Machine Learning Department
School of Computer Science
Carnegie Mellon University

State Space Models

Matt Gormley & Pat Virtue
Lecture 22
Apr. 9, 2025

Reminders

- Project Midway Report
  - Due: Thu, Apr-17 at 11:59pm
- HW623
  - Only for students registered in 10-623
  - Due: Mon, Apr-21 at 11:59pm

Motivation

- Transformers are slow at test time: they require a KV cache that grows quadratically in size with the sequence length
- State space models (SSMs) are fast at test time: they only hold a fixed size hidden state in memory (like RNNs)
- But we'll see that SSMs can also be trained efficiently with the right tricks
- As well, they elegantly transition between different granularities of representation for the input (e.g. sound at 16Hz vs. 8Hz

Motivation

- https://www.isattentionallyouneed.com/

Is Attention All You Need?

Current Status: Yes

Time Remaining: 631d 22h 55m 11s

Proposition:

On January 1, 2027, a Transformer-like model will continue to hold the state-of-the-art position in most benchmarked tasks in natural language processing.

Motivation

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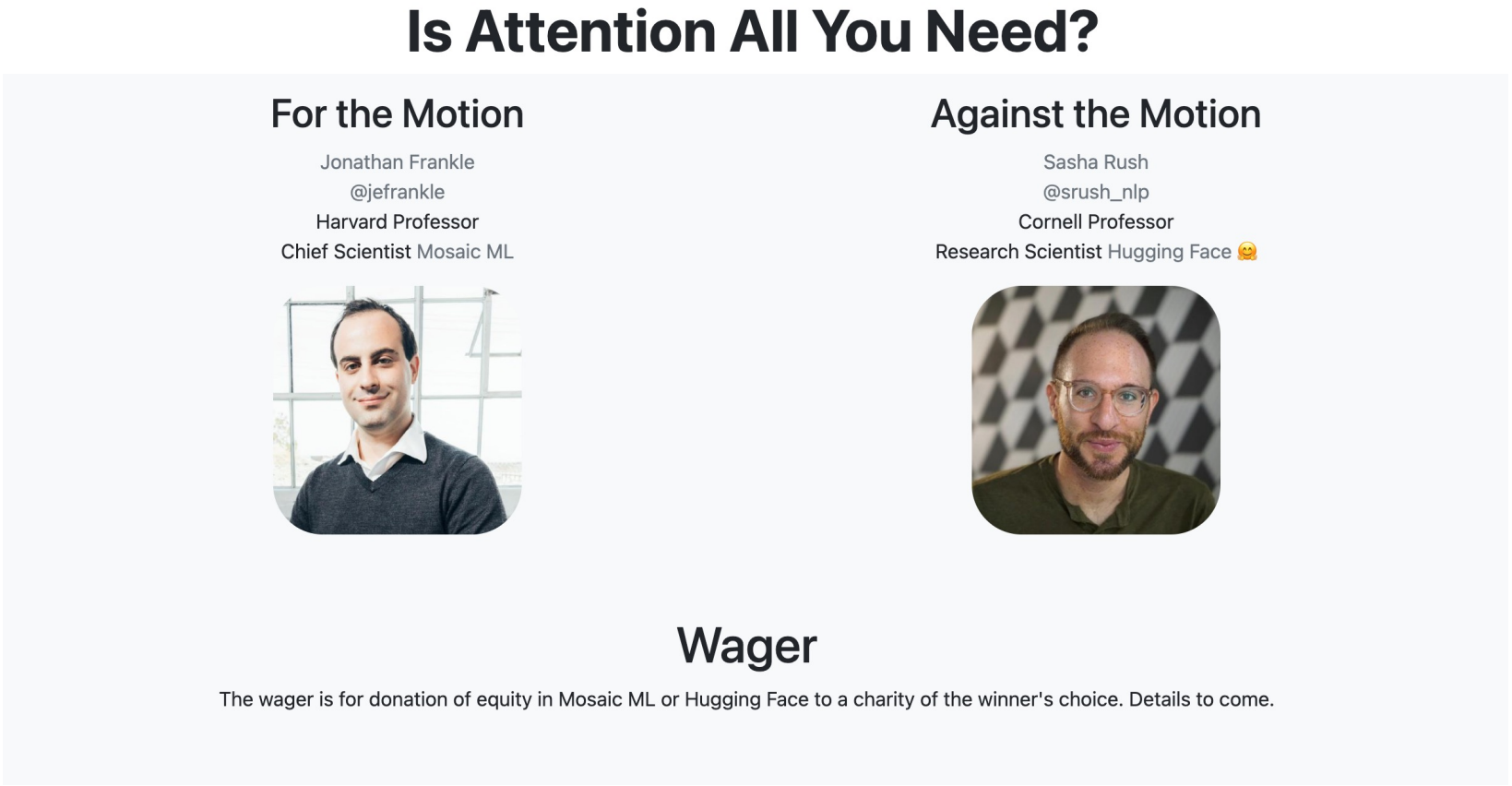
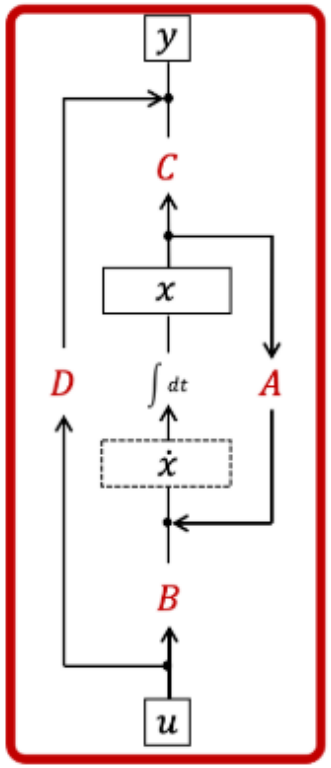


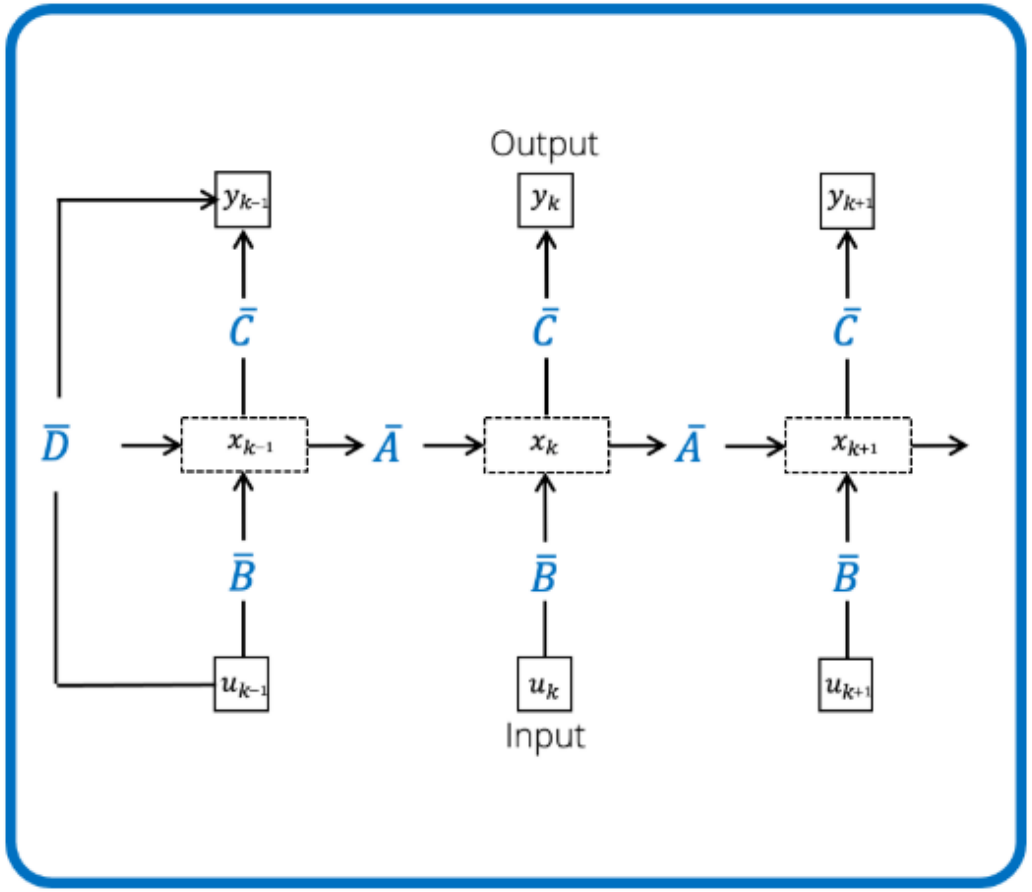
STATE SPACE MODEL (SSM)

Three Representations of a State Space Model

Continuous



Recurrent



Convolutional

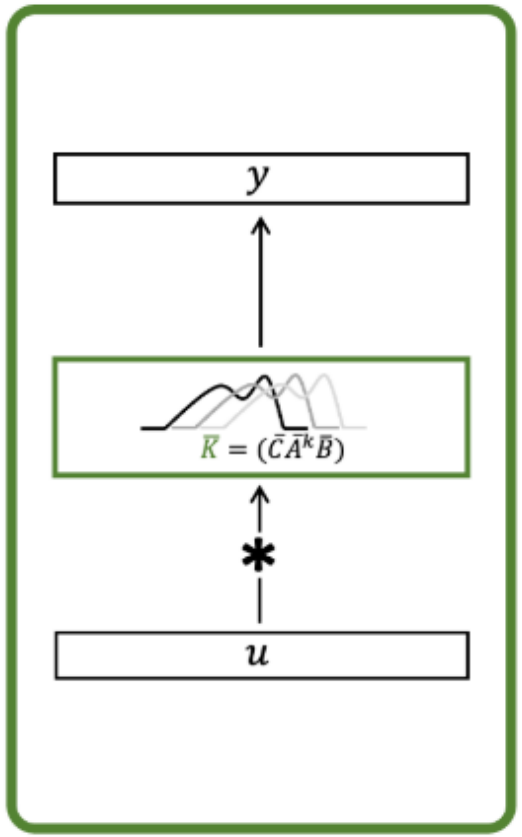
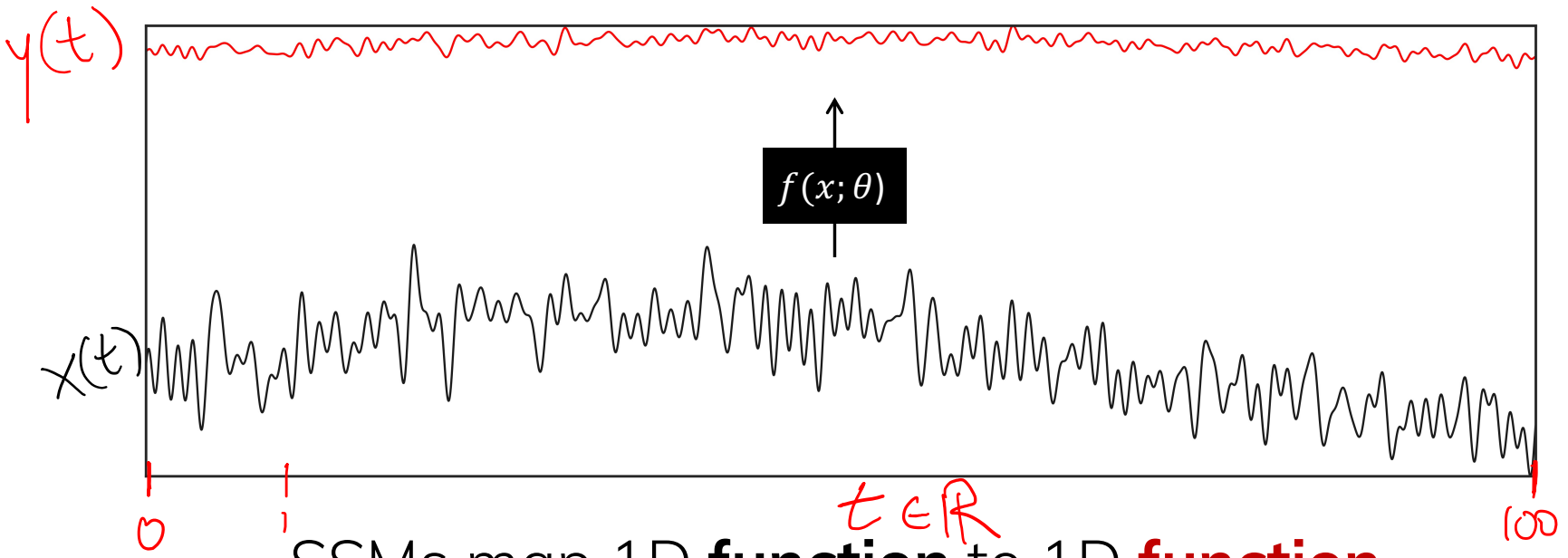


Figure from https://hazyresearch.stanford.edu/blog/2022-01-14-s4-3

SSM: 1D Continuous Representation

$x(t) \in \mathbb{R}$
$y(t) \in \mathbb{R}$
$h(t) \in \mathbb{R}^N$
$A \in \mathbb{R}^{N \times N}$
$B \in \mathbb{R}^{N \times 1}$
$C \in \mathbb{R}^{1 \times N}$
$D \in \mathbb{R}^{1 \times 1}$

$$h'(t) = \mathbf{A}h(t) + \mathbf{B}x(t) = \frac{\partial}{\partial t} h(t)$$
$$y(t) = \mathbf{C}h(t) + \mathbf{D}x(t)$$



SSMs map 1D **function** to 1D **function**

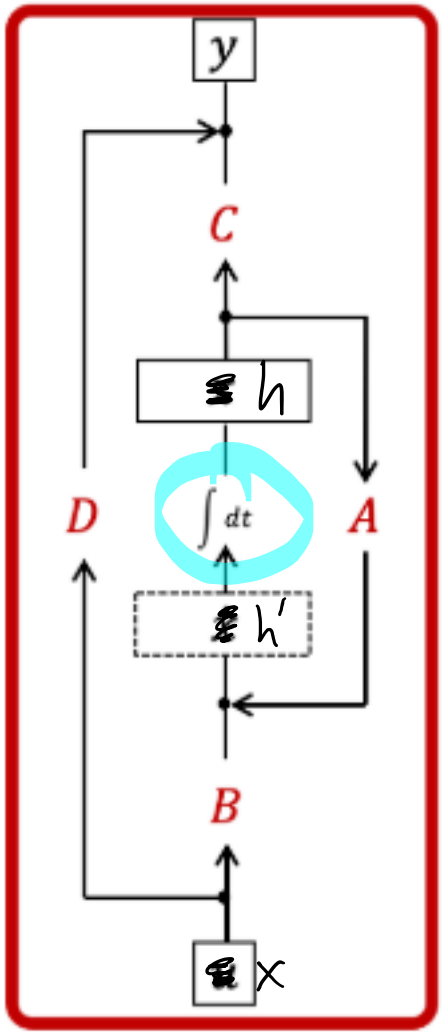


Figure from Albert Gu's F24 guest lecture in 10-423/623

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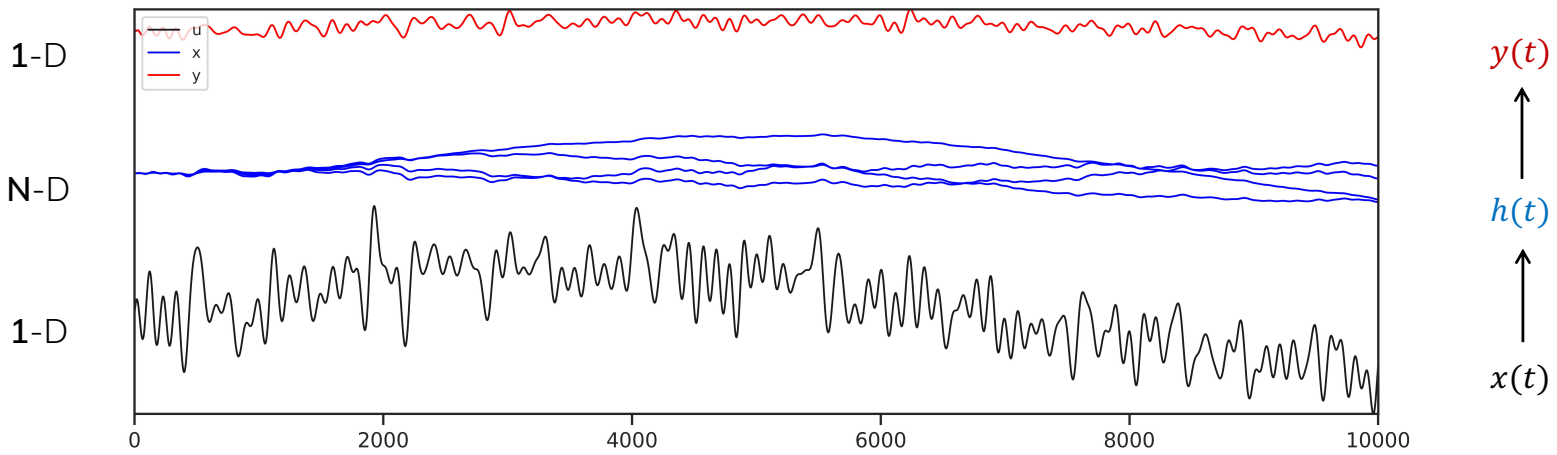

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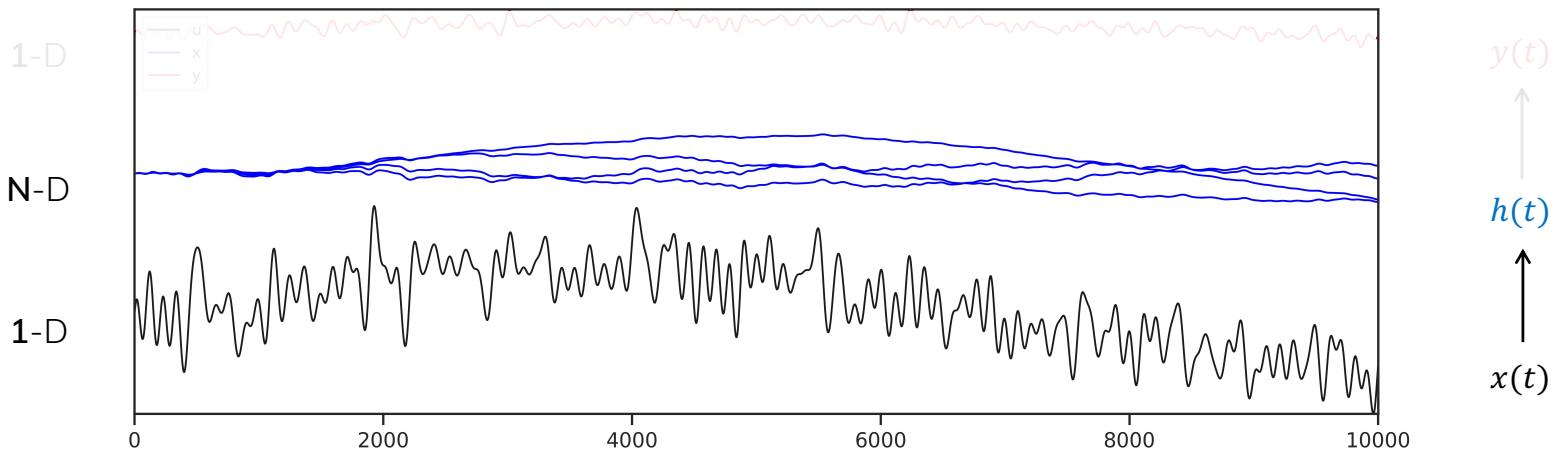

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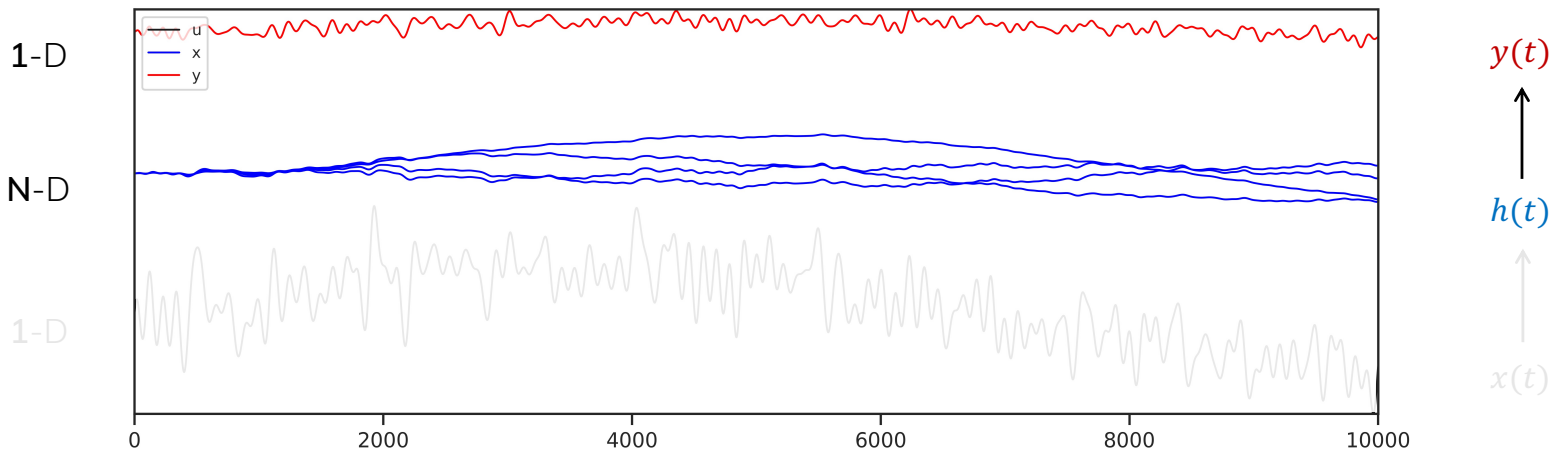

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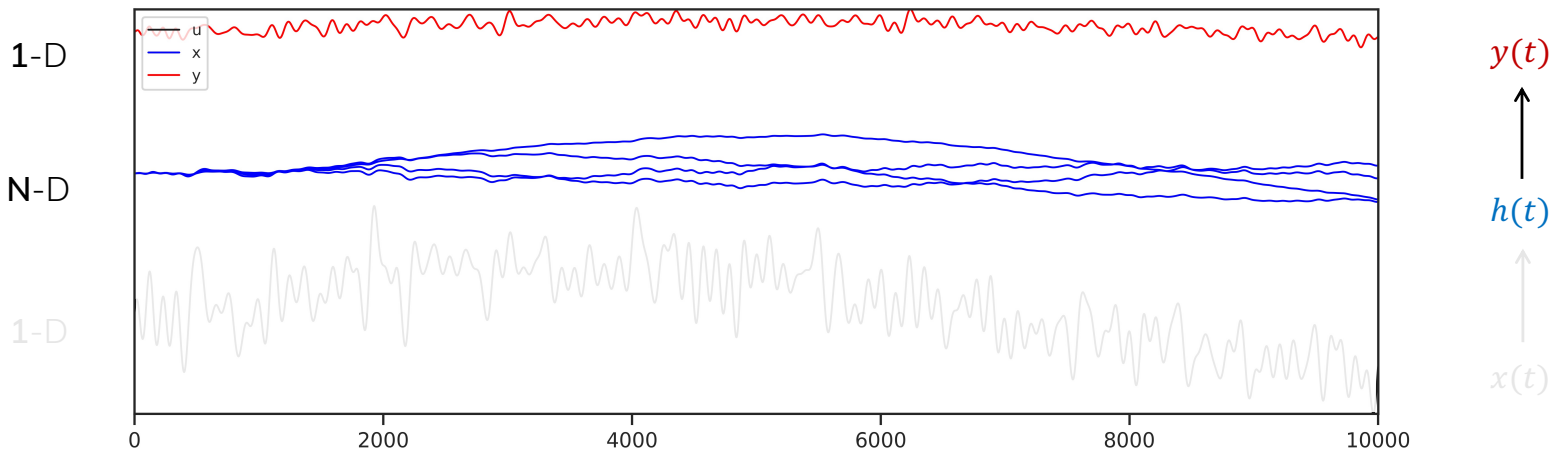

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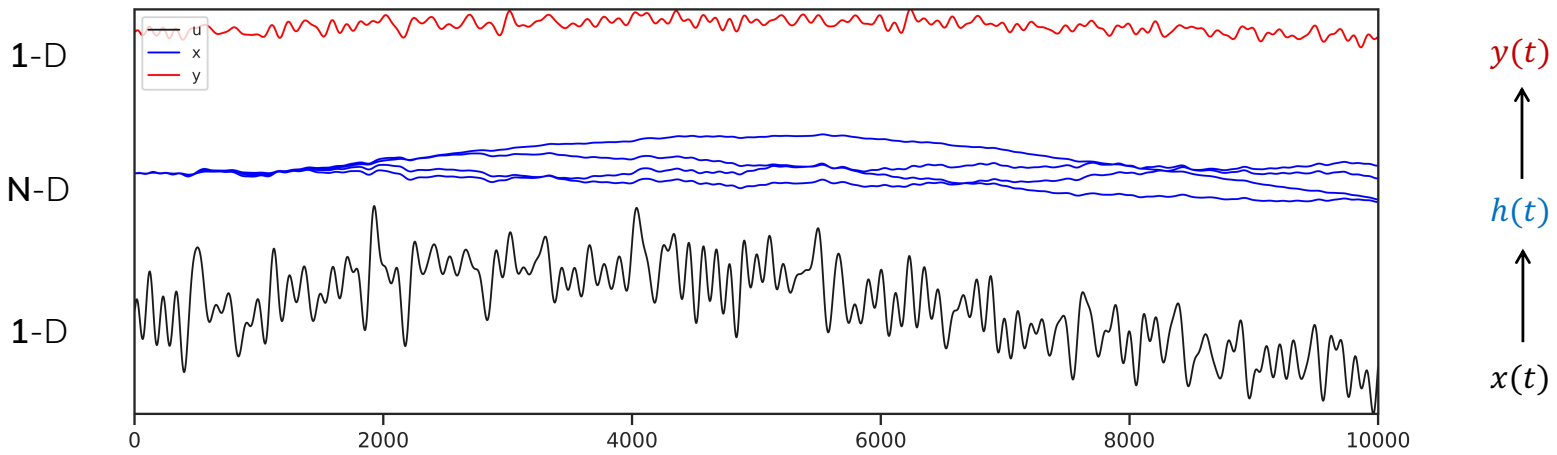
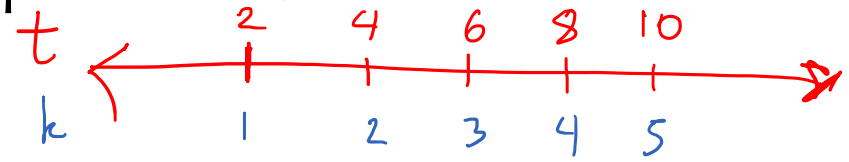


Figure from Albert Gu's F24 guest lecture in 10-423/623

SSM: 1D Discrete Recurrent Representation

Continuous Representation

- Uses parameters we will actually work with in the end
- Seamlessly represents any continuous 1D → 1D function
- Impractical for real data



$$h'(t) = \mathbf{A}h(t) + \mathbf{B}x(t)$$

$$y(t) = \mathbf{C}h(t) + \mathbf{D}x(t)$$

$t \in \mathbb{R}^+$

Discrete Recurrent Representation

- A discrete approximation using different parameters which are **functions** of the original parameters A,B,C,D
- Allows us to work with real data

$$h_{k+1} = \bar{\mathbf{A}}h_k + \bar{\mathbf{B}}x_k$$

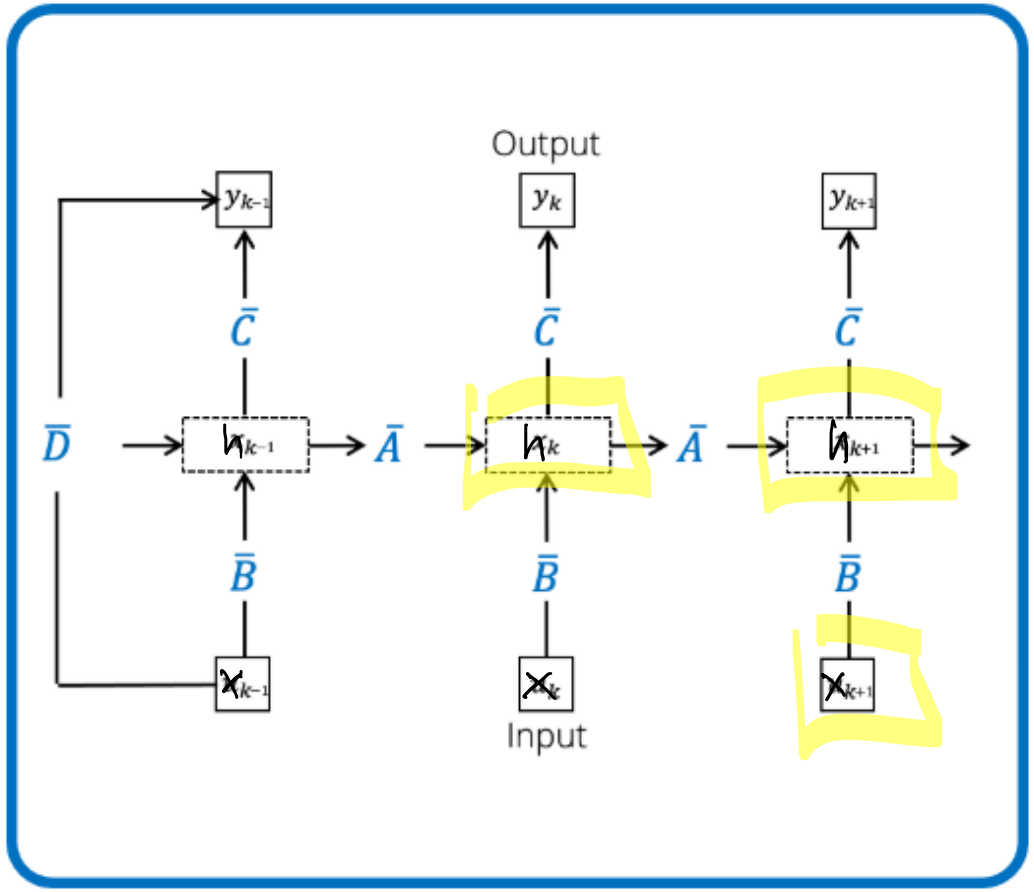
$$y_k = \bar{\mathbf{C}}h_k + \bar{\mathbf{D}}x_k$$

$k \in \mathbb{N}^+$

SSM: 1D Discrete Recurrent Representation

Q: How is this different from an RNN?

Question: How can we depict this recurrent computation?



Discrete Recurrent Representation

- A discrete approximation using different parameters which are **functions** of the original parameters A,B,C,D
- Allows us to work with real data

no activation function

no bias term

$$h_{k+1} = \bar{\mathbf{A}} h_k + \bar{\mathbf{B}} x_k$$

$$y_k = \bar{\mathbf{C}} h_k + \bar{\mathbf{D}} x_k$$

residual connection

are 1D

How to discretize a continuous SSM?

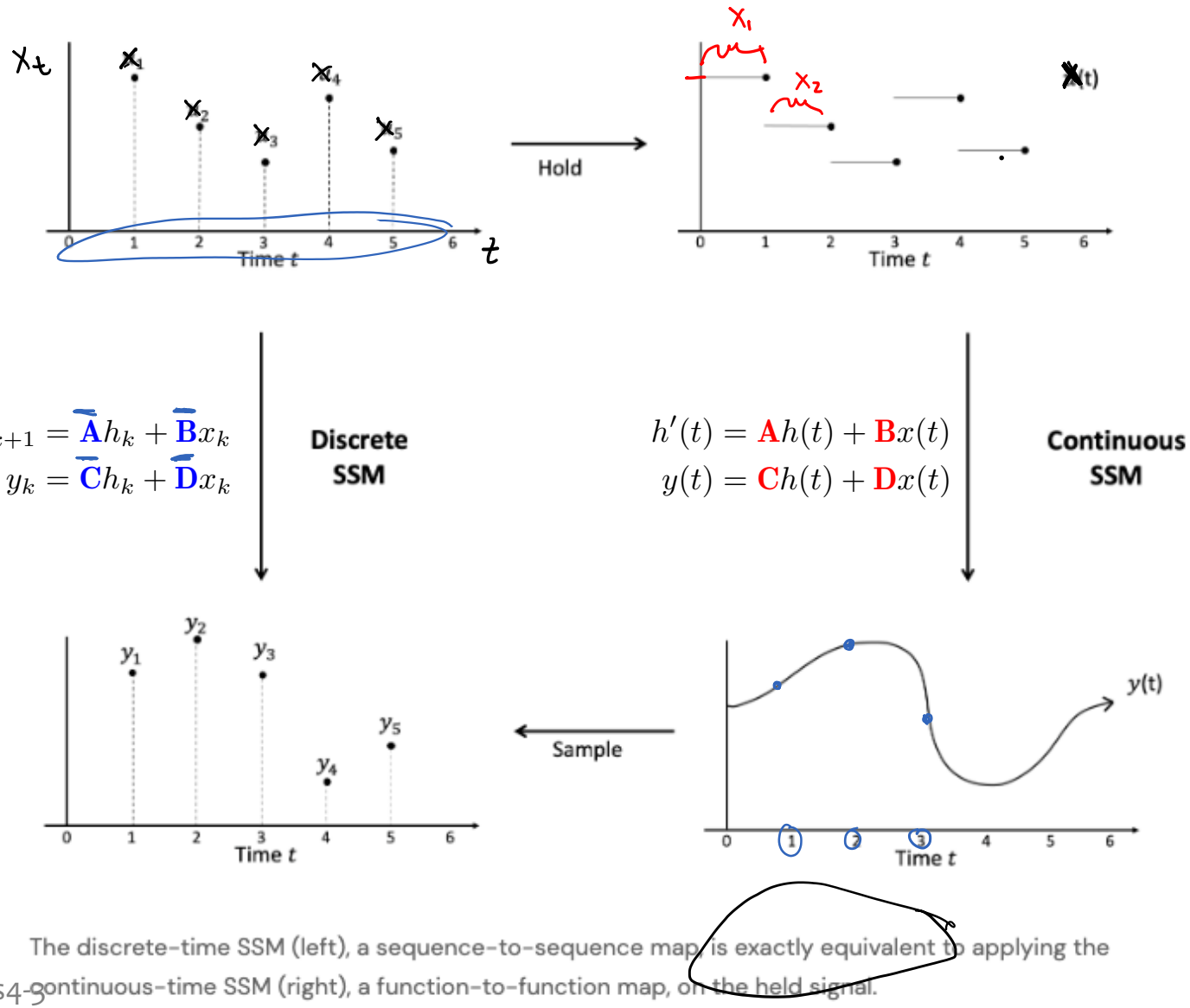
S4 uses a bilinear transformation to discretize the continuous SSM

$$\overline{\boldsymbol{A}} = e^{\Delta \boldsymbol{A}}$$
$$\overline{\boldsymbol{B}} = \boldsymbol{A}^{-1}(e^{\Delta \boldsymbol{A}} - \boldsymbol{I})\boldsymbol{B}$$
$$\overline{\boldsymbol{C}} = \boldsymbol{C}$$
$$\overline{\boldsymbol{D}} = \boldsymbol{D}$$

Δ = step size

The bilinear transformation uses a first order Pade approximation:

$$e^x \approx \frac{1+x/2}{1-x/2}$$



The discrete-time SSM (left), a sequence-to-sequence map, is exactly equivalent to applying the continuous-time SSM (right), a function-to-function map, on the held signal.

Figure from https://hazyresearch.stanford.edu/blog/2022-01-14-s4-3

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The bilinear transformation uses a first order Pade approximation:

$$e^x \approx \frac{1+x/2}{1-x/2}$$

$$\overline{\boldsymbol{A}} = (\boldsymbol{I} - \frac{\Delta}{2} \cdot \boldsymbol{A})^{-1}(\boldsymbol{I} + \frac{\Delta}{2} \cdot \boldsymbol{A})$$
$$\overline{\boldsymbol{B}} = (\boldsymbol{I} - \frac{\Delta}{2} \cdot \boldsymbol{A})^{-1}\Delta \boldsymbol{B}$$
$$\overline{\boldsymbol{C}} = \boldsymbol{C}$$
$$\overline{\boldsymbol{D}} = \boldsymbol{D}$$

think of this as a PyTorch layer.
b/c we will learn A, B, C, D and not $\bar{A}, \bar{B}, \bar{C}, \bar{D}$

Figure from https://hazyresearch.stanford.edu/blog/2022-01-14-s4-3

SSM: 1D Convolutional Representation

We unroll the recurrent computation as:
Assume a zero initial state: $h_{-1} = 0$.

$$h_0 = \bar{\mathbf{B}}x_0$$
$$h_1 = \bar{\mathbf{A}}\bar{\mathbf{B}}x_0 + \bar{\mathbf{B}}x_1$$
$$h_2 = \bar{\mathbf{A}}^2\bar{\mathbf{B}}x_0 + \bar{\mathbf{A}}\bar{\mathbf{B}}x_1 + \bar{\mathbf{B}}x_2$$
$$\vdots$$
$$y_0 = \bar{\mathbf{C}}\bar{\mathbf{B}}x_0$$
$$y_1 = \bar{\mathbf{C}}\bar{\mathbf{A}}\bar{\mathbf{B}}x_0 + \bar{\mathbf{C}}\bar{\mathbf{B}}x_1$$
$$y_2 = \bar{\mathbf{C}}\bar{\mathbf{A}}^2\bar{\mathbf{B}}x_0 + \bar{\mathbf{C}}\bar{\mathbf{A}}\bar{\mathbf{B}}x_1 + \bar{\mathbf{C}}\bar{\mathbf{B}}x_2$$
$$\vdots$$
$$y_k = \bar{\mathbf{C}}\bar{\mathbf{A}}^k\bar{\mathbf{B}}x_0 + \bar{\mathbf{C}}\bar{\mathbf{A}}^{k-1}\bar{\mathbf{B}}x_1 + \cdots + \bar{\mathbf{C}}\bar{\mathbf{A}}^1\bar{\mathbf{B}}x_{k-1} + \bar{\mathbf{C}}\bar{\mathbf{A}}^0\bar{\mathbf{B}}x_k$$

each x_t is multiplied by $\bar{C}\bar{A}^{(t-k)}\bar{B}$

We can represent this as a *global* convolution computation:

$$\mathbf{y} = \bar{\mathbf{K}} * \mathbf{x}$$

L = sequence length

where the SSM convolution kernel is:

$$\bar{\mathbf{K}} \in \mathbb{R}^L = (\bar{\mathbf{C}}\bar{\mathbf{A}}^0\bar{\mathbf{B}}, \bar{\mathbf{C}}\bar{\mathbf{A}}^1\bar{\mathbf{B}}, \ldots, \bar{\mathbf{C}}\bar{\mathbf{A}}^{L-2}\bar{\mathbf{B}}, \bar{\mathbf{C}}\bar{\mathbf{A}}^{L-1}\bar{\mathbf{B}})$$

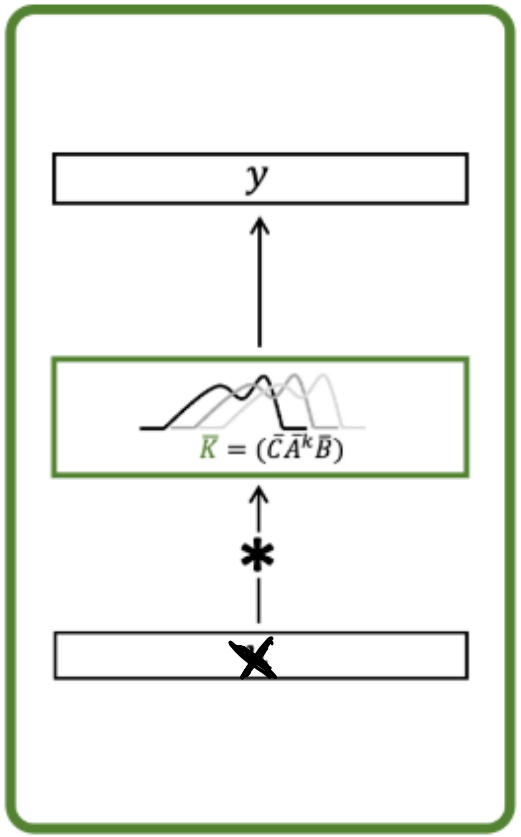
$\in \mathbb{R}$ $\in \mathbb{R}$

$h_{k+1} = \bar{A}h_k + \bar{B}x_k$

$y_k = \bar{C}h_k$

leave out $\bar{D}x_k$

SSM: 1D Convolutional Representation



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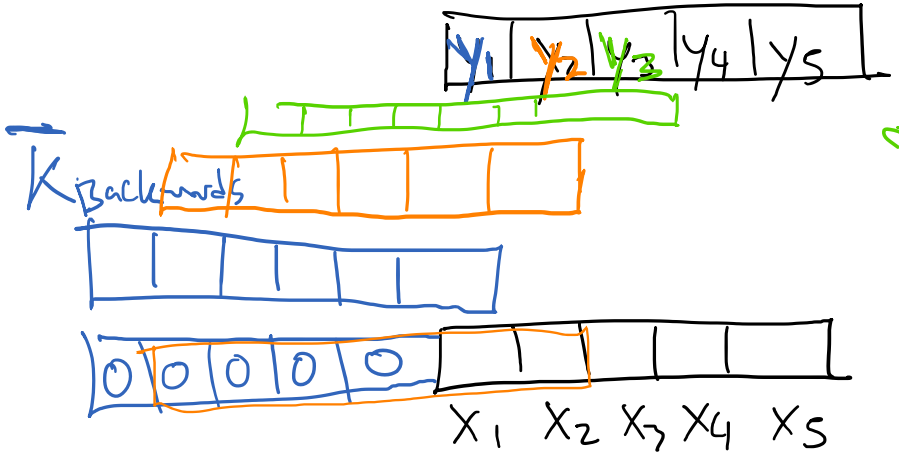
$$\bar{\mathbf{K}} \in \mathbb{R}^L = \left(\bar{\mathbf{C}}\bar{\mathbf{A}}^0\bar{\mathbf{B}},\ \bar{\mathbf{C}}\bar{\mathbf{A}}^1\bar{\mathbf{B}},\ \ldots,\ \bar{\mathbf{C}}\bar{\mathbf{A}}^{L-2}\bar{\mathbf{B}}, \bar{\mathbf{C}}\bar{\mathbf{A}}^{L-1}\bar{\mathbf{B}}\right)$$



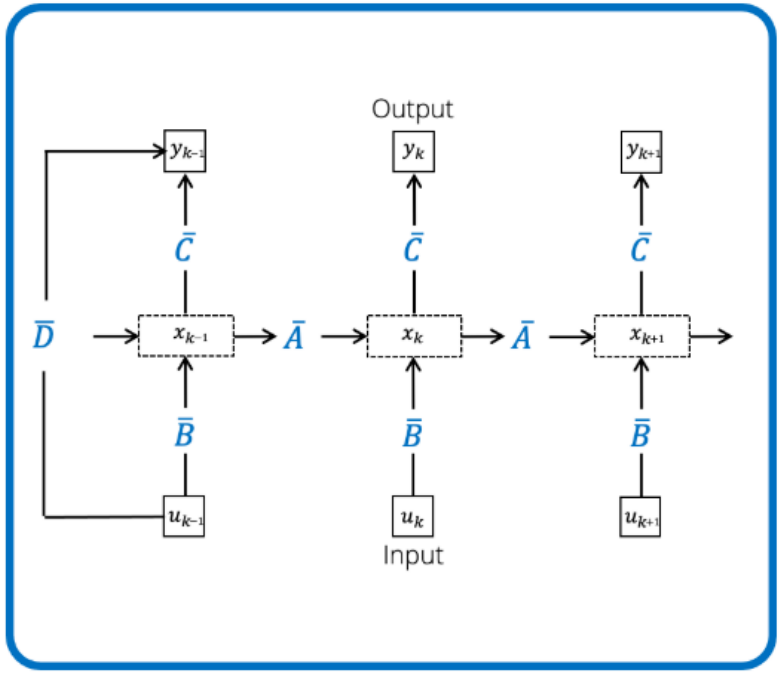
★ can compute all $y_k \ \forall k \in \{1, \ldots, L\}$ in parallel

$$y_k = \bar{\mathbf{C}}\bar{\mathbf{A}}^k\bar{\mathbf{B}}x_0 + \bar{\mathbf{C}}\bar{\mathbf{A}}^{k-1}\bar{\mathbf{B}}x_1 + \cdots + \bar{\mathbf{C}}\bar{\mathbf{A}}^1\bar{\mathbf{B}}x_{k-1} + \bar{\mathbf{C}}\bar{\mathbf{A}}^0\bar{\mathbf{B}}x_k$$

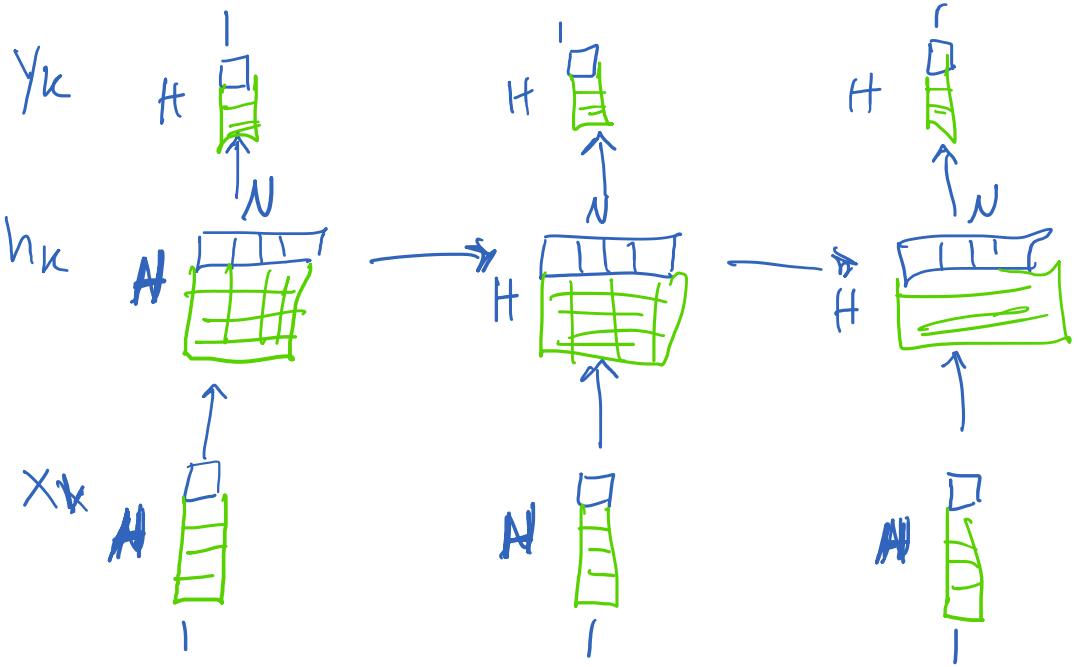
THE STRUCTURED STATE SPACE SEQUENCE MODEL (S4)

SSM as a Neural Network Layer

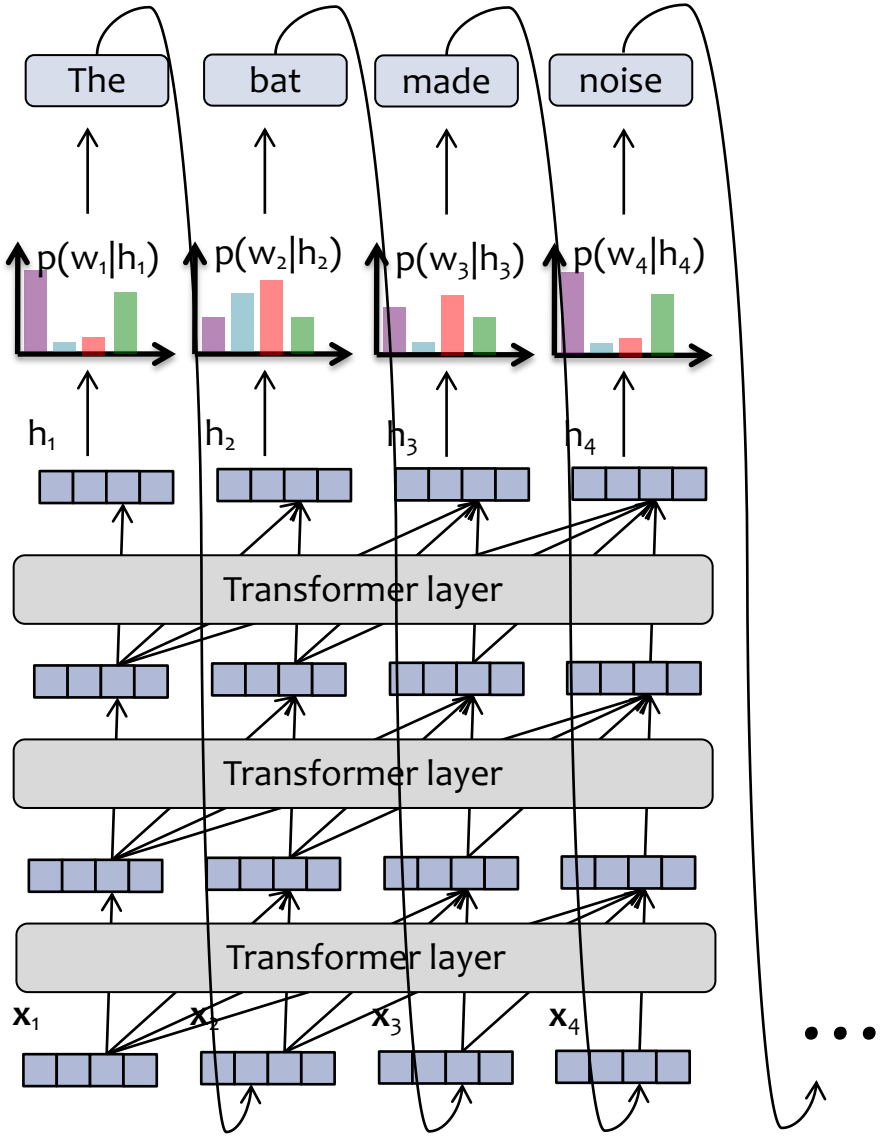
- We can take H copies of the 1D recurrent representation
- Let each copy have its own parameters



- This is just like multiple (indep.) heads in Attention
- And just like multiple (indep.) channels in Convolution
- So we get…



Transformer Language Model



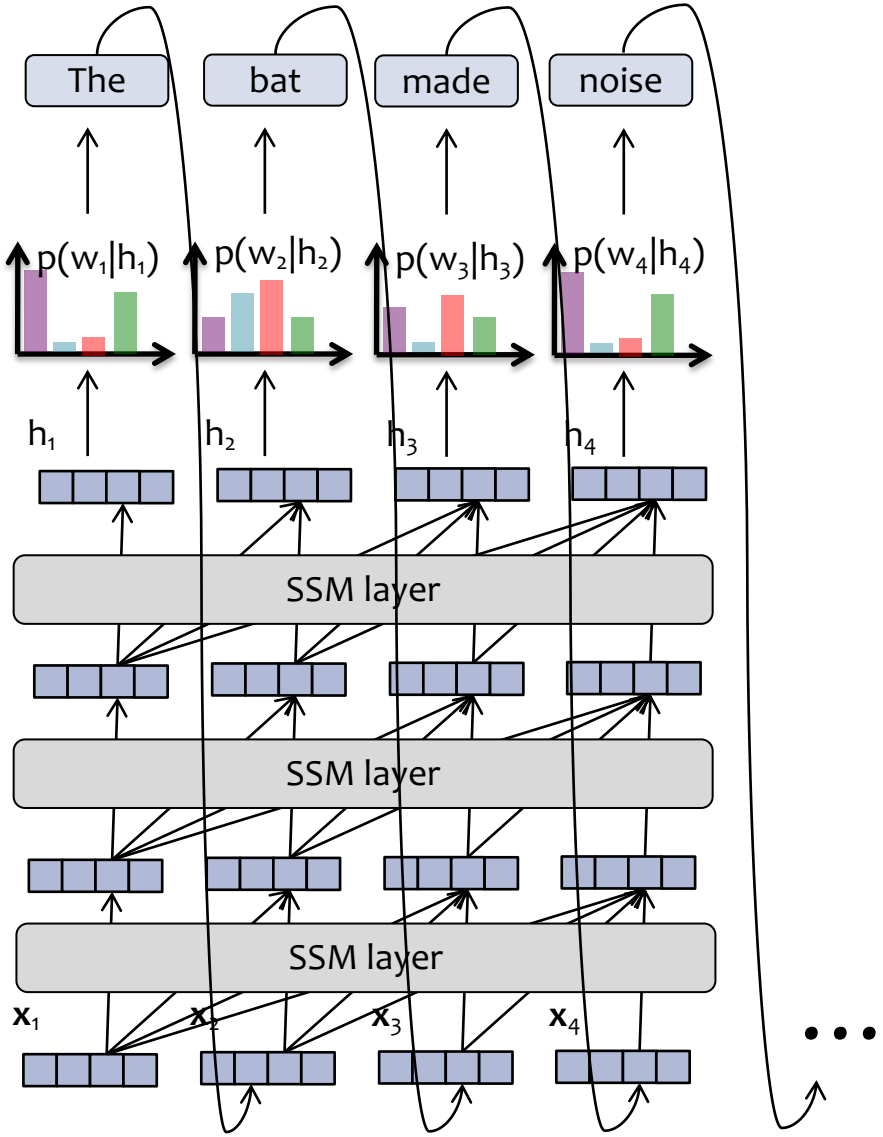
Each layer of a Transformer LM consists of several **sublayers:**

1. attention
2. feed-forward neural network
3. layer normalization
4. residual connections

Each hidden vector looks back at the hidden vectors of the **current and previous timesteps in the previous layer.**

The language model part is just like an RNN-LM.

SSM inside a Deep Language Model



Each layer of an S4 LM consists of several **sublayers as well** including an SSM, nonlinearity, etc.

Each hidden vector looks back at the hidden vectors of the **current and previous timesteps in the previous layer.**

The language model part is just like an RNN-LM or Transformer-LM

Efficiency of SSM, RNN, & Transformer

For SSMs:

1. At test time, generation does NOT need a KV-cache in our **Recurrent representation**, so we can effortlessly generate truly long sequences (unlike Transformers, but just like RNNs)
2. At train time, we can use the **Convolution representation** to do fast parallel training (just like Transformers, but unlike RNNs)

	Train	Test
Recurrence	Slow	Fast
Attention	Fast	Slow
SSM	Fast	Fast

S4 Model

We need several additional tricks to get training to work well:

- HiPPO Matrix
  - we initialize the matrix A very carefully
- Efficient computation
  - we decompose A so that we can compute the kernel K very efficiently and in a numerically stable way

$$h_{k+1} = \bar{A} h_k + \bar{B} x_k$$

$$h'(t) = A h(t) + B x(t)$$

$$K = (\ldots, \bar{C}\bar{A}^{L-1}\bar{B})$$

unstable

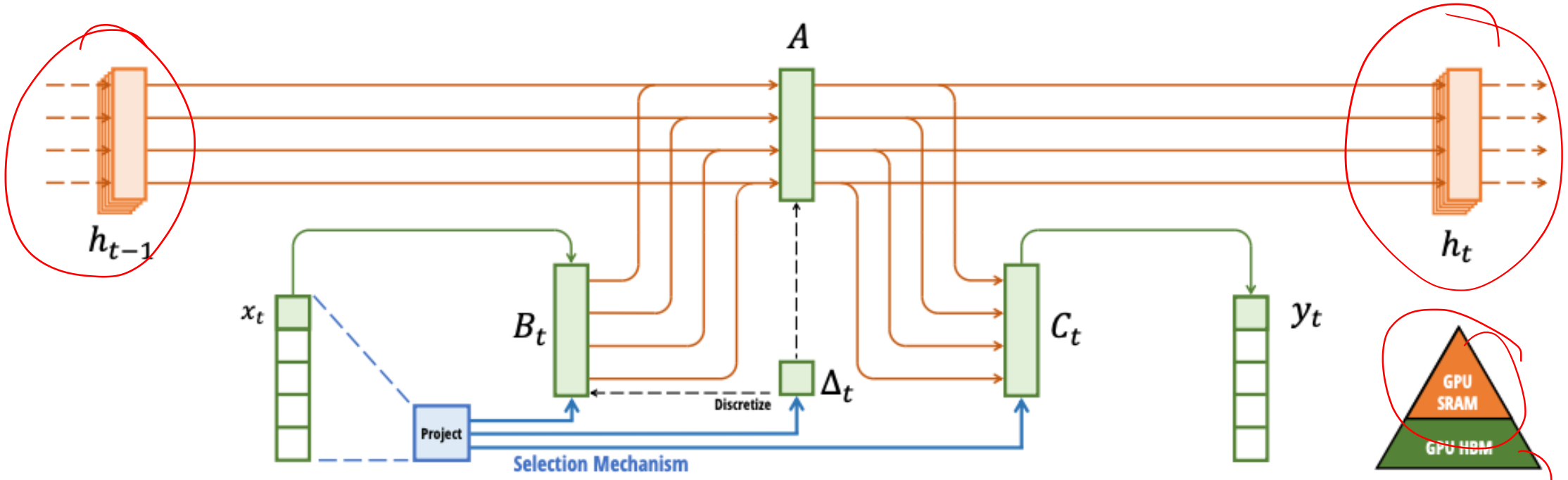
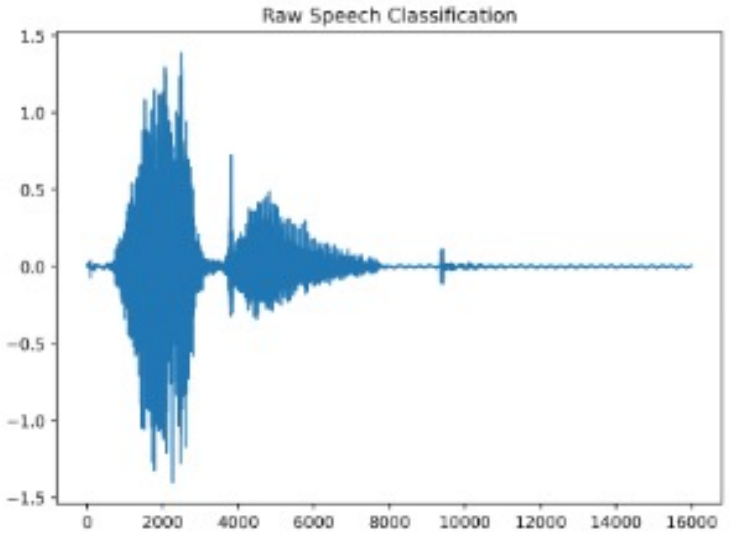


Figure 1: (**Overview**.) Structured SSMs independently map each channel (e.g. $D = 5$) of an input x to output y through a higher dimensional latent state h (e.g. $N = 4$). Prior SSMs avoid materializing this large effective state (DN, times batch size B and sequence length L) through clever alternate computation paths requiring time-invariance: the (Δ, $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$) parameters are constant across time. Our selection mechanism adds back input-dependent dynamics, which also requires a careful hardware-aware algorithm to only materialize the expanded states in more efficient levels of the GPU memory hierarchy.

Figure from https://arxiv.org/pdf/2312.00752

S4 Results: Train and test on different input granularities



		Train: 16K Hz	Test: 8K Hz
	MFCC	Raw	0.5×
Transformer	90.75	✗	✗
Performer	80.85	30.77	30.68
ODE-RNN	65.9	✗	✗
NRDE	89.8	16.49	15.12
ExpRNN	82.13	11.6	10.8
LipschitzRNN	88.38	✗	✗
CKConv	**95.3**	71.66	65.96
WaveGAN-D	✗	96.25	✗
LSSL	93.58	✗	✗
S4	93.96	**98.32**	**96.30**

Figure from https://hazyresearch.stanford.edu/blog/2022-01-14-s4-3

MAMBA

Selective State Space Models

Algorithm 1 SSM (S4)

Input: $x : (\mathsf{B}, \mathsf{L}, \mathsf{D})$
Output: $y : (\mathsf{B}, \mathsf{L}, \mathsf{D})$
1: $\boldsymbol{A} : (\mathsf{D}, \mathsf{N}) \leftarrow$ Parameter
▷ Represents structured $N \times N$ matrix
2: $\boldsymbol{B} : (\mathsf{D}, \mathsf{N}) \leftarrow$ Parameter
3: $\boldsymbol{C} : (\mathsf{D}, \mathsf{N}) \leftarrow$ Parameter
4: $\Delta : (\mathsf{D}) \leftarrow \tau_\Delta(\text{Parameter})$
5: $\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}} : (\mathsf{D}, \mathsf{N}) \leftarrow \text{discretize}(\Delta, \boldsymbol{A}, \boldsymbol{B})$
6: $y \leftarrow \text{SSM}(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}, \boldsymbol{C})(x)$
▷ Time-invariant: recurrence or convolution
7: **return** y

Algorithm 2 SSM + Selection (S6)

Input: $x : (\mathsf{B}, \mathsf{L}, \mathsf{D})$
Output: $y : (\mathsf{B}, \mathsf{L}, \mathsf{D})$
1: $\boldsymbol{A} : (\mathsf{D}, \mathsf{N}) \leftarrow$ Parameter
▷ Represents structured $N \times N$ matrix
2: $\boldsymbol{B} : (\mathsf{B}, \mathsf{L}, \mathsf{N}) \leftarrow s_B(x)$ = linear(W_B, x)
3: $\boldsymbol{C} : (\mathsf{B}, \mathsf{L}, \mathsf{N}) \leftarrow s_C(x)$ = linear(W_C, x)
4: $\Delta : (\mathsf{B}, \mathsf{L}, \mathsf{D}) \leftarrow \tau_\Delta(\text{Parameter} + s_\Delta(x))$
5: $\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}} : (\mathsf{B}, \mathsf{L}, \mathsf{D}, \mathsf{N}) \leftarrow \text{discretize}(\Delta, \boldsymbol{A}, \boldsymbol{B})$
6: $y \leftarrow \text{SSM}(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}, \boldsymbol{C})(x)$
▷ Time-varying: recurrence (*scan*) only
7: **return** y

- **Selective** state space models differ from S4 in that they let the parameters B and C vary at each timestep

Selective State Space Models

Algorithm 1 SSM (S4)

Input: $x : (\mathsf{B}, \mathsf{L}, \mathsf{D})$
Output: $y : (\mathsf{B}, \mathsf{L}, \mathsf{D})$

$\boldsymbol{A} : (\mathsf{D}, \mathsf{N}) \leftarrow$ Parameter
▷ Represents structured $N \times N$ matrix
$\boldsymbol{B} : (\mathsf{D}, \mathsf{N}) \leftarrow$ Parameter
$\boldsymbol{C} : (\mathsf{D}, \mathsf{N}) \leftarrow$ Parameter
$\Delta : (\mathsf{D}) \leftarrow \tau_\Delta(\text{Parameter})$
$\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}} : (\mathsf{D}, \mathsf{N}) \leftarrow \text{discretize}(\Delta, \boldsymbol{A}, \boldsymbol{B})$
$y \leftarrow \text{SSM}(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}, \boldsymbol{C})(x)$
▷ Time-invariant: recurrence or convolution
return y

Algorithm 2 SSM + Selection (S6)

Input: $x : (\mathsf{B}, \mathsf{L}, \mathsf{D})$
Output: $y : (\mathsf{B}, \mathsf{L}, \mathsf{D})$

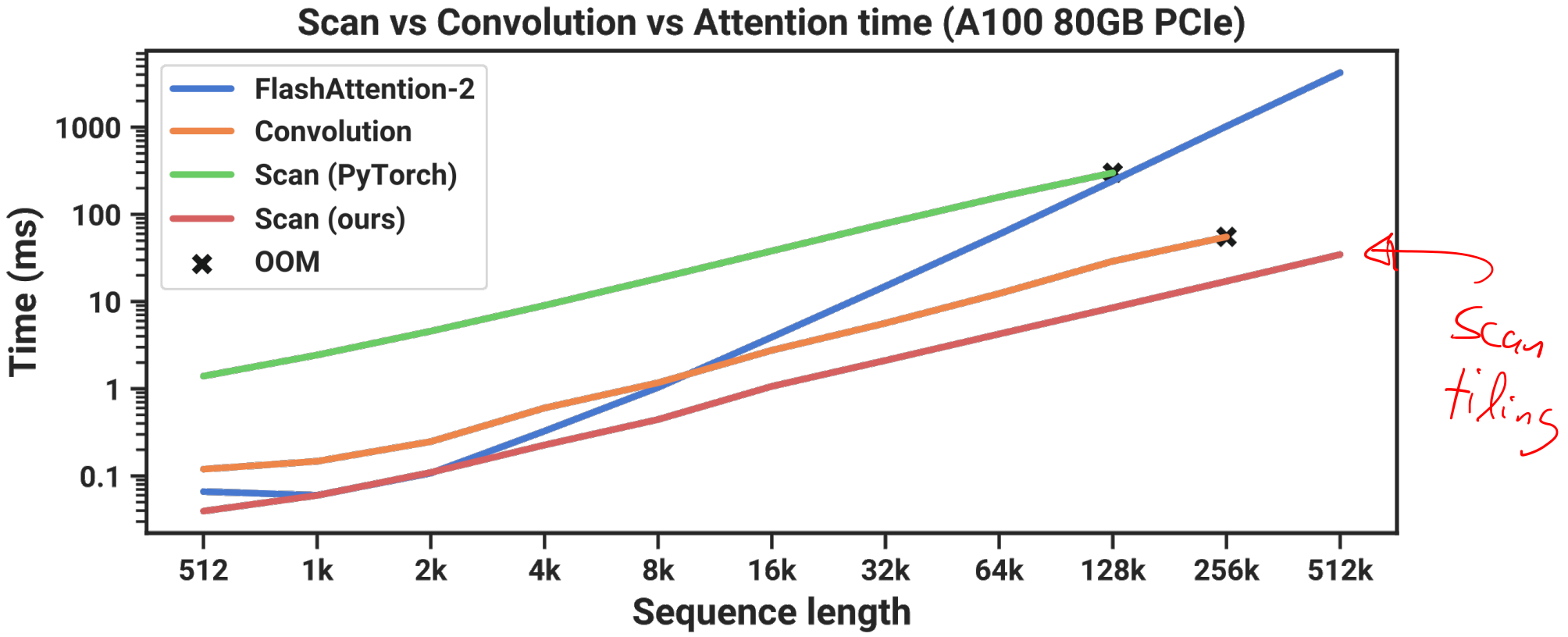
$\boldsymbol{A} : (\mathsf{D}, \mathsf{N}) \leftarrow$ Parameter
▷ Represents structured $N \times N$ matrix
$\boldsymbol{B} : (\mathsf{B}, \mathsf{L}, \mathsf{N}) \leftarrow s_B(x)$
$\boldsymbol{C} : (\mathsf{B}, \mathsf{L}, \mathsf{N}) \leftarrow s_C(x)$
$\Delta : (\mathsf{B}, \mathsf{L}, \mathsf{D}) \leftarrow \tau_\Delta(\text{Parameter} + s_\Delta(x))$
$\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}} : (\mathsf{B}, \mathsf{L}, \mathsf{D}, \mathsf{N}) \leftarrow \text{discretize}(\Delta, \boldsymbol{A}, \boldsymbol{B})$
$y \leftarrow \text{SSM}(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}, \boldsymbol{C})(x)$
▷ Time-varying: recurrence (*scan*) only
return y

$$h_t = Ah_{t-1} + Bx_t$$
$$y_t = C^\top h_t$$

$$h_t = A_t h_{t-1} + B_t x_t$$
$$y_t = C_t^\top h_t$$

Mamba's Scan Implementation

- We can no longer compute the kernel K once up front
- Instead we perform an efficient scan implementation



https://jackcook.com/2024/02/23/mamba.html

Mamba Results

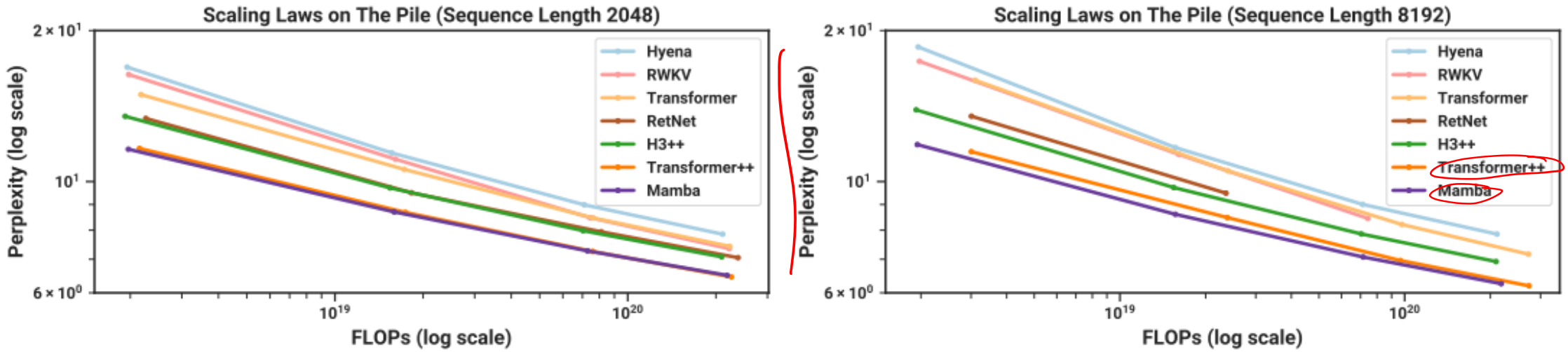
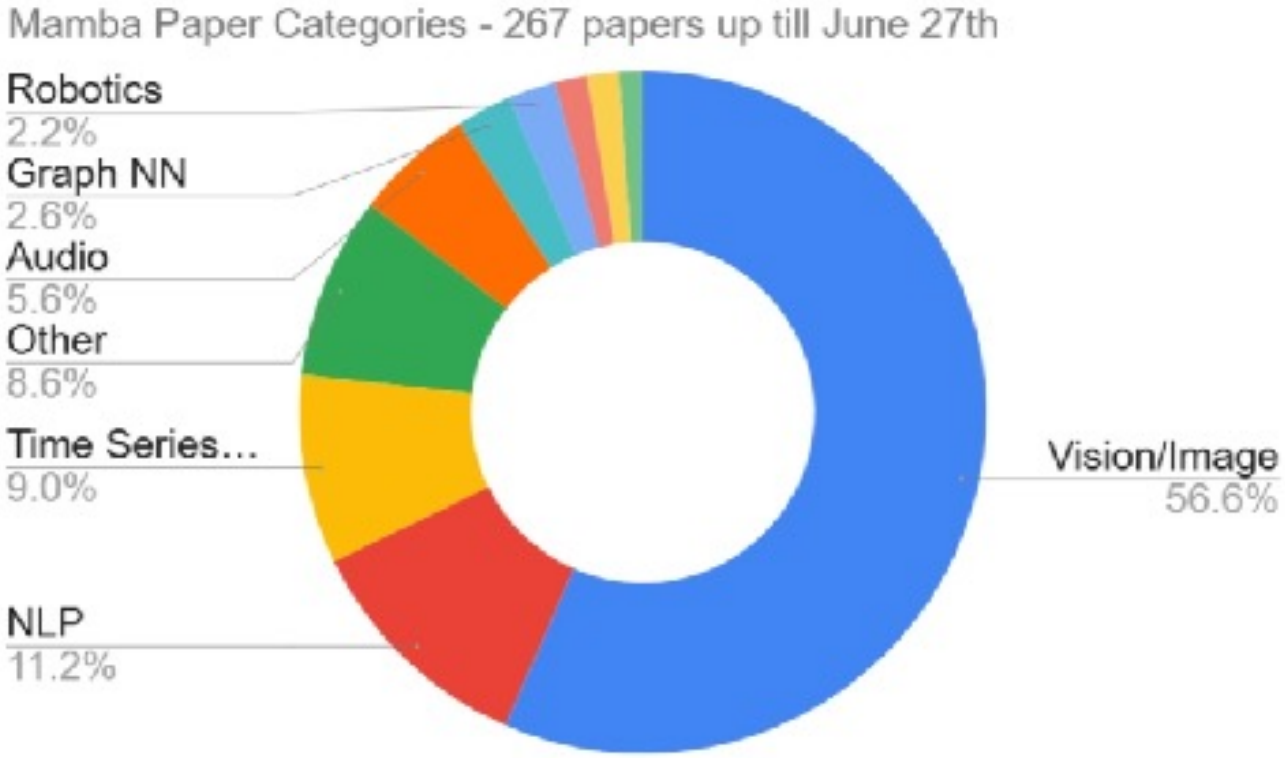


Figure 4: (**Scaling Laws.**) Models of size $\approx 125M$ to $\approx 1.3B$ parameters, trained on the Pile. Mamba scales better than all other attention-free models and is the first to match the performance of a very strong "Transformer++" recipe that has now become standard, particularly as the sequence length grows.

- Main takeaway: Mamba is the first (only?) non-attention based LM to challenge a Transformer

https://arxiv.org/pdf/2312.00752

Mamba Use in the Real World



Strong out-of-the-box on **general modalities**

(not just language!)

Figure from Albert Gu's F24 guest lecture in 10-423/623

Motivation

- https://www.isattentionallyouneed.com/

Is Attention All You Need?

Current Status: Yes

Time Remaining: 631d 22h 55m 11s

Proposition:

On January 1, 2027, a Transformer-like model will continue to hold the state-of-the-art position in most benchmarked tasks in natural language processing.