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A LOWER BOUND FOR LEBESGUE'S UNIVERSAL COVER PROBLEM

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ABSTRACT

In the following we show that any convex set that contains a congruent copy of any set of diameter one (universal cover) has area at least 0.832. This considerably improves the lower bound for Lebesgue's universal cover problem, using a combination of computer search and geometric bounds.

Keywords: Universal cover; Lebesgue's universal cover problem.

1. Introduction and Result

The universal cover problem was first stated 1914 in a personal communication by Lebesgue to Pál[1]; Lebesgue asked for the minimum area of a convex set U in the plane such that for each set C of diameter 1 there is a congruent copy C' contained in U. So U is a universal cover for the family of sets of diameter 1, under congruence, and we wish to determine the minimum area of a convex set with that property.

This became the prototype for a large family of problems, with the parameters

- the family of sets to be covered,
- the sets allowed as covers (convex or nonconvex, special types of sets),
- the size measure to be minimized (area, perimeter, diameter, etc.),
- the allowed transformations (congruence or translation).

The other well-known problem of this type is Moser's worm problem, which asks for a convex minimum-area universal cover for curves of length one; but many other variants have been studied, see Brass, Moser, Pach,[2] chapter 11.4, for a survey, and Wetzel[3] for an update on the worm problem. In this note we will study only Lebesgue's classical version.

An easy example of a universal cover for sets of diameter 1 is the circle of radius $\frac{1}{\sqrt{3}}$; Jung[4] proved that the smallest ball that contains all sets of diameter 1 is the ball circumscribed to the equilateral simplex of diameter 1 (a different proof for the planar case was also given by Jung[5]). This circle has area $\frac{\pi}{3} \approx 1.047$. The unit square is a smaller universal cover, and it is also a universal cover even under translation.

Pál constructed a sequence of better and better universal covers in his paper,[1] culminating in his truncated hexagon, a regular hexagon circumscribed to the unit circle, with two corners cut off (see Fig. 1); this universal cover has the area 0.8454. Further improvements on this cover were made by cutting off very small pieces of a corner by Sprague,[6] and Hansen[8,9,10]; Duff[7] showed that the convexity of the cover is an important restriction by constructing a significantly smaller nonconvex universal cover; and Eggleston[11] studied universal covers minimal under set inclusion, and observed that the set obtained as union of a Reuleaux triangle of diameter 1 and a circle of diameter 1, when the triangle vertices are antipodal points of the circle, is a universal cover. But all progress was small, and after Sprague[6] almost infinitesimal, the papers aimed only to show that each previous cover could still be improved. Sprague did not even compute the area of his cover, this was done by Meschkowski,[12] who popularized the problem by inclusion in one of his very successful books for mathematical amateurs. The smallest currently known universal cover has area 0.844.

As lower bound, Pál[1] observed that any set that contains congruent copies of all sets of diameter 1 must contain at least congruent copies of the circle and equilateral triangle of diameter 1; if the set is additionally convex, the area is at least the minimum area of the convex hull of a circle and a triangle of diameter 1. Pál shows that this minimum is reached when circle and triangle are concentric; that set has area $\frac{\pi}{8} + \frac{\sqrt{3}}{4} \approx 0.8257$. This lower bound could be improved if one could add further sets of diameter 1 to this family, for which the area of the convex hull is minimized. This was already observed by Pál, but he found unsurmountable

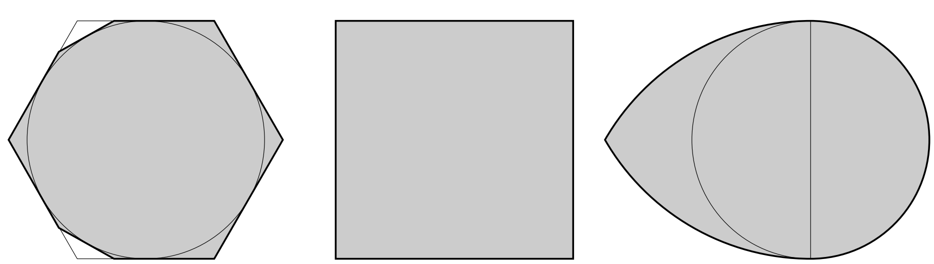

Fig. 1. Pál's universal cover, two other universal covers for the same diameter.

difficulties in extending his method from two sets (disc and triangle) to three sets. This step was finally taken by Elekes,[13] more than seventy years later, when Elekes showed that the smallest convex hull of a circle, and all regular 3^i-gons, all of diameter 1, is reached if all these sets are concentric and equally aligned; this raised the lower bound to ≈ 0.8271.

The improvement was comparatively small since the next set included in this sample, the regular 9-gon, is already very near a circle, and the improvement decreases fast with the number of vertices. It would have been much more efficient if one could have taken circle, equilateral triangle, and regular fivegon, of diameter 1; but the analytic methods do not extend to this situation. It is the aim of this paper to use instead computational methods to bound the minimum area of the convex hull of a circle, triangle, and fivegon, as a lower bound for the minimum area of a universal cover for sets of diameter 1.

Theorem 1. *A convex set in the plane that contains a congruent copy of each set of diameter one has area at least 0.832.*

Figure 2 shows the placement of triangle, fivegon and circle that gives the smallest convex hull we know of. It appears quite irregular. Part of this irregularity is due to sampling, we did not make any search for the best placement, but obtained this only as the best among the centers of boxes covering the space of placements in our lower bound computation. The only relevance of this example is to bound the potential for improvement of our lower bound which could be reached by considering the same sets; the set in figure 2 is *not* itself a universal cover. But it does show that in the optimal placement, the sets are not concentric, which was crucial for the proofs by Pál[1] and Elekes.[13] This suggests that the analytic methods for finding the minimizing position are not applicable anymore.

The search space of possible placements of the three sets is five-dimensional (the circle is fixed, the triangle might be rotated to be axis-aligned, only the five-gon has three degrees of freedom); adding another set would raise the dimension of the search space to eight and make our approach again infeasible.

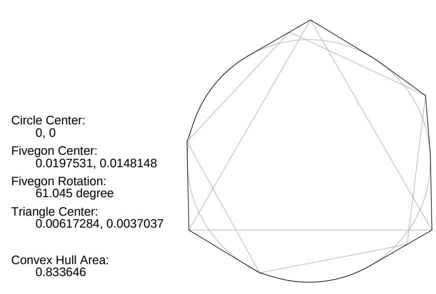

Fig. 2. The smallest known convex hull of circle, triangle and fivegon.

2. Proof of the Theorem

To prove the theorem, we wish to show that the convex hull of a circle of diameter one, an equilateral triangle of diameter one, and a regular fivegon of diameter one has always area at least 0.8323. So we want to show that for any choice of three rigid motions μ_1, μ_2, μ_3 applied to the sets $\bigcirc$, $\triangle$, $⬟$, holds

$$\text{area}\Big(\text{conv}\big(\mu_1(\bigcirc) \cup \mu_2(\triangle) \cup \mu_3(⬟)\big)\Big) \geq 0.832.$$

Each rigid motion is described by three parameters, but without loss of generality we can assume that the circle is centered at $(0, 0)$, and the triangle is oriented with one side axis-parallel, so the remaining parameters are just the translation of the triangle, and translation and rotation of the fivegon. Thus we have a function

$$f_\rho(x_3, y_3, x_5, y_5) = \text{area}\Big(\text{conv}\big(\bigcirc \cup (\triangle + (x_3, y_3)) \cup (\text{rot}(⬟, \rho) + (x_5, y_5))\big)\Big),$$

which we want to bound from below.

We first observe that there is an easy Lipschitz bound for this function. If X is the set obtained as convex hull of one placement, and $\tilde{X}$ is a convex hull for a different placement, where each point of the circle, triangle, and fivegon generating $\tilde{X}$ has distance at most δ to the corresponding point of the circle, triangle, and fivegon generating X, then $\tilde{X}$ is contained in the outer parallel body of X at distance δ. So the area can increase by at most $\delta\, \text{peri}(X) + \pi\delta^2$. Any relevant set X is certainly contained in a disc of radius $\frac{3}{2}$, since the triangle and the fivegon will at least have a common point with the circle; since the perimeter of convex sets is a monotone function under set inclusion, we have $\text{peri}(X) \leq 3\pi$. This bound would be sufficient, if we could subdivide the search region fine enough, but since the parameter space is five-dimensional this turns out to lead to an unfeasibly large number of cases.

We do use the bound, however, to discretize the rotations. It is sufficient to prove a slightly stronger lower bound for all the fixed rotation angles $\frac{i}{36000}\pi$ (multiples of $0.005°$) for $i = 0 \ldots, 14400$. If the claimed inequality holds for each of these fixed-rotation cases for all translations, then the correct angle for the extremal case differs from the nearest of our discretized angles by at most $\frac{1}{72000}\pi$. By this rotation, the vertices of the fivegon would move at most $\frac{1}{72000}\pi\frac{1}{2\sin\frac{2}{5}\pi} \approx 0.000023$; so by considering only our discretized angles, we overestimate the area by at most $3\pi 0.000023 + \pi(0.000023)^2 < 0.00022$.

The main step is to prove the bound in the pure translative case, for a fixed rotation angle. We observe that $f_\rho(x_3, y_3, x_5, y_5) > 0.83222$ is certainly satisfied if one of the translation components x_3, y_3, x_5, y_5 is larger than 0.19. For if the triangle is translated by more than 0.19 in some direction, then the vertex of the triangle with the smallest angle to the translation direction (at most $\frac{\pi}{3}$) will be moved to a distance of at least 0.692192 from the center of the circle, and the convex hull of that vertex and the circle already has area at least 0.8338. In the same way, if the center of the fivegon is moved by more than 0.19, then the fivegon vertex with the smallest angle to the translation direction (at most $\frac{\pi}{5}$) will be moved to a distance of at least 0.68856 from the center of the circle, and the convex hull of that vertex and the circle has already area at least 0.8325.

So we can restrict ourselves to the cube $[-0.19, 0.19]^4$ as possible values for the translations. To bound the function over this region, we divide it into cells, which are small cubes for the translations. Then we evaluate the function in the center of the cell, and compute a lower bound for the function in that cell. If that lower bound is larger than 0.83222, we have proved the required lower bound on the function for that cell. Else we subdivide the cell. The key to the applicability of the method is that lower bound.

We construct a bound similar to a first derivative plus error terms, which has the advantage of being a good bound near the minimum. We will drop the fixed rotation angle from our notation. The bound for each search cube with center (x_3, y_3, x_5, y_5) and edgelength 2δ has the form

$$
\begin{aligned}
f(x_3 + \delta_{x3}, y_3 + \delta_{y3}, x_5 + \delta_{x5}, y_5 + \delta_{y5}) \\
&\geq f(x_3, y_3, x_5, y_5) + c_{x3}\delta_{x3} + c_{y3}\delta_{y3} + c_{x5}\delta_{x5} + c_{y5}\delta_{y5} - \gamma \\
&\geq f(x_3, y_3, x_5, y_5) - \delta\Big(|c_{x3}| + |c_{y3}| + |c_{x3}| + |c_{y3}|\Big) - \gamma,
\end{aligned}
$$

so for each cell we have to construct the set

$$X = \mathrm{conv}\big(\text{⬤} \cup (\text{▲} + (x_3, y_3)) \cup (\mathrm{rot}(\text{⬟}, \rho) + (x_5, y_5))\big),$$

compute the numbers $c_{x3}, c_{y3}, c_{x5}, c_{y5}, \gamma$ and the area $\mathrm{area}(X) = f(x_3, y_3, x_5, y_5)$, and then check whether

$$f(x_3, y_3, x_5, y_5) - \delta\Big(|c_{x3}| + |c_{y3}| + |c_{x3}| + |c_{y3}|\Big) - \gamma \geq 0.83222;$$

if the inequality is satisfied, then our bound is proved for that cell, otherwise we subdivide it, cutting each coordinate interval in thirds.

We now have to define the numbers $c_{x3}, c_{y3}, c_{x5}, c_{y5}, \gamma$. For this, consider the boundary of the set for the center (x_3, y_3, x_5, y_5). The boundary is a sequence of segments and circular arcs, these are separated by points that are vertices of the

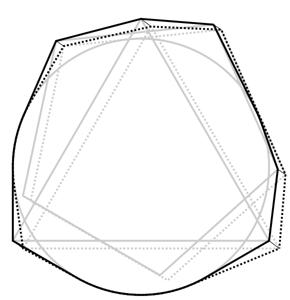

Fig. 3. A configuration and a change by a small translation.

triangle, and of the fivegon, and points on the circle that are starting point of a tangent to the circle through a triangle or fivegon vertex. Consider now the set with the same boundary structure, but where the fivegon and triangle vertices are replaced by vertices of the triangle at $(x_3 + \delta_{x3}, y_3 + \delta_{y3})$ and the fivegon at $(x_5 + \delta_{x5}, y_5 + \delta_{y5})$. The area of this set is a lower bound for $f(x_3 + \delta_{x3}, y_3 + \delta_{y3}, x_5 + \delta_{x5}, y_5 + \delta_{y5})$, since it is contained in the convex hull of those translates of circle, triangle and fivegon. The change of area of this set, in dependence of $(\delta_{x3}, \delta_{y3}, \delta_{x5}, \delta_{y5})$ is easy to bound, since the set changes only at its boundary, but that has the same combinatorial structure. We compute the change for each part of the boundary, and sum these contributions.

The boundary of the convex hull consists of the following parts:

- edges between two triangle vertices (3-3 edges)
- edges between two fivegon vertices (5-5 edges)
- edges between a triangle and a fivegon vertex (3-5 edges)
- tangents from a triangle vertex to the circle (3-t-edges)
- tangents from a fivegon vertex to the circle (5-t-edges)
- circular arcs

There is no change at the circular arcs, since we do not change the tangent points. For 3-3-edges, 5-5-edges, 3-t-edges, and 5-t-edges the change in area is actually a linear function of $(\delta_{x3}, \delta_{y3}, \delta_{x5}, \delta_{y5})$. If pq is a 3-3-edge, with $p = (p_x, p_y), q = (q_x, q_y)$, then the area change along that edge is $\delta_{x3}(q_y - p_y) + \delta_{y3}(q_x - p_x)$ If pt is a 3-t-edge, with $p = (p_x, p_y), t = (t_x, t_y)$, then the area change along that edge is $\frac{1}{2}\delta_{x3}(t_y - p_y) + \frac{1}{2}\delta_{y3}(t_x - p_x)$. The other cases are similar.

Only the change of area at a 3-5 edge is not linear. Let pq be a 3-5 edge, and $p' = p + (\delta_{x3}, \delta_{y3})$, $q' = q + (\delta_{x5}, \delta_{y5})$. Then the change of area is the area of the fourgon $pqq'p'$, which we can write as the union of the triangles pqq' and $pq'p'$. The area of pqq' is again a linear function in δ_{x5}, δ_{y5}. The area of $pq'p'$ differs from the area of pqp' by at most δ^2, since the common side pp' has length at most $\sqrt{2}\delta$, and the other endpoints q, q' also have distance at most $\sqrt{2}\delta$. The area of pqp' is a linear function in δ_{x3}, δ_{y3}. So we compute for each side its contribution to the coefficients $c_{x3}, c_{y3}, c_{x5}, c_{y5}$, and for 3-5-sides we also add δ^2 to γ.

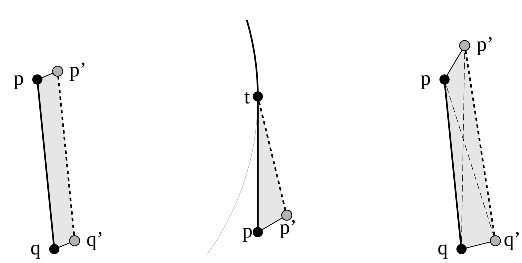

Fig. 4. Change at 3-3 side, 3-t side and 3-5 side.

This completes the construction of the error bound for the cube with center (x_3, y_3, x_5, y_5) and edgelength 2δ . Applying this bound, starting with center $(0,0,0,0)$ and $\delta = 0.19$, and refining whenever the inequality was not satisfied, we proved the claimed inequality. Our program did this for 14400 different orientations of the fivegon; the easiest did not require any subdivision, the most difficult evaluated 5671 cubes. In total 53118162 cubes were evaluated, of these 655602 required subdivision.

3. Related Problems

This problem is just one of a large family of universal cover problems, obtained by varying the parameters mentioned in the introduction. Many of these questions have actually been studied, and this technique could be used for a lower bound in each of them. It just needs a sufficiently strong local lower bound; but that bound has to be quite quite strong, since we work in the 10^{-4} error region, and at that resolution a five-dimensional parameter space is already very large. The technique is only reasonable if the extremal set is quite irregular, so there is no direct attack on that problem. For the translative analogue of our problem, e.g., there is a good conjecture on the extremal set.[14]

One could use a much larger family of test sets if one could determine the minimum for the translations directly; then the recursive subdivision of the parameter space as used here would be unnecessary. It would be interesting to find an algorithm for the following problem: given polygons $P_1, \ldots, P_k$ with a total of n vertices, find the translations $t_1, \ldots, t_k$ that minimize

$$\operatorname{area}\Big(\operatorname{conv}\Big((P_1 + t_1) \cup \cdots \cup (P_k + t_k)\Big)\Big).$$

It was observed by G. Rote that this area function in the translations is convex for $k = 2$, but not for $k \geq 3$; if $P_1 = P_2 = P_3$ just consists of one point, then for $(t_1, t_2, t_3) = \big((0,0),(1,0),(0,0)\big)$ one has area 0, and the same for $\big((0,0),(0,0),(0,1)\big)$, but for $\big((0,0),(\frac{1}{2},0),(0,\frac{1}{2})\big)$ the area is $\frac{1}{8}$. So the set of minimizing translations need not have a nice structure.

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