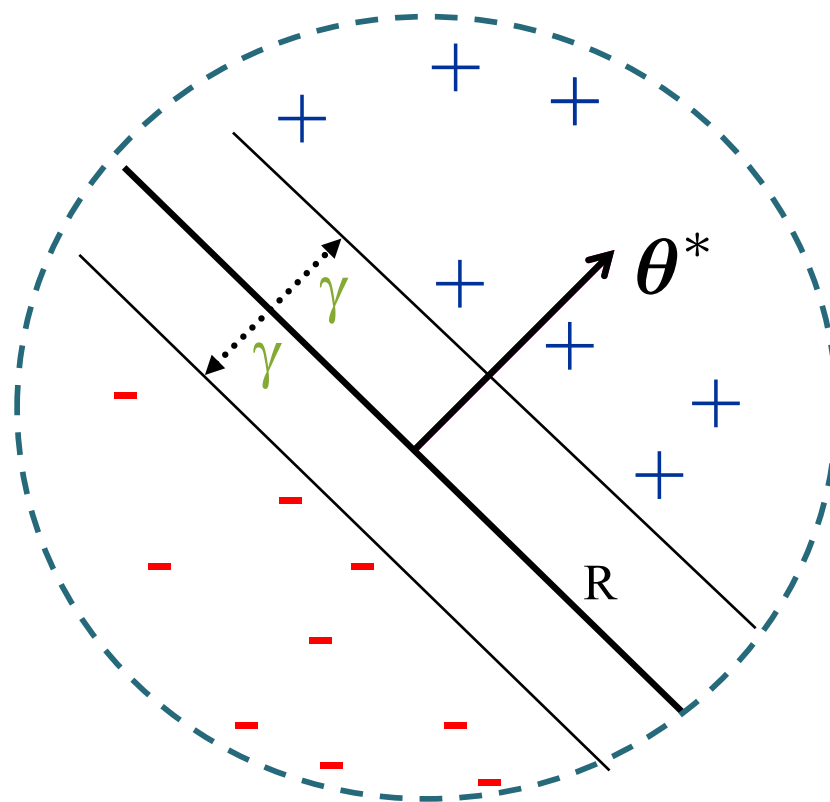


# Analysis: Perceptron

## Perceptron Mistake Bound

**Guarantee:** If data has margin  $\gamma$  and all points inside a ball of radius  $R$ , then Perceptron makes  $\leq (R/\gamma)^2$  mistakes.

(Normalized margin: multiplying all points by 100, or dividing all points by 100, doesn't change the number of mistakes; algo is invariant to scaling.)

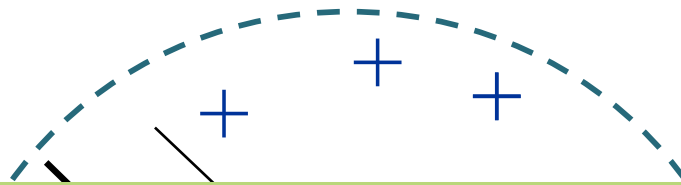


# Analysis: Perceptron

## Perceptron Mistake Bound

**Guarantee:** If data has margin  $\gamma$  and all points inside a ball of radius  $R$ , then Perceptron makes  $\leq (R/\gamma)^2$  mistakes.

(Normalized margin: multiplying all points by 100, or dividing all points by 100, doesn't change the number of mistakes; algo is invariant to scaling.)



**Def:** We say that the (batch) perceptron algorithm has **converged** if it stops making mistakes on the training data (perfectly classifies the training data).

**Main Takeaway:** For **linearly separable** data, if the perceptron algorithm cycles repeatedly through the data, it will **converge** in a finite # of steps.

# Analysis: Perceptron

## Perceptron Mistake Bound

**Theorem 0.1** (Block (1962), Novikoff (1962)).

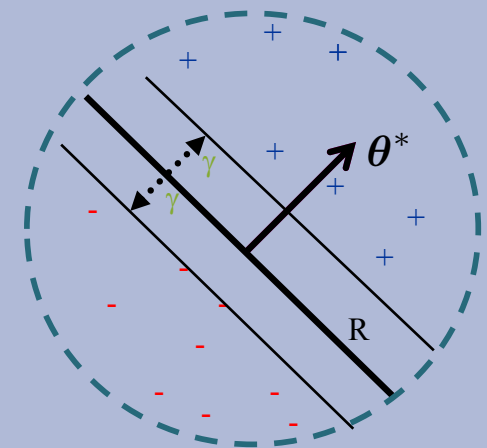
Given dataset:  $\mathcal{D} = \{(\mathbf{x}^{(i)}, y^{(i)})\}_{i=1}^N$ .

Suppose:

1. Finite size inputs:  $\|\mathbf{x}^{(i)}\| \leq R$
2. Linearly separable data:  $\exists \boldsymbol{\theta}^*$  s.t.  $\|\boldsymbol{\theta}^*\| = 1$  and  $y^{(i)} (\boldsymbol{\theta}^* \cdot \mathbf{x}^{(i)}) \geq \gamma, \forall i$

Then: The number of mistakes made by the Perceptron algorithm on this dataset is

$$k \leq (R/\gamma)^2$$



# Analysis: Perceptron

**Common Misunderstanding:**

The radius is centered at the origin, not at the center of the points.

## Perceptron Mistake Bound

**Theorem 0.1** (Block (1962), Novikoff (1962))

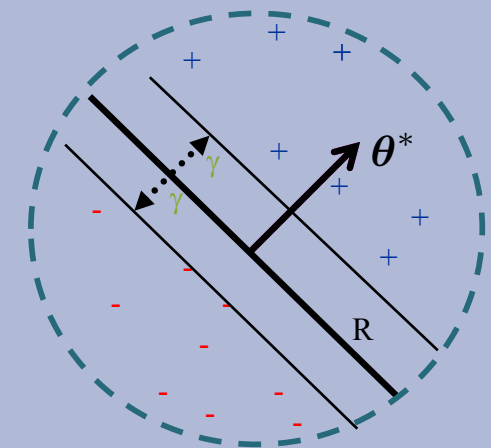
Given dataset:  $\mathcal{D} = \{(\mathbf{x}^{(i)}, y^{(i)})\}_{i=1}^N$

Suppose:

1. Finite size inputs:  $\|\mathbf{x}^{(i)}\| \leq R$
2. Linearly separable data:  $\exists \boldsymbol{\theta}^*$  s.t.  $\|\boldsymbol{\theta}^*\| = 1$  and  $y^{(i)}(\boldsymbol{\theta}^* \cdot \mathbf{x}^{(i)}) \geq \gamma, \forall i$

Then: The number of mistakes made by the Perceptron algorithm on this dataset is

$$k \leq (R/\gamma)^2$$



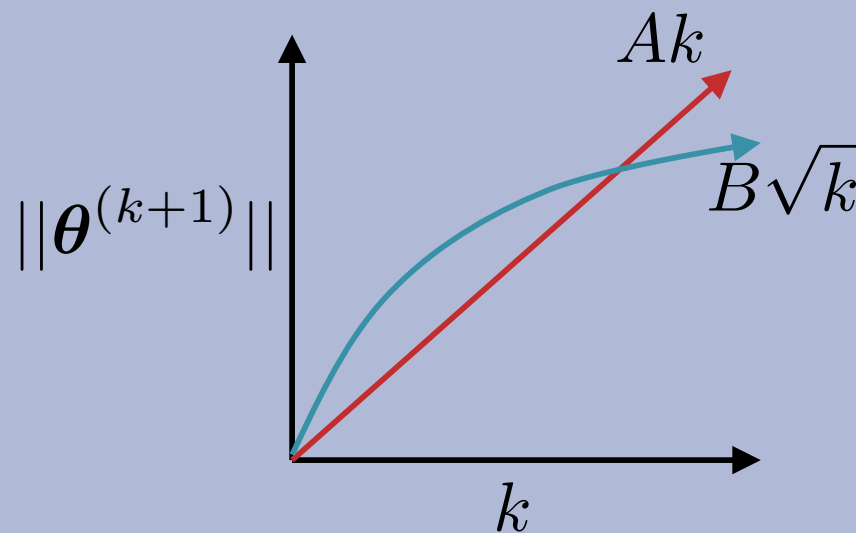
# Analysis: Perceptron

Covered in Recitation

## Proof of Perceptron Mistake Bound:

We will show that there exist constants  $A$  and  $B$  s.t.

$$Ak \leq \|\boldsymbol{\theta}^{(k+1)}\| \leq B\sqrt{k}$$





# Analysis: Perceptron

Covered in Recitation

**Theorem 0.1** (Block (1962), Novikoff (1962)).

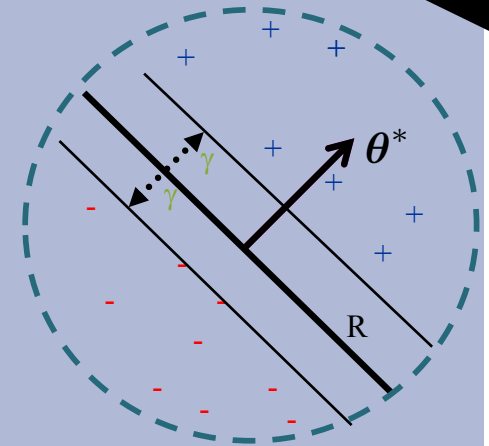
Given dataset:  $\mathcal{D} = \{(\mathbf{x}^{(i)}, y^{(i)})\}_{i=1}^N$ .

Suppose:

1. Finite size inputs:  $\|\mathbf{x}^{(i)}\| \leq R$
2. Linearly separable data:  $\exists \boldsymbol{\theta}^*$  s.t.  $\|\boldsymbol{\theta}^*\| = 1$  and  $y^{(i)}(\boldsymbol{\theta}^* \cdot \mathbf{x}^{(i)}) \geq \gamma, \forall i$

Then: The number of mistakes made by the Perceptron algorithm on this dataset is

$$k \leq (R/\gamma)^2$$



## Algorithm 1 Perceptron Learning Algorithm (Online)

```
1: procedure PERCEPTRON( $\mathcal{D} = \{(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \dots\}$ )
2:    $\boldsymbol{\theta} \leftarrow \mathbf{0}, k = 1$  ▷ Initialize parameters
3:   for  $i \in \{1, 2, \dots\}$  do ▷ For each example
4:     if  $y^{(i)}(\boldsymbol{\theta}^{(k)} \cdot \mathbf{x}^{(i)}) \leq 0$  then ▷ If mistake
5:        $\boldsymbol{\theta}^{(k+1)} \leftarrow \boldsymbol{\theta}^{(k)} + y^{(i)} \mathbf{x}^{(i)}$  ▷ Update parameters
6:        $k \leftarrow k + 1$ 
7:   return  $\boldsymbol{\theta}$ 
```

# Analysis: Perceptron

Covered in Recitation

## Proof of Perceptron Mistake Bound:

Part 1: for some  $A$ ,  $Ak \leq \|\theta^{(k+1)}\|$

$$\theta^{(k+1)} \cdot \theta^* = (\theta^{(k)} + y^{(i)} \mathbf{x}^{(i)}) \theta^*$$

by Perceptron algorithm update

$$= \theta^{(k)} \cdot \theta^* + y^{(i)} (\theta^* \cdot \mathbf{x}^{(i)})$$

$$\geq \theta^{(k)} \cdot \theta^* + \gamma$$

by assumption

$$\Rightarrow \theta^{(k+1)} \cdot \theta^* \geq k\gamma$$

by induction on  $k$  since  $\theta^{(1)} = \mathbf{0}$

$$\Rightarrow \|\theta^{(k+1)}\| \geq k\gamma$$

since  $\|\mathbf{w}\| \times \|\mathbf{u}\| \geq \mathbf{w} \cdot \mathbf{u}$  and  $\|\theta^*\| = 1$

Cauchy-Schwartz inequality

# Analysis: Perceptron

Covered in Recitation

## Proof of Perceptron Mistake Bound:

Part 2: for some  $B$ ,  $\|\boldsymbol{\theta}^{(k+1)}\| \leq B\sqrt{k}$

$$\|\boldsymbol{\theta}^{(k+1)}\|^2 = \|\boldsymbol{\theta}^{(k)} + y^{(i)}\mathbf{x}^{(i)}\|^2$$

by Perceptron algorithm update

$$= \|\boldsymbol{\theta}^{(k)}\|^2 + (y^{(i)})^2\|\mathbf{x}^{(i)}\|^2 + 2y^{(i)}(\boldsymbol{\theta}^{(k)} \cdot \mathbf{x}^{(i)})$$

$$\leq \|\boldsymbol{\theta}^{(k)}\|^2 + (y^{(i)})^2\|\mathbf{x}^{(i)}\|^2$$

since  $k$ th mistake  $\Rightarrow y^{(i)}(\boldsymbol{\theta}^{(k)} \cdot \mathbf{x}^{(i)}) \leq 0$

$$= \|\boldsymbol{\theta}^{(k)}\|^2 + R^2$$

since  $(y^{(i)})^2\|\mathbf{x}^{(i)}\|^2 = \|\mathbf{x}^{(i)}\|^2 = R^2$  by assumption and  $(y^{(i)})^2 = 1$

$$\Rightarrow \|\boldsymbol{\theta}^{(k+1)}\|^2 \leq kR^2$$

by induction on  $k$  since  $(\boldsymbol{\theta}^{(1)})^2 = 0$

$$\Rightarrow \|\boldsymbol{\theta}^{(k+1)}\| \leq \sqrt{k}R$$

# Analysis: Perceptron

Covered in Recitation

## Proof of Perceptron Mistake Bound:

Part 3: Combining the bounds finishes the proof.

$$k\gamma \leq \|\boldsymbol{\theta}^{(k+1)}\| \leq \sqrt{k}R$$

$$\Rightarrow k \leq (R/\gamma)^2$$

The total number of mistakes  
must be less than this

# Analysis: Perceptron

What if the data is *not* linearly separable?

1. Perceptron will **not converge** in this case (it can't!)
2. However, Freund & Schapire (1999) show that by projecting the points (hypothetically) into a higher dimensional space, we can achieve a similar bound on the number of mistakes made on **one pass** through the sequence of examples

**Theorem 2.** Let  $\langle (\mathbf{x}_1, y_1), \dots, (\mathbf{x}_m, y_m) \rangle$  be a sequence of labeled examples with  $\|\mathbf{x}_i\| \leq R$ . Let  $\mathbf{u}$  be any vector with  $\|\mathbf{u}\| = 1$  and let  $\gamma > 0$ . Define the deviation of each example as

$$d_i = \max\{0, \gamma - y_i(\mathbf{u} \cdot \mathbf{x}_i)\},$$

and define  $D = \sqrt{\sum_{i=1}^m d_i^2}$ . Then the number of mistakes of the online perceptron algorithm on this sequence is bounded by

$$\left(\frac{R + D}{\gamma}\right)^2.$$

# Perceptron Exercises

## Question:

Unlike Decision Trees and K-Nearest Neighbors, the Perceptron algorithm **does not suffer from overfitting** because it does not have any hyperparameters that could be over-tuned on the training data.

- A. True
- B. False
- C. True and False

# Summary: Perceptron

- Perceptron is a **linear classifier**
- **Simple learning algorithm:** when a mistake is made, add / subtract the features
- Perceptron will converge if the data are **linearly separable**, it will **not** converge if the data are **linearly inseparable**
- For linearly separable and inseparable data, we can **bound the number of mistakes** (geometric argument)
- **Extensions** support nonlinear separators and structured prediction



# Perceptron Learning Objectives

*You should be able to...*

- Explain the difference between online learning and batch learning
- Implement the perceptron algorithm for binary classification [CIML]
- Determine whether the perceptron algorithm will converge based on properties of the dataset, and the limitations of the convergence guarantees
- Describe the inductive bias of perceptron and the limitations of linear models
- Draw the decision boundary of a linear model
- Identify whether a dataset is linearly separable or not
- Defend the use of a bias term in perceptron