Automated Program Verification and Testing 15414/15614 Fall 2016 Lecture 20: Explicit-State Model Checking, Part 2

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Today's Lecture

LTL Model Checking

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Algorithm based on automata operations

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- Refresher on basic automata theory

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- Introduce automata for languages of infinite words

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Algorithm based on automata operations

- Refresher on basic automata theory
- Introduce automata for languages of infinite words
- See how to apply them to model checking

Finite Automata: Refresher

A Nondeterministic Finite Automaton (NFA) is a tuple $(Q, \Sigma, \delta, Q_0, F)$:

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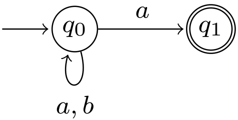

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Every NFA can be converted to a DFA accepting the same language

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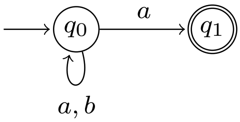

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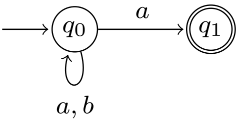

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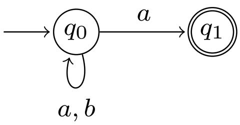

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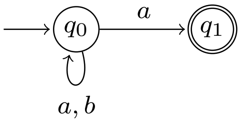

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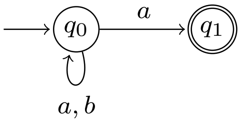

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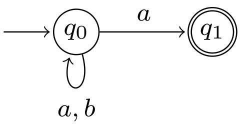

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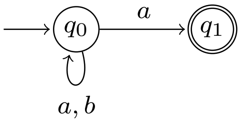

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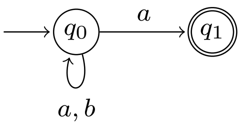

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$L(A) = \{w \mid w$contains arbitrary sequence of a, b ending with $a\}$

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This can be decided in $O(|A|)$ by depth-first search

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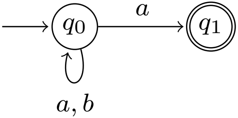

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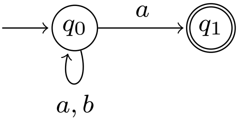

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Given NFAs representing a language, we can construct NFAs corresponding to the application of these operations

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- Like Kleene $*$, but ad infinitum

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If L doesn't contain ϵ, L^ω is an infinite language

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The “syntax” is the same as NFAs; obviously the semantics is different

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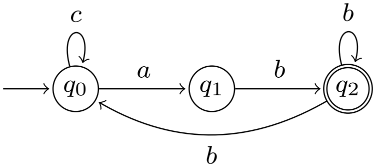
A language is ω-regular language iff it is recognizable by an NBA

Example



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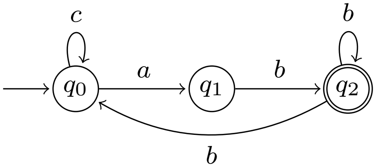

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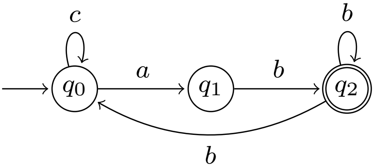

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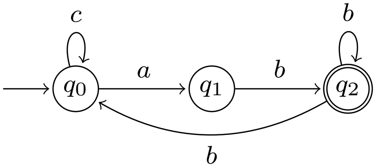

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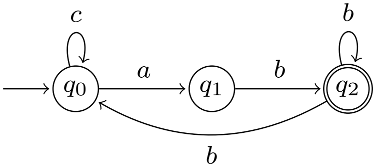

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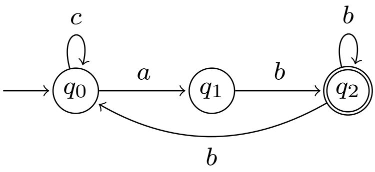

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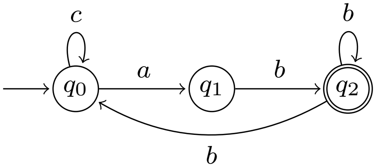

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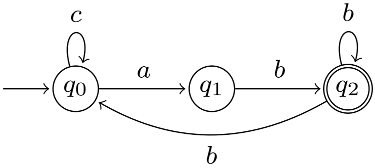

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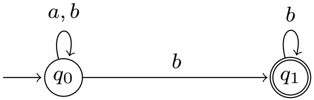

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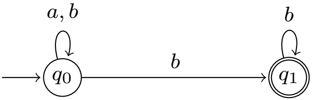

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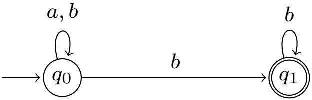

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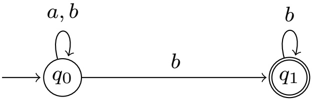

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We'll be a bit sloppy, and let Σ be formulas over $\mathit{Send}, \mathit{Read}$

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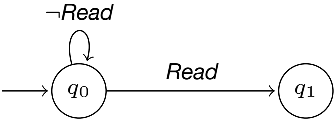

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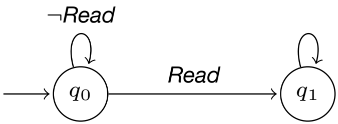

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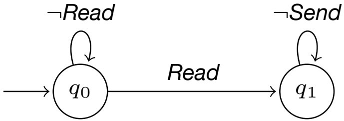

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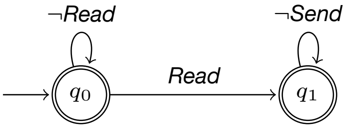

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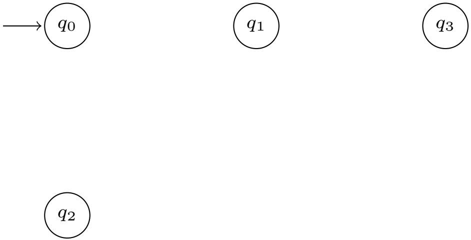

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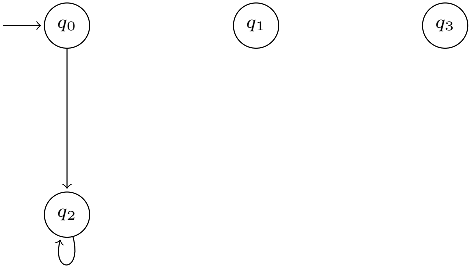

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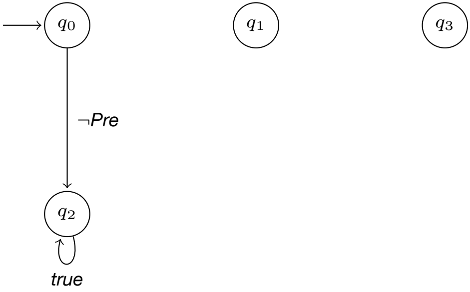

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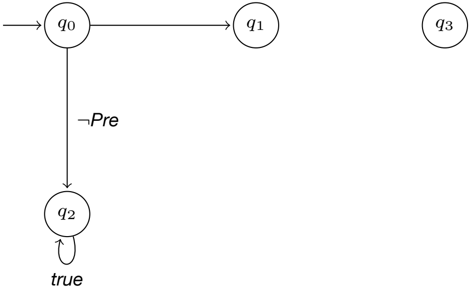

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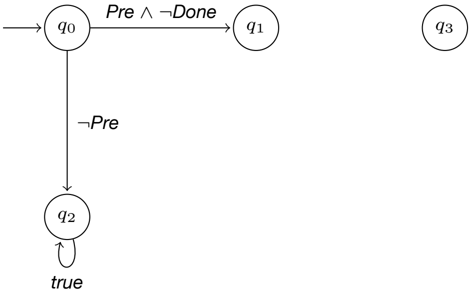

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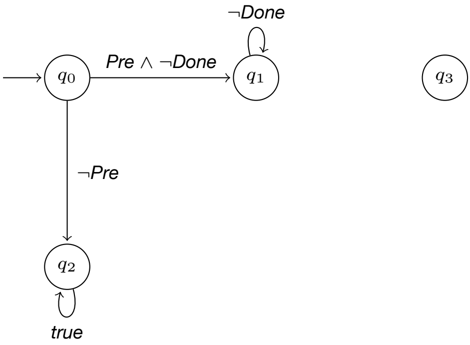

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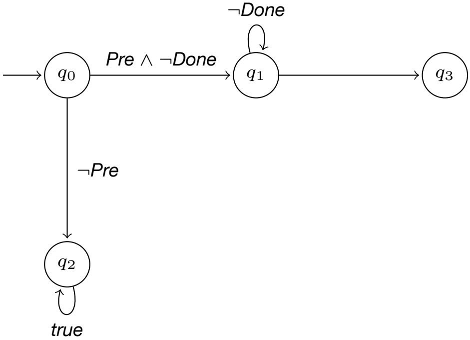

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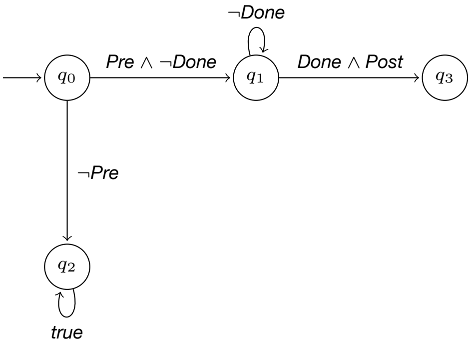

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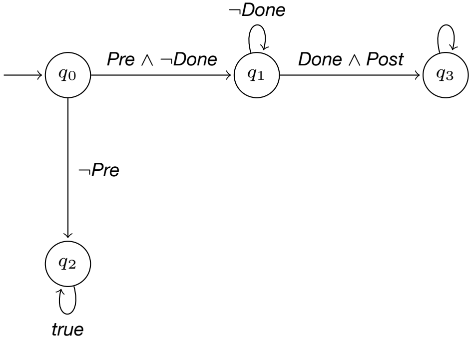

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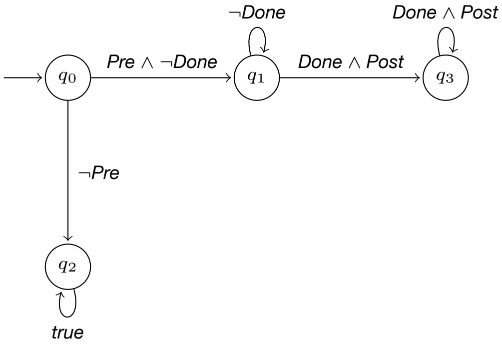

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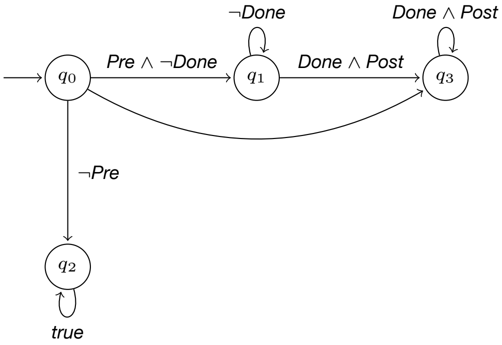

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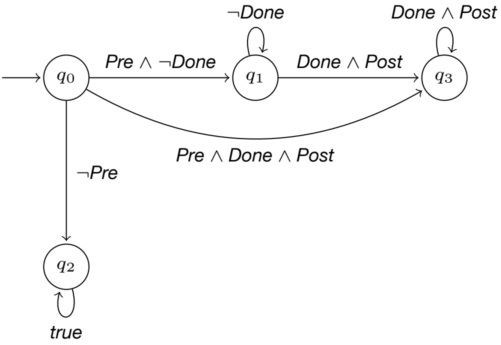

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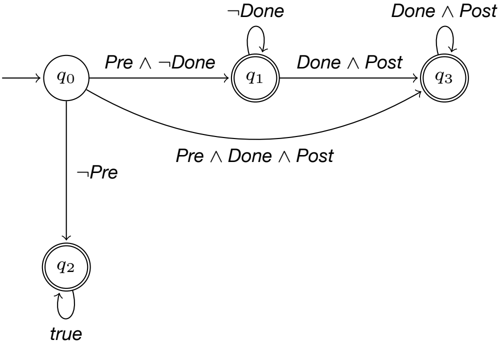

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- This is important for model checking, as we'll see

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Let A be an NBA representing some computation

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The set of traces in A is contained in the set of "good" computations

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$$L(A) \subseteq L(S) \Leftrightarrow L(A) \cap \overline{L(A_\phi)} = \varnothing$$

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> *Given a Kripke structure M and LTL formula ϕ, decide whether $M, \pi \models \phi$ for each π starting in an initial state.*

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A Kripke structure $M = (P, S, I, L, R)$ consists of:

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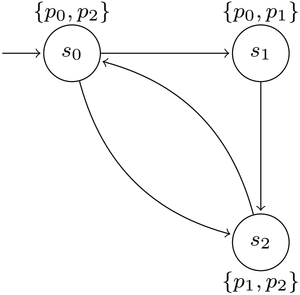

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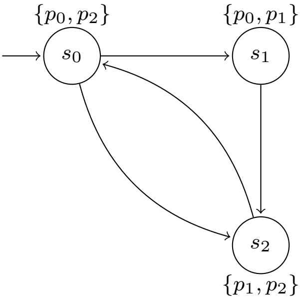

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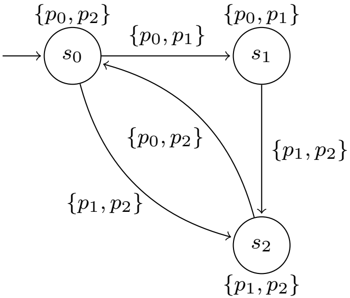

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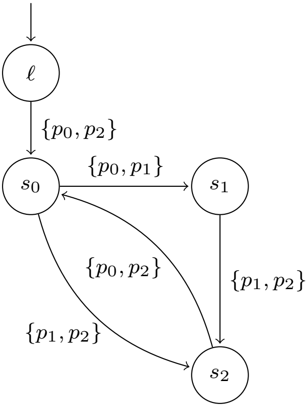

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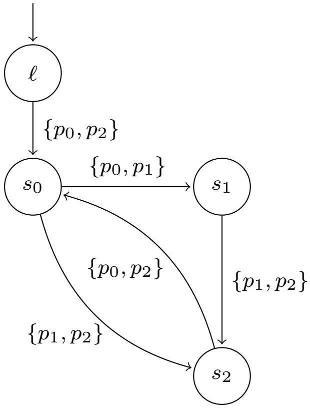

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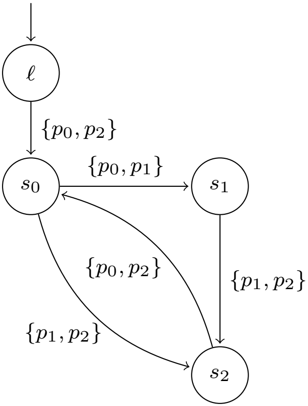

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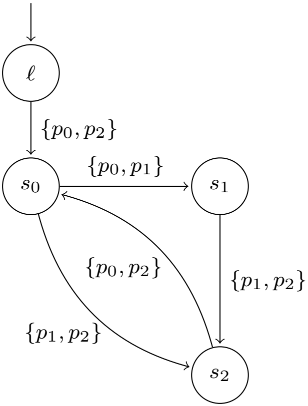

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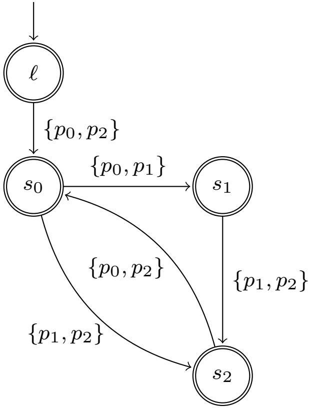

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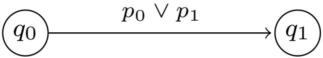
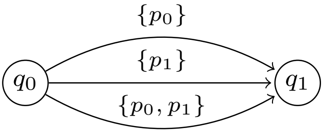


Then this is shorthand for:



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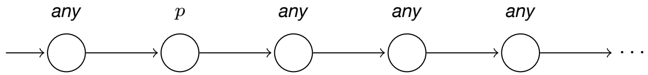
X p



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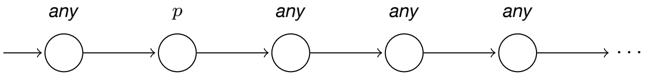

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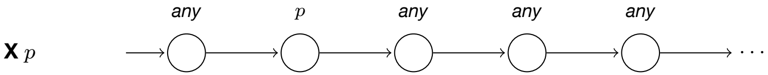

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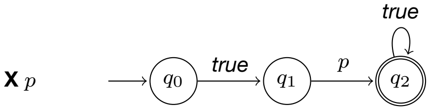

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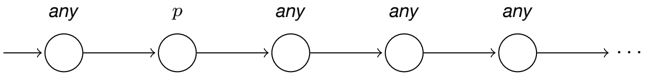

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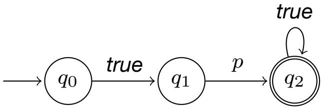

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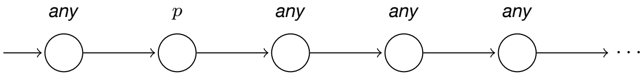

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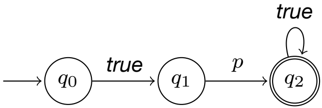

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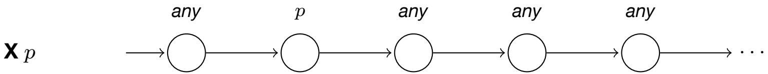

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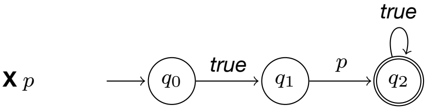

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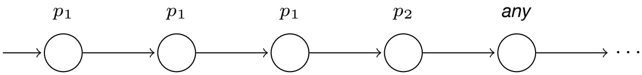

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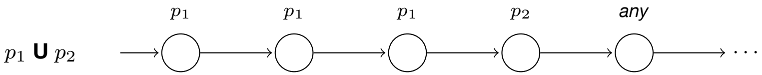

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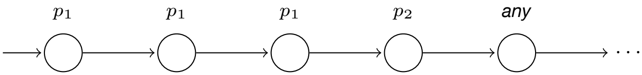

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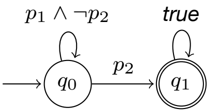

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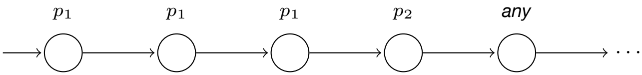

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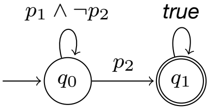

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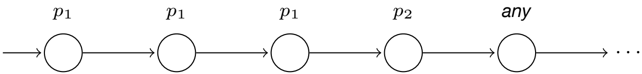

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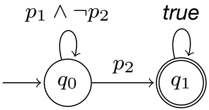

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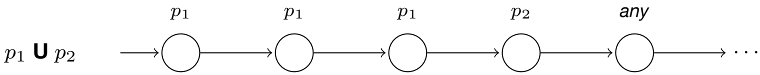

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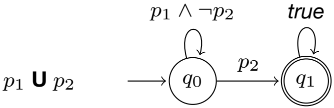

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- LTL to NBA produces A_ϕ of size $2^{|\phi|}$

Summary: Automata-Based LTL Model Checking

Given a Kripke structure M and LTL ϕ:

1. Convert M into Buchi automaton A, ϕ into A_ϕ
2. Negate ϕ by building complement $\overline{A_\phi}$
   - **Note**: Complement can blow up exponentially!
   - In practice, negate ϕ before building NBA
3. Check emptiness of $L(A \cap \overline{A_\phi})$
4. If not empty, return a word (prefix) $w \in L(A \cap \overline{A_\phi})$

Worst case complexity: $O(|M| \cdot 2^{|\phi|})$

- Intersection $A_1 \cap A_2$ produces automaton of size $|A_1| \cdot |A_2|$
- LTL to NBA produces A_ϕ of size $2^{|\phi|}$
- Emptiness check is depth-first search – linear time

On-the-fly model checking

The expensive part of this algorithm is in constructing $A \cap \overline{A_\phi}$

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1. Construct property automaton A_ϕ first

On-the-fly model checking

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In practice, the search can proceed with the construction

1. Construct property automaton A_ϕ first
2. Begin taking intersection at initial states of A

On-the-fly model checking

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In practice, the search can proceed with the construction

1. Construct property automaton A_ϕ first
2. Begin taking intersection at initial states of A
3. Perform DFS incrementally at each step

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4. Whenever DFS needs a state that hasn't been built, add it

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This works because bugs are often easy to find – software is buggy!

Next Lecture

- Symbolic model checking
- If time: introduce a model-checking tool