Thesis Proposal: Hybrid Resource-Bound Analyses of Programs

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Abstract

Resource-bound analysis aims to infer symbolic bounds of worst-case resource usage (e.g., running time, memory, and energy) of programs as functions of program inputs. Resource analysis has numerous applications, including job scheduling in cloud computing and prevention of side-channel attacks. Various resource analysis technique have been developed, and they have unique strengths and weaknesses that complement each other. (Automatic) static resource analysis, which analyzes the source code of programs, is *sound*: if it successfully infers a cost bound, it is guaranteed to be a valid bound. However, due to undecidability of resource analysis in general, every static analysis technique is *incomplete*: there exists a program that the analysis technique cannot handle. Meanwhile, data-driven analysis, which statistically analyzes cost measurements obtained by running programs on many inputs, can infer a candidate cost bound for any program. However, it does not guarantee soundness of inference results.

To overcome limitations of individual analysis techniques, I propose *hybrid resource analysis*, which integrates two complementary analysis techniques to retain their strengths while mitigating their respective weaknesses. The user first specifies which analysis techniques are used to analyze which code fragments and quantities. Hybrid analysis then performs its constituent analysis techniques on their respective code fragments and quantities. Finally, their inference results are combined into an overall cost bound.

The development of hybrid resource analysis has been driven by the desire to go beyond Automatic Amortized Resource Analysis (AARA), a state-of-the-art type-based static resource analysis technique. I start by proving polynomial-time completeness of AARA. I next introduce Bayesian data-driven analysis, which conducts Bayesian inference on cost measurements to infer a posterior distribution of symbolic cost bounds. I then present the first hybrid resource analysis, *Hybrid AARA*, followed by a discussion of its limitations. To overcome these limitations, I introduce the second hybrid resource analysis, *resource decomposition*. I additionally describe Swiftlet, which instantiates the resource-decomposition framework with AARA and Bayesian resource analysis.

For proposed work, my collaborators and I plan to develop data-driven-analysis for statistically inferring not only a worst-case *symbolic cost bound* but also a worst-case *input generator*, which is a program generating worst-case program inputs of various sizes. In existing data-driven analyses, program inputs used for recording cost measurements are usually either generated randomly or assumed to be representative of real-world workload. Consequently, it is challenging to statistically infer worst-case bounds of those programs (e.g. quicksort) whose average-case complexity is significantly lower than the worst-case complexity. By testing programs with various input generators and inferring worst-case input generators, we can improve the inference quality of data-driven resource analysis.

Contents

1 Introduction

In this section, I first introduce resource-bound analysis of programs and review resource analysis techniques studied in the literature. I then propose hybrid resource analysis, which integrates two complementary resource analysis techniques to retain their strengths while mitigating their respective weaknesses. Lastly, I outline three completed works and a proposed work for the thesis.

1.1 Resource Analysis

Given a program P, resource-bound analysis aims to infer a symbolic bound $f(x)$ on the worst-case resource usage (e.g., running time, memory, and energy) of the program P as a function of a program input x. The symbolic bound $f(x)$ must be a valid upper bound on the computational cost of executing $P(x)$ for any input x. Hence, the bound $f(x)$ inferred by resource analysis is more precise than the result of asymptotic complexity analysis, which only concerns sufficiently large inputs and disregards constant factors.

Resource analysis of programs has a number of applications. For example, in cloud computing [49, 75, 78], a cloud-service provider seeks to avoid over-provisioning resources, which would waste resources and hence reduce profits, and under-provisioning resources, which could violate service-level agreements. To this end, the cloud-service provider can perform resource analysis to infer cost bounds of the clients' programs. Other applications of resource analysis include worst-case input generation to identify potential algorithmic complexity attacks [13, 56, 59, 69, 83, 87], ensuring constant resource usage to prevent side-channel attacks [9, 14, 67], and detecting performance bugs for programmers [21, 27].

Three approaches to resource analysis exist in the literature: (automatic) static analysis, data-driven analysis, and interactive analysis. *Static resource analysis* examines the source code of a program and reasons about all possible behaviors of the program, including its worst-case behaviors, to automatically infer a cost bound. Since the pioneering work of Wegbreit [85], numerous static resource analysis techniques have been developed: type systems [6, 20, 23, 25, 36, 55, 82], recurrence relations [1–3, 10, 24, 31, 37, 51, 52, 58, 76, 77], term rewriting [7, 8, 46, 66], ranking functions [11, 12, 19, 28, 29, 79], and invariant generation [34, 35, 89].

Data-driven resource analysis first runs a program on many inputs of varying sizes and records execution costs. It then analyzes the dataset of cost measurements to statistically infer a cost bound. To collect cost measurements, most existing works [22, 26, 30, 48, 50, 73, 74, 88] use randomly generated program inputs or representative workloads, which do not necessarily reveal worst-case behaviors of the program. Also, to statistically infer bounds from cost measurements, these works perform optimization (e.g., polynomial regression) without quantifying statistical uncertainty or incorporating the user's domain knowledge into the statistical model.

In *interactive resource analysis*, the user manually writes formal proofs of cost bounds, and the proofs are machine-checked for correctness. Existing works include those based on separation logic [4, 18, 33, 63, 65] and dependent types [5, 25, 32, 68].

Static and data-driven analyses have their own strengths and weaknesses that complement each other. Static resource analysis is *sound*: whenever it successfully returns an inference result, it is guaranteed to be a valid worst-case cost bound of the program. However, static analysis is *incomplete*: for any technique, there exists a program whose cost bound cannot be automatically inferred even if the bound is expressible in the language of symbolic bounds supported by the technique. The incompleteness is due to the undecidability of resource analysis for a Turing-complete programming language.

In contrast to static analysis, data-driven analysis can infer a candidate cost bound for any program. Another advantage is that data-driven analysis only needs a black-box access to the source code, making the analysis applicable to third-party programs whose source code is not publicly available. However, data-driven analysis does not guarantee soundness of inferred cost bounds, because the analysis does not rigorously reason about worst-case behaviors of the program. Also, a finite dataset of cost measurements used in data-driven analysis may not contain worst-case inputs, making it challenging to infer the true worst-case costs.

1.2 Hybrid Resource Analysis

In this thesis, to overcome the limitations of individual analysis techniques, I propose and develop *hybrid resource analysis*, which integrates two (or more) resource analysis techniques with complementary strengths and weaknesses. In hybrid resource analysis, the user first specifies which techniques should analyze which code fragments and quantities in the source code. Next, hybrid analysis performs the two constituent techniques on their designated code fragments and quantities. Finally, the two inference results are combined into an overall cost bound of the entire program. By integrating two complementary analysis techniques, hybrid resource analysis retains their strengths while mitigating their respective weaknesses.

The primary technical challenge of hybrid analysis lies in the design of the interface between the two constituent analysis techniques. The interface specifies (i) representations of cost bounds inferred by the two analyses and (ii) what information (if any) is exchanged between the two analyses during their inference of cost bounds. Firstly, the cost bounds inferred by the two analyses must have compatible representations such that they can be composed together to yield an overall cost bound. Secondly, some resource analysis techniques impose numerical constraints on cost bounds to define a set of accepted bounds. Consequently, to successfully compose such a cost bound with another cost bound inferred by a different analysis, the latter bound must satisfy the numeric constraints imposed by the former bound. Therefore, the two analyses must take into account each other's numeric constraints on bounds.

To illustrate the benefit of hybrid resource analysis in a real-world use case, let us consider a cloud-service provider who wishes to estimate resource usage of a client's program to optimize resource allocation on the cloud. One reasonable choice of resource analysis techniques is static resource analysis as it offers soundness guarantees. The soundness guarantees are beneficial to the service provider because they can rest assured that they will never accidentally under-provision resources. Without the soundness guarantees, it is possible that the client's program consumes more resources than anticipated. In such a case, the provider may need to rerun the program from scratch with more resources, violating service-level agreements on timely execution of the program. However, any single static analysis technique cannot be used to automatically infer cost bounds of all programs due to the incompleteness of static resource analysis. If a static analysis technique of the provider's choice fails to infer a cost bound for a program, the cloud-service provider is left with no clues to guide the resource allocation and scheduling for the program.

Data-driven resource analysis, on the other hand, can always infer a candidate cost bound for any program from its finitely many cost measurements. These measurements are often readily available, especially when the same program is repeatedly executed on many inputs (e.g., the serverless cloud service AWS Lambda). However, data-driven analysis provides no soundness guarantees of inferred cost bounds. Even if the statistical model adds an extra buffer on top of maximum observed costs in the dataset, it may still fail to yield a sufficiently conservative cost bound desired by the cloud-service provider.

Hybrid resource analysis lets the cloud-service provider integrate static and data-driven analyses, thereby striking a desirable balance between soundness (achieved by static analysis) and completeness (achieved by data-driven analysis). For example, the provider can apply static analysis to all code fragments that are amenable to static analysis, and data-driven analysis to the rest of the source code. The two cost bounds by static and data-driven analyses are then combined into an overall bound. Oftentimes, even if static analysis fails to analyze the entire program, it is still capable of analyzing a non-trivial amount of code fragments. So it makes sense to apply static analysis wherever possible in the source code, retaining its soundness guarantees as much as possible. Meanwhile, data-driven analysis yields reasonable (but not necessarily sound) cost bounds for those code fragments that cannot be handled by static analysis. Even though cost bounds inferred by data-driven analysis are not guaranteed to be sound, they are sensible inference results derived using mathematically principled methods (e.g., Bayesian statistics) from observed cost measurements and a statistical model incorporating the user's domain knowledge. Thus, data-driven resource analysis is no less useful than, for example, weather forecasting from observed data and a scientific model, where forecasts never come with guarantees but are nonetheless helpful in our lives. As I empirically demonstrate in this thesis, hybrid resource analysis returns more accurate cost bounds (i.e., the inferred cost bounds are closer to the ground-truth bound) than purely data-driven analysis, thanks to the integration of static analysis. In summary, hybrid resource analysis can infer cost bounds for program that purely static analysis cannot handle, while obtaining more accurate bounds than purely data-driven analysis.

The thesis statement is therefore:

Thesis Statement *Hybrid resource analysis, which integrates two resource analysis techniques with complementary strengths and weaknesses, enables the analysis of programs and inference of cost bounds that are beyond the reach of individual analysis techniques.*

1.3 Roadmap of the Thesis

AARA For the static-analysis part of hybrid resource analysis, I focus on Automatic Amortized Resource Analysis (AARA) [39, 41, 43, 45], which is a type-based static resource analysis technique that automatically infers polynomial cost bounds of functional programs. The thesis starts by introducing background information on AARA (§2.1).

AARA is sound: whenever it returns a polynomial cost bound, the bound is guaranteed to be sound. However, AARA is incomplete: there exists a (terminating) polynomial-cost program for which AARA cannot infer any polynomial cost bound. The incompleteness is not unique to AARA. Every static analysis technique suffers incompleteness because resource analysis is undecidable for a Turing-complete programming language in general.

Polynomial-time completeness of AARA In the first completed work (§2.2), I prove that the typable fragment of AARA is *polynomial-time complete* [70]: for every polynomial-time function f, there exists a polynomial-cost program P that (i) simulates the function f with the same computational cost and (ii) is typable in AARA (i.e., a polynomial cost bound of the program P can be inferred by AARA). The proof creates a functional program with a (polynomial-length) list, where an element is removed from the list whenever the polynomial-time Turing machine of the function f consumes one unit of time. This idea of adding an extra program variable to represent a certain symbolic bound will later be exploited in the development of the second hybrid resource analysis.

Hybrid AARA I describe the first hybrid resource analysis, *Hybrid AARA* [72], which integrates AARA and data-driven analysis. For the data-driven-analysis part of Hybrid AARA, my collaborators and I have developed two Bayesian data-driven analyses (§2.3). They perform Bayesian inference to infer posterior distributions of cost bounds from observed cost measurements according to a user-customizable probabilistic model. Hybrid AARA (§2.4) integrates AARA with data-driven analyses, including the two newly developed Bayesian resource analyses as well as an optimization-based baseline from the literature. The primary technical challenge is the design of an interface between AARA, which runs an optimization algorithm (i.e., linear-program solver) to infer cost bounds, and data-driven analysis, which runs a sampling-based probabilistic inference algorithm to infer cost bounds. I propose novel interface designs to overcome the challenge.

Because Hybrid AARA reuses types from AARA to capture polynomial potential functions assigned to program variables, Hybrid AARA inherits two limitations from AARA. Firstly, Hybrid AARA cannot apply two constituent resource analyses to infer quantities of different resource metrics. For instance, given a recursive function, its total cost is given by the product of (i) the cost (e.g., running time and memory) of a single recursive call and (ii) the number of recursive calls. However, because these two quantities have different units (i.e., resource metrics), Hybrid AARA cannot perform resource analyses on the two quantities separately and then combine their inferred bounds. Secondly, due to the use of polynomial potential functions in Hybrid AARA, it cannot express non-polynomial cost bounds (e.g., $O(n \log n)$ for merge sort).

Resource decomposition To address the limitations of Hybrid AARA, my collaborators and I have designed the second hybrid-resource analysis, *resource decomposition* [71] (§2.5), which integrates two complementary analyses using a different interface from Hybrid AARA. The key idea of the resource-decomposition framework is to extend an input program with an additional numeric input variable called a *resource guard* r. The resource guard r captures a user-specified quantity, such as the cost and recursion depth of a code fragment. We perform some resource analysis technique (e.g., data-driven analysis) to infer a symbolic bound $g(x)$ of the resource guard r parametric in a program input x. Next, we run a different resource analysis technique (e.g., AARA) to infer an overall cost bound $f(x, r)$ of the entire program parametric in both the original program input x and the newly inserted program input r. Finally, we substitute the symbolic bound $g(x)$ for the resource guard r in the overall cost bound $f(x, r)$, obtaining a bound $f(x, g(x))$ parametric only in the original input x. In addition to formalizing resource decomposition and proving its soundness theorem, the thesis presents a concrete instantiation of the resource-decomposition framework, called Swiftlet (§2.6), integrates AARA and Bayesian data-driven analysis.

Inference of Program Input Generators In existing data-driven analyses, program inputs used for recording cost measurements are either generated randomly or assumed to be representative of real-world workload. Either case, the user has no control over the process of generating program inputs (and their associated cost measurements)—the dataset of program inputs and cost measurements is fixed before data-driven analysis. It is challenging to statistically infer worst-case cost bounds from a fixed dataset of cost measurements, especially when the dataset rarely contains worst-case inputs. For example, if we use a random input generator in data-driven resource analysis of QuickSort, it is difficult to correctly infer a worst-case $O(n^2)$ cost bound from a dataset where cost measurements highly concentrate around the average-case $O(n \log n)$ complexity.

For proposed work, my collaborators and I plan to develop a data-driven-resource-analysis methodology that statistically infers not only a worst-case *symbolic cost bound* but also a worst-case *program input generator* (§2.7). A program input generator is a (domain-specific) program that generates program inputs of various sizes conforming to a certain pattern (e.g., sorted lists and balanced trees). In the proposed methodology, the user has control over the data-collection process: they are enabled to run an input program with various input generators, instead of working with a fixed input generator. By testing various input generators, we are more likely able to find a more desirable input generator that yields worse program inputs than a random input generator, hence inferring a more accurate worst-case cost bound.

1.4 Publications

This thesis proposal is based on the following three publications:

1. Long Pham and Jan Hoffmann. Typable Fragments of Polynomial Automatic Amortized Resource Analysis [70]. Published at CSL 2021.
2. Long Pham, Feras A. Saad, and Jan Hoffmann. Robust Resource Bounds with Static Analysis and Bayesian Inference [72]. Published at PLDI 2024.
3. Long Pham, Yue Niu, Nathan Glover, Feras A. Saad, and Jan Hoffmann. Integrating Resource Analyses via Resource Decomposition [71]. Under submission.

The first paper proves polynomial-time completeness of AARA. The second paper presents Bayesian data-driven analysis and the first hybrid resource analysis, Hybrid AARA. The third paper (under submission) presents the second hybrid resource analysis, resource decomposition, and its instantiation, Swiftlet, which integrates AARA and Bayesian data-driven analysis.

Yue Niu and I collaborated on the theoretical foundation of resource decomposition. I first formulated resource decomposition and proved its soundness in an operational semantics of a first-order language. Yue Niu then recast the formulation in a domain-theoretic denotational semantics of a higher-order language, proving the soundness using a logical relation.

2 Overview

This section first introduces the static resource analysis that I focus on throughout the thesis. It then describes three completed works and a proposed work for the thesis.

2.1 Background

Resource analysis Given a program P, the goal of resource analysis is to infer a symbolic bound $f(x)$ on the worst-case resource usage of the program P as a function of a program input x. For every input x, the predicted cost $f(x)$ must be a valid upper bound of the computational cost of executing $P(x)$.

Resource analysis generally supports a wide range of resource metrics, such as running time, memory, and energy. To specify a resource metric of interest, the user (manually or automatically) inserts a program construct tick q for $q \in \mathbb{Q}$ throughout the source code. The construct tick q increments a cost counter by $q \in \mathbb{Q}$ and returns the unit element $\langle \rangle$. If $q \geq 0$, it means q units of resources are consumed. Otherwise, if $q < 0$, it means $|q|$ units of resources are freed up (and become available to be reused later).

If all tick q satisfy $q \geq 0$, the resource metric is said to be *monotone*. Running time of programs is a monotone resource metrics because we cannot reuse time. By contrast, memory usage is a non-monotone resource metric because memory can be freed up as well as consumed. In non-monotone resource metrics, we have two notions of costs: *net costs* and *high-water-mark costs*. The net cost refers to the net amount of resources consumed after program execution, while the high-water-mark cost refers to the maximum cost at any point during program execution. In monotone resource metrics, these two notions of costs coincide.

Programming language Throughout this thesis, I consider a call-by-value functional programming language. A program $\mathcal{P}$ is a finite set of (mutually recursive) function definitions of the form $P(x) = e$, where P is a function name, x is the function input, and e is the function body that is allowed to mention function names (including P itself) defined in the program $\mathcal{P}$. For resource analysis, the user specifies which function $(P(x) = e) \in \mathcal{P}$ is to be analyzed. Given a program $\mathcal{P}$ (i.e., a finite set of function definitions), the cost semantics of an expression e under an environment V (i.e., a mapping from variables to values) is given by a judgment

$$V \vdash_{\mathcal{P}} e \Downarrow^{h,r} v, \tag{2.1}$$

where v is the output value, h is the high-water-mark cost, and r is the remaining potential. The net cost is given by $h - r$.

AARA For the static-analysis part of hybrid resource analysis, I focus on a state-of-the-art static analysis technique, Automatic Amortized Resource Analysis (AARA) [39, 41, 43, 45] and its implementation Resource-Aware ML (RaML) [42, 44] for OCaml programs. AARA is a type-based technique that automatically infers polynomial cost bounds. AARA automates the potential method from amortized analysis of algorithms and data structures by Sleator and Tarjan [80, 81]. Given a functional program, all variables in the program are assigned *polynomial potential function* parametric in the input size such that (i) the potential is always non-negative throughout the program execution and (ii) for every step of computation, the pre-state potential is larger than or equal to the the post-state potential plus the cost of computation. These two conditions ensure that the initial total potential of the programs is a valid upper bound on the total computational cost.

To encode polynomial potential functions, AARA augments standard datatypes from functional programming with polynomial coefficients of potential functions, resulting in *resource-annotated types*. To illustrate the types, consider the `partition` function that partitions an integer list around a pivot. Our goal is to derive a worst-case bound on the number of comparisons during an evaluation of `partition`, namely n, where n is the input list length. In AARA, we type an expression `partition` (p, x), where p is a pivot and x is an input list, as follows:

$$\{p : \texttt{int}, x : L^1(\texttt{int})\}; 0 \vdash \texttt{partition}\,(p, x) : \langle L^0(\texttt{int}) \times L^0(\texttt{int}), 0\rangle. \tag{2.2}$$

The resource-annotated type $L^1(\texttt{int})$ assigns the potential function $\Phi(v : L^1(\texttt{int})) = 1 \cdot |v|$ to an input list v. Also, the annotation 0 in the typing context (i.e., the left-hand side of the turnstile $\vdash$) indicates that 0 additional constant potential is stored in the context. The typing judgment (2.2) states that, if we start with the linear input potential $1 \cdot |x|$, the expression `partition` (p, x) can be successfully evaluated without running out of potential, with zero leftover potential stored in the output of the expression.

AARA is naturally compositional because resource annotated types not only capture computational costs but also implicitly track size changes of data structures. This is achieved by assigning potential functions to the output as well as the input of an expression, where the potential functions are parametric in the output and input sizes, respectively. Assume we have two nested calls to `partition` as in the following function `f`:

```
f(x) = let (x1,x2) = partition (42,x) in partition (1,x1)
```

In the second function call `partition` $(1, x_1)$, we can use the previous type for partition. However, for the first function call `partition` $(42, x)$, we use the typing judgment

$$\{p : \texttt{int}, x : L^2(\texttt{int})\}; 0 \vdash \texttt{partition}\,(p, x) : \langle L^1(\texttt{int}) \times L^1(\texttt{int}), 0\rangle. \tag{2.3}$$

It assigns a resource-annotated type $L^1(\texttt{int})$ to the two output lists such that the they have enough potential to pay for the subsequent computation. Let v, v_1, v_2 be values of variables x, x_1, x_2, respectively. The intuition is that the input potential $\Phi(v : L^2(\texttt{int})) = 2 \cdot |v|$ of the typing judgment (2.3) is used to cover both the cost (i.e., $1 \cdot |v|$) and the potential of the result (i.e., $1 \cdot |v_1| + 1 \cdot |v_2|$). It relies on the fact $|v_1| + |v_2| = |v|$, which AARA's type system implicitly figures out. The potential $\Phi(v_1 : L^1(\texttt{int})) = 1 \cdot |v_1|$ stored in the first output list covers the cost of the second function call `partition` $(1, x_1)$. In general, the resource-annotated type of `partition` (p, x) can be expressed with linear constraints:

$$\{p : \texttt{int}, x : L^{q_1}(\texttt{int})\}; q_0 \vdash \texttt{partition}\,(p, x) : \langle L^{r_1}(\texttt{int}) \times L^{r_2}(\texttt{int}), r_0\rangle \tag{2.4}$$

$$\text{subject to } q_1 \geq 1 + q',\ q' \geq r_1,\ q' \geq r_2,\ q_0 \geq r_0. \tag{2.5}$$

Similar constraints are emitted by the type system of AARA during the type inference. The constraints, which are all linear, are then solved with an off-the-shelf linear-program (LP) solver.

If the linear constraints are solvable, the solution translates to a polynomial potential function, which in turn serves as a polynomial cost bound.

Although the resource-annotated type of the `partition` function only stores linear potential, AARA can also encode polynomial potential functions and therefore polynomial cost bounds while retaining compositionality and type inference with linear constraint solving [40, 43]. In (polynomial) AARA, resource annotations inside resource-annotated types record polynomial coefficients of potential functions. Since it is not necessary to understand the details of how polynomial potential functions are encoded in AARA, I omit the details.

A typing judgment of an expression e in (polynomial) AARA has the form

$$\Gamma; p \vdash e : \langle a, q \rangle, \tag{2.6}$$

where Γ is a resource-annotated typing context (i.e., a mapping from variables to resource-annotated types), $p \in \mathbb{Q}_{\geq 0}$ is constant potential of the context, and a is a resource-annotated type of the output with constant potential $q \in \mathbb{Q}_{\geq 0}$. Let $\Phi(V : \Gamma)$ denote the amount of potential stored in an environment V with a resource-annotated typing context Γ, and likewise $\Phi(v : a)$ denote the amount of potential stored in a value v of resource-annotated type a. The typing judgment (2.6) means, given an environment V that carries potential $p + \Phi(V : \Gamma)$, if the expression e evaluates to a value v, then v carries $q + \Phi(v : a)$ much potential.

The soundness of cost bounds inferred by AARA is formally stated in Thm. 2.1 [40].

Theorem 2.1 (Soundness of AARA)**.** *Given a program $\mathcal{P}$, consider an expression e such that $V \vdash_{\mathcal{P}} e \Downarrow^{h,r} v$. If we have a resource-annotated typing judgment $\Gamma; p \vdash e : \langle a, q \rangle$, then we have $\Phi(V : \Gamma) + p \geq h$ (i.e., the initial potential is a bound of the high-water mark cost) and $\Phi(V : \Gamma) + p - \Phi(v : a) - q \geq h - r$ (i.e., the net potential consumption is a bound on the net cost).*

However, AARA is incomplete: there exists a polynomial-cost program for which AARA cannot infer a polynomial cost bound. This incompleteness, which every static analysis technique suffers, stems from not only the unsupported language features of AARA but also its inadequate reasoning power for complicated recursion patterns.

Bayesian inference Bayesian inference is a paradigm of statistical analysis where a probabilistic model is conditioned on observed data (by Bayes' rule) to derive an inference result. Let θ be a *latent variable* (i.e., a random variable that we wish to infer) and y be an *observed variable* (i.e., a random variable whose values can be observed). The user provides a probabilistic generative model that specifies the joint probability distribution $\pi(\theta, y)$. The probabilistic model encodes the user's domain knowledge. It follows from Bayes' rule that the *posterior distribution* of the latent variable θ given observed data $\mathcal{D}$ is

$$\pi(\theta \mid y = \mathcal{D}) = \frac{\pi(\theta, y = \mathcal{D})}{\pi(y = \mathcal{D})} = \frac{\pi(\theta, y = \mathcal{D})}{\int_\theta \pi(\theta, y = \mathcal{D}) \, \mathrm{d}\theta}. \tag{2.7}$$

The denominator $\int_\theta \pi(\theta, y = \mathcal{D}) \, \mathrm{d}\theta$ is an integral over the space of the latent variable θ, which is usually a high-dimensional space. Hence, it is computationally intractable to compute the exact integral. Instead, in practice, a sampling-based probabilistic inference algorithm (e.g., Hamiltonian Monte Carlo (HMC)) is employed to draw a large number of samples $\theta'_1, \ldots, \theta'_M$ from a Markov chain that is carefully crafted to converge to the target posterior distribution. The posterior samples $\theta'_1, \ldots, \theta'_M$ then serve as an approximation of the posterior distribution.

2.2 Polytime Completeness of AARA (Completed)

Despite the incompleteness of AARA, the typable fragments of AARA is *polynomial-time complete* [70]. That is, for any function $f : \mathbb{N} \to \mathbb{N}$, if it is polynomial-time (i.e., there exists a polynomial-time Turing machine that computes the function f), there is a program that (i) computes the same function f and (ii) is typable in AARA.

Thm. 2.2 formally states the polynomial-time completeness of the typable fragment of AARA.

Theorem 2.2 (Polynomial-time completeness of AARA)**.** *Given a finite alphabet Σ, let M be a polynomial-time one-tape Turing machine that take in and returns bit strings from Σ^*. Then there exists a functional program $P : \Sigma^* \to \Sigma^*$ such that*

- *For every input $w \in \Sigma^*$, we have $M(w) = P(w)$;*
- *The computational cost of the functional program P (according to the tick resource metrics) is larger than or equal to the running time of the Turing machine M;*
- *AARA can infer a resource-annotated type of the functional program P.*

That is, the set of functional programs typable in AARA is complete *with respect to polynomial-time functions.*

To prove Thm. 2.2, given a polynomial-time one-tape Turing machine M, we assume that a polynomial time bound $p(n)$ of the Turing machine M is known, where n is the input size of M (i.e., the length of an input bit string). We translate the Turing machine M to a functional program P that first creates a list $\ell_{\text{potential}}$ whose length is equal to the known time bound $p(n)$. The list $\ell_{\text{potential}}$ acts as a reservoir of potential, storing one unit of potential in each list element. The program P then simulates the Turing machine M, consuming one element of the list every time the simulated Turing machine M moves its tape head. In every consumption of the list elements of $\ell_{\text{potential}}$, the program P runs the expression tick 1, which can be paid by the potential stored in the list. Thus, the program P has the same cost (and also the output) as the Turing machine M. Furthermore, the program P constructs the list $\ell_{\text{potential}}$ of size $p(n)$ in such a way AARA's type system can infer a polynomial potential function for the list. Hence, the program P is typable in AARA. The idea of adding a list to a program that explicitly encodes a known cost bound will later be exploited in the development of hybrid resource analysis (§2.5).

Turing machines Consider a one-tape Turing machine $M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, q_0, q_{\text{final}})$. Here, Q is a finite set of states, Σ is a finite alphabet of input symbols, and $\Gamma \supset \Sigma$ is a finite alphabet of tape symbols (including input symbols). The tape symbol $\vdash \in \Gamma \setminus \Sigma$ is the left end marker, and $\sqcup \in \Gamma \setminus \Sigma$ is the blank symbol on the tape. Finally, $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\text{left}, \text{right}\}$ is the transition function, q_0 is the initial state, and q_{final} is the final state.

In the initial configuration of a Turing machine, an input string $w \in \Sigma^*$ is placed immediately after the left end marker $\vdash$ on the tape. The state of the machine is initially q_0, and the read/write head is positioned on the first symbol of the input string w. The rest of the tape is filled with the blank symbol $\sqcup$.

The Turing machine first (i) reads the content of the cell currently under the tape head and (ii) identifies the current state of the machine. The machine then overwrites the current cell (if necessary), updates the machine's state, and moves the head to the left or right according to the transition function δ. The machine terminates when it enters q_{final}. Upon termination, the

Algorithm 1 Operational working of the target functional program P

```
Require: Input string $w \in \Sigma^*$
 1: procedure $P(w)$
 2:     Create a singleton list $\ell_{\text{left}} := [\vdash]$ and a list $\ell_{\text{right}} := [\sqcup, \ldots, \sqcup]$ of length $p(|w|)$
 3:     Prepend the list $\ell_{\text{right}}$ with the input string $w$
 4:     Create a list $\ell_{\text{potential}}$ of size $p(|w|)$          ▷ Reservoir of potential
 5:     $s \leftarrow q_0$                                              ▷ Initialize the current state
 6:     while $s \neq q_{\text{final}} \land \ell_{\text{potential}} \neq [\,]$ do
 7:         $\ell_{\text{potential}} \leftarrow$ tail $\ell_{\text{potential}}$                    ▷ Potential is released
 8:         Run tick 1 to consume one unit of resources
 9:         Compute $\delta(s, \ell_{\text{right}}[0])$
10:         Update $s$ and $\ell_{\text{right}}[0]$ appropriately
11:         Update the tape head's position by moving the head of $\ell_{\text{left}}$ or $\ell_{\text{right}}$ to the other
12:     return append(reverse $\ell_{\text{left}}, \ell_{\text{right}}$)
```

content of the tape before the first blank symbol $\sqcup$ is considered as the machine's output. The running time is defined as the number of steps the Turing machine makes before termination.

Simualtion of polynomial-time Turing machines I show how to translate a polynomial-time Turing machine to a functional program while preserving the semantics and cost. Fix a polynomial-time one-tape Turing machine $M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, q_0, q_{\text{final}})$. Assume that the running time of M is bounded above by $p(n)$ for some polynomial $p : \mathbb{N} \to \mathbb{N}$. Let P denote the functional program obtained by translating the Turing machine M.

Alg. 1 describes how the program P simulates the Turing machine M given an input string $w \in \Sigma^*$. In §2.2, the list ℓ_{left} represents the region on the Turing machine M's tape to the left of the tape head (in the reverse order and excluding the cell where the tape head is currently on), and ℓ_{right} represents the region to the right of the head (including the current cell). Since it is assumed that $p(|w|)$, where $|w|$ denotes the length of the input string w, is a bound on the running time of the Turing machine M, we are assured that the machine M requires at most $p(|w|)$ many cells on the tape. Hence, we safely initialize ℓ_{left} to a list of length $p(|w|)$ without running out of the space later. In §2.2, the list $\ell_{\text{potential}}$ acts as a reservoir of potential, storing one unit of potential in each element. As the head of $\ell_{\text{potential}}$ is removed in §2.2, the potential stored in this element is released and is used to pay for tick 1 in §2.2.

2.3 Bayesian Data-Driven Resource Analyses (Completed)

This section presents *Bayesian data-driven analysis* [72], which infers a cost bound by Bayesian inference on a dataset of cost measurements obtained by running a program on many inputs. Bayesian data-driven analysis offers two advantages over existing data-driven resource analyses, which are mostly optimization-based:

1. Bayesian inference lets the user express their domain knowledge (e.g., how conservative inferred cost bounds should be with respect to the observed costs) in the form of

probabilistic models. By contrast, most data-driven analyses in the literature, which are optimization-based, do not let the user customize statistical models for data analysis.

2. Bayesian inference returns a distribution of inferred cost bounds, providing richer information about statistical uncertainty of inferred bounds than optimization-based methods.

Throughout this section, I consider monotone resource metrics (e.g., running time) for simplicity. Because high-water-mark costs coincide with net costs in monotone resource metrics, I simplify the cost-semantic judgment by only indicating one cost:

$$V \vdash_{\mathcal{P}} e \Downarrow^{c} v. \tag{2.8}$$

I later discuss how to adapt data-driven analysis to non-monotone resource metrics.

Code annotations for data-driven analysis To specify which code fragments are to be analyzed by data-driven analysis, the user annotates them. Let $\mathcal{L}$ be a countable set of labels. To indicate that an expression e inside the source code is subject to data-driven analysis, the user annotates the expression as $\mathsf{stat}_{\ell}\ e$, where $\ell \in \mathcal{L}$ is a label that uniquely identifies a site of data-driven analysis. In fully data-driven analysis, which is a special case of hybrid analysis, if a function $P(x) = e$ is to be analyzed, the entire function body e is annotated as $\mathsf{stat}_{\ell}\ e$.

Runtime cost datasets Consider a program $\mathcal{P}$ where code fragments subject to data-driven analysis are annotated with with stat_{ℓ} for $\ell \in \mathcal{L}$. Let $P(x)$ be the main function of of the program. We first prepare $N > 0$ many environments $U_i = \{x : u_i\}$ $(i = 1, \ldots, N)$ and run the main function $P(x)$ each environment U_i as input. Let $L' \subset \mathcal{L}$ denote the finite set of labels of all stat_{ℓ} subexpressions in the program $\mathcal{P}$. During the execution of $P(u_i)$, suppose we encounter an annotated code fragment $\mathsf{stat}_{\ell}\ e_{\ell}$ for $N_i^{\ell} \geq 0$ times. Let the evaluation judgments of the expression e_{ℓ} while evaluating $P(u_i)$ be

$$V_{i,j}^{\ell} \vdash_{\mathcal{P}} e_{\ell} \Downarrow^{c_{i,j}^{\ell}} \tilde{v}_{i,j}^{\ell} \qquad (i = 1, \ldots, N, j = 1, \ldots, N_i^{\ell}). \tag{2.9}$$

In (2.9), the input u_i to the main function P, all values in the environment $V_{i,j}^{\ell}$ for the expression e^{ℓ}, and the output value $\tilde{v}_{i,j}^{\ell}$ are all required to have non-arrow types. Otherwise, higher-order functions would raise technical challenges in program-input generation and data collection.

A dataset $\mathcal{D}_{\ell}$ of cost measurements for the code fragment $\mathsf{stat}_{\ell}\ e_{\ell}$ is constructed by recording three components: inputs $V_{i,j}^{\ell}$, outputs $\tilde{v}_{i,j}^{\ell}$, and costs $c_{i,j}^{\ell}$. An overall dataset $\mathcal{D}$ is then given by aggregating all $\mathcal{D}_{\ell}$ for $\ell \in L'$. Formally, datasets $\mathcal{D}_{\ell}$ and $\mathcal{D}$ are defined as

$$\mathcal{D}_{\ell} \coloneqq \{(\ell, V_{i,j}^{\ell}, \tilde{v}_{i,j}^{\ell}, c_{i,j}^{\ell}) \mid 1 \leq i \leq N, 1 \leq j \leq N_i^{\ell}\} \quad (\ell \in L') \qquad \mathcal{D} \coloneqq \bigcup_{\ell \in L'} \mathcal{D}_{\ell}. \tag{2.10}$$

Setting the stage To set the stage for fully data-driven analysis, consider a program $\mathcal{P}$ where a function $(P(x) = e) \in \mathcal{P}$ is the target of resource analysis. Let the dataset of cost measurements of e be $\mathcal{D} = \{(V_i, \tilde{v}_i, c_i)\}_{i=1}^{N}$. To simplify the presentation, we assume that P takes as input a length-n integer list, returns a length $\xi(n)$-integer list, and contains no free variables. Since

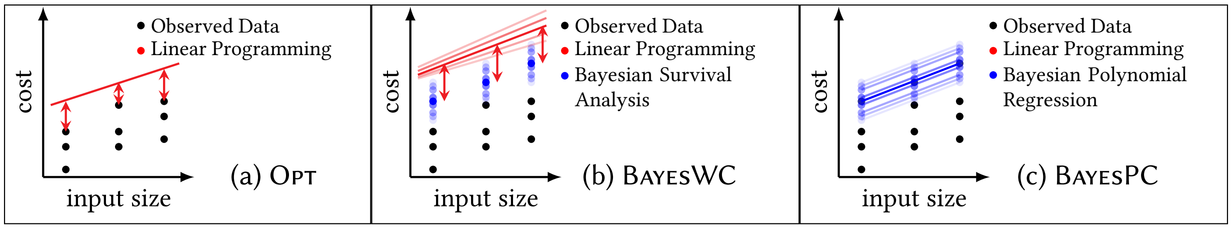


Figure 1: Three approaches to data-driven resource analysis. (a) Opt uses linear programming to fit a polynomial curve that lies above the runtime data while minimizing the distance to the observed worst-case cost at each input size. (b) BayesWC uses a two-step approach: first, Bayesian survival analysis is used to infer a posterior distribution over the worst-case cost at each input size; second, linear programming is used to fit polynomial curves with respect to samples from the inferred distribution of worst-case costs. (c) BayesPC uses Bayesian polynomial regression to infer the coefficients of polynomial curves that lie above the observed runtime data.

$V_i \equiv \{x : v_i\}$ holds for each $i, \dots, N$, I denote the measurements more concisely as $(v_i, \tilde{v}_i, c_i)$. A cost bound of P is described by a resource-annotated typing judgment

$$\{x : L^{\vec{p}}(\texttt{int})\}, p_0 \vdash P\,x : \langle L^{\vec{q}}(\texttt{int}), q_0 \rangle, \tag{2.11}$$

where $\vec{p}$ adn $\vec{q}$ are vector of polynomial coefficients (except for degree-zero coefficients) of input and output potential functions, respectively, and p_0 and q_0 are constant potential in the input and output, respectively. This typing judgment is sound if, for all lists $v : L(\texttt{int})$ such that $\{x : v\} \vdash_{\mathcal{P}} P\,x \Downarrow^c \tilde{v}$, the input potential is enough to pay for the cost and output potential:

$$[\Phi(v : L^{\vec{p}}(\texttt{int})) + p_0] - [\Phi(\tilde{v} : L^{\vec{q}}(\texttt{int})) + q_0] \equiv \left[\Psi(|v|; p_0, \vec{p}) - \Psi(|\tilde{v}|; q_0, \vec{q})\right] \geq c, \tag{2.12}$$

where I have introduced the function $\Psi(n; p_0, \vec{p}) := \sum_{i=1}^{|\vec{p}|} p_i \binom{n}{i} + p_0 (n \in \mathbb{N})$ to evaluate the amount of potential for input size n with coefficients p_0 and $\vec{p}$ of polynomial potential functions. Unlike Conventional AARA, which derives (2.11) by static analysis of e and linear programming, in data-driven resource analysis, we will infer the parameters $(p_0, \vec{p})$ and $(q_0, \vec{q})$ using the dataset $\mathcal{D}$.

2.3.1 Optimization-Based Data-Driven Analysis

Before presenting Bayesian inference, I consider a simple optimization-based baseline (adapted from the literature [22, 30, 88]) to ensure that (2.12) is satisfied with respect to the runtime data $\mathcal{D}$, i.e.,

$$\forall i = 1, \dots, N.\ \Psi(|v_i|; p_0, \vec{p}) \geq c_i + \Psi(|\tilde{v}_i|; q_0, \vec{q}). \tag{2.13}$$

We seek the tightest bound among all $p_0, \vec{p}, q_0, \vec{q}$ that minimizes the nonnegative cost gaps between the predicted and observed costs in the dataset $\mathcal{D}$. Letting

$$N_{\mathcal{D}} := \{|v_i|; i = 1, \dots, N\} \qquad \text{set of unique input sizes appearing in } \mathcal{D} \tag{2.14}$$

$$\hat{c}_n^{\max} := \max\{c_i \mid i = 1, \dots, N; |v_i| = n\} \qquad \text{max. observed cost for input size } n \in N_{\mathcal{D}} \tag{2.15}$$

$$c_n^{\max} := \max\{\text{cost}(f\,v) \mid v : L(\texttt{int}), |v| = n\} \qquad \text{true worst-case cost for input size } n \in N_{\mathcal{D}}, \tag{2.16}$$

I define the following linear program:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N} \left[\Psi(|v_i|; p_0, \vec{p}) - \Psi(|\tilde{v}_i|; q_0, \vec{q})\right] - \hat{c}^{\max}_{|v_i|} \qquad (\textsc{Opt-LP}) \\ \text{subject to} & \Psi(|v_i|; p_0, \vec{p}) \geq \Psi(|\tilde{v}_i|; q_0, \vec{q}) + \hat{c}^{\max}_{|v_i|} \; (i = 1, \dots, N); \quad p_0, p_1, \dots, p_{|\vec{p}|}, q_0, q_1, \dots, q_{|\vec{q}|} \geq 0. \end{array}$$

An example of this approach, which I call Opt, is shown in Fig. 1a. While any solution $\hat{p}_0, \hat{\vec{p}}, \hat{r}_0, \hat{\vec{q}}$ to (Opt-LP) is guaranteed to satisfy (2.13), even a conservative estimate $\Psi(n; \hat{p}_0, \hat{\vec{p}})$ of the worst-case cost may lie below the true value $c^{\max}_n$ in Eq. (2.16) (which I assume is finite). This shortcoming occurs because Opt uses the point estimate $\hat{c}^{\max}_n$ given in Eq. (2.15) as a proxy for $c^{\max}_n$, which is not robust in cases where the data $\mathcal{D}$ is such that $\hat{c}^{\max}_n < c^{\max}_n$ for some $n \in N_{\mathcal{D}}$.

2.3.2 Bayesian Inference on Worst-Case Costs

Overview The first approach to addressing the aforementioned limitation of Opt is *Bayesian inference on worst-case costs* (BayesWC). Whereas Opt uses the data $\mathcal{D}$ to form a point estimate $\hat{c}^{\max}_n$ of the worst-case cost $c^{\max}_n$ for each input size $n \in N_{\mathcal{D}}$ in the linear program, BayesWC instead leverages $\mathcal{D}$ to learn an entire probability distribution μ_n that characterizes our uncertainty about $c^{\max}_n$. I identify two requirements that the inferred worst-case cost distributions μ_n must satisfy:

$$\mu_n([\hat{c}^{\max}_n, \infty)) = 1, \qquad \forall \epsilon > 0, w > \hat{c}^{\max}_n. \, \mu_n([w - \epsilon, w + \epsilon]) > 0. \tag{2.17}$$

The left expression guarantees soundness (2.13) with respect to $\mathcal{D}$ and the right expression ensures robustness with respect to the true worst-case cost $c^{\max}_n$. The latter property is not satisfied by Opt.

If we have access to probability distributions μ_n $(n \in N_{\mathcal{D}})$ over worst-case costs, we can use them to robustly estimate bounds by generating $|N_{\mathcal{D}}|$ batches of $M > 0$ i.i.d. samples

$$(c'_{n,1}, \dots c'_{n,M}) \sim \mu_n \qquad (n \in N_{\mathcal{D}}). \tag{2.18}$$

Reorganizing these $|N_{\mathcal{D}}| \times M$ samples into M lists $\mathbf{c}'_j := (c'_{n,j}; n \in N_{\mathcal{D}})$ $(j = 1, \dots, M)$ each of length $N_{\mathcal{D}}$, we obtain posterior samples of coefficients $p_0, \vec{p}, q_0, \vec{q}$ by solving M linear programs parametrized by the random samples $\mathbf{c}'_j$:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N} \left[\Psi(|v_i|; p_0, \vec{p}) - \Psi(|\tilde{v}_i|; q_0, \vec{q})\right] - c'_{|v_i|,j} \qquad (\textsc{BayesWC-LP}) \\ \text{subject to} & \Psi(|v_i|; p_0, \vec{p}) \geq \Psi(|\tilde{v}_i|; q_0, \vec{q}) + c'_{|v_i|,j} \; (i = 1, \dots, N); \quad p_0, p_1, \dots, p_{|\vec{p}|}, q_0, q_1, \dots, q_{|\vec{q}|} \geq 0. \end{array}$$

Fig. 1b shows an example of BayesWC, where the blue dots above a given input size n represents the samples $c'_{n,j}$ from the worst-case cost distribution μ_n. The solutions of the corresponding linear programs (BayesWC-LP) are shown in red. Whereas Opt delivers a single bound using from one LP, BayesWC delivers a posterior samples of bounds using multiple randomly generated LPs.

Sampling worst-case costs via Bayesian inference To obtain distributions μ_n over worst-case costs that satisfy Eq. (2.17), we perform Bayesian inference as follows. Let $\mathbf{v} := (v_1, \ldots, v_N)$ denote the observed inputs in $\mathcal{D}$ and $\mathbf{c} := (c_1, \ldots, c_N)$ the corresponding costs. Let C_n be a random variable representing the cost of P applied to a size-n input v (the randomness is taken over the (unknown) distribution of v). Define $\mathbf{C} := (C_{|v_1|}, \ldots, C_{|v_N|})$ as a vector of random variables representing the costs of running P on inputs of sizes $|v_i|$ ($i = 1, \ldots, N$). We design a Bayesian model indexed by $\mathbf{v}$:

$$\pi_{\mathbf{v}}(\theta, \mathbf{C}) := h(\theta) \prod_{i=1}^{N} g(c_i; \theta, |v_i|), \tag{2.19}$$

where $h(\theta)$ is a prior distribution of latent parameters θ and $g(c_i; \theta, |v_i|)$ is the likelihood of c_i given the latent parameters θ and the input size $|v_i|$. The Bayesian model encodes the user's domain knowledge about how conservative inferred cost bounds should be relative to maximum observed costs in the dataset $\mathcal{D}$.

To infer worst-case costs $c_n^{\max}$ ($n \in N_{\mathcal{D}}$), we approximate the posterior distribution $\pi_{\mathbf{v}}(\theta \mid \mathbf{c})$ by running a sampling-based probabilistic inference algorithm and drawing posterior samples $\theta_1, \ldots, \theta_M$. Next, for each posterior sample θ_j ($j = 1, \ldots, M$), we sample inferred worst-case cost $c'_{n,j}$ ($n \in N_{\mathcal{D}}$) from the distribution $g(\cdot; \theta_j, n)$ truncated to the interval $[\hat{c}_n^{\max}, \infty)$, where $\hat{c}_n^{\max}$ is the maximum observed cost of size n present in the dataset $\mathcal{D}$ (Eq. (2.15)). Finally, we insert the posterior samples of worst-case costs into the linear programs (BayesWC-LP) and solve them.

2.3.3 Bayesian Inference on Polynomial Coefficients

Whereas BayesWC performs Bayesian inference on worst-case costs and composes the results with (BayesWC-LP) to deliver bounds, my collaborators and I develop another approach that bypasses LP solving and directly performs inference over the unknown coefficients $p_0, \vec{p}, q_0, \vec{q}$ in the resource-annotated typing judgment (2.11).

In this approach, which I call *Bayesian inference on polynomial coefficients* (BayesPC), a Bayesian model is again indexed by the input instances $\mathbf{v}$ and defines a probability distribution $\pi_{\mathbf{v}}(\theta, p_0, \vec{p}, q_0, \vec{q}, \mathbf{c})$ over a set of auxiliary latent parameters θ, resource coefficients $p_0, \vec{p}, q_0, \vec{q}$, and observable costs $\mathbf{c}$. Conditioned on $\mathbf{c}$, we sample

$$p'_0, \vec{p}', q'_0, \vec{q}' \sim \pi_{\mathbf{v}}(p_0, \vec{p}, q_0, \vec{q} | \mathbf{c}), \tag{2.20}$$

which define the posterior bound $\lambda n.\ \Psi(n; p'_0, \vec{p}') - \Psi(\xi(n); q'_0, \vec{q}')$. Fig. 1c illustrates this idea: the blue curves represent posterior samples of cost bounds and the blue dots show samples c'_n of inferred worst-case costs that estimate the true value $c_n^{\max}$ at each input size $n \in N_{\mathcal{D}}$. As in BayesWC, BayesPC delivers posterior samples of both worst-case costs and cost bounds, but it rests on a different modeling and inference approach that bypasses linear programming entirely.

Remark 2.3. *The main challenge to posterior inference in BayesPC is the fact that the polynomial coefficients $p_0, \vec{p}, q_0, \vec{q}$ are constrained to the linear regions $c_i \sqsubseteq \Psi(|v_i|; p_0, \vec{p}) - \Psi(\xi(|v_i|); q_0, \vec{q})$ for $i = 1, \ldots, N$. That is, the coefficients must be sound at least with respect to the cost measurements in the dataset $\mathcal{D}$. Coefficients outside this region have zero posterior probability density.*

Whereas traditional Markov chain Monte Carlo algorithms struggle in this setting, we leverage "reflective" Hamiltonian Monte Carlo (reflective HMC) sampling [15, 17, 53, 64] for posterior inference in BayesPC, where simulated trajectories reflect at the boundaries of the convex polytopes. A high-quality implementation is available in the C++ library Volesti [16]. «

2.3.4 Extension to Non-Monotone Resource Metrics

I discuss how to extend data-driven resource analysis to non-monotone resource metrics where resources can be freed up as well as consumed (e.g., memory). To ensure the soundness of AARA (Thm. 2.1) with respect to a runtime cost dataset $\mathcal{D}$, we record not only net costs $c^{\ell}_{i,j}$ but also high-water-mark costs $h^{\ell}_{i,j}$ during data collection (2.9). In Opt, we then incorporate into the linear program (Opt-LP) the linear constraints that the input potential $\Psi(|v_i|; p_0, \vec{p})$ must be larger than or equal to the observed high-water marks $h^{\ell}_{i,j}$. The objective function of the linear program (Opt-LP) can remain unchanged, but the user is allowed to adjust it.

For BayesWC to handle non-monotone resource metrics, we infer worst-case high-water-mark costs $h'_n \geq 0$ ($n \in N_{\mathcal{D}}$) by Bayesian inference, in addition to worst-case net costs c'_n. Furthermore, $h'_n \geq c'_n$ must hold for all $n \in N_{\mathcal{D}}$, and the net costs c'_n are now allowed to be negative because resources can be freed up. The linear program (BayesWC-LP) is then modified by incorporating the inferred worst-case high-water-mark costs h'_n into linear constraints.

Finally, for BayesPC, the Bayesian model is modified by incorporating the observed high-water-mark costs $h^{\ell}_{i,j}$ into (i) the joint probability distribution and (ii) the linear constraints defining the region of positive density to draw posterior samples from.

2.4 Hybrid AARA (Completed)

This section presents *Hybrid AARA* [72], the first hybrid resource analysis technique that integrates static and data-driven analyses. Specifically, Hybrid AARA integrates AARA with one of the three data-driven analysis methods: Opt, BayesWC, and BayesPC. To run Hybrid AARA, the user first annotates the source code to specify which code fragments are to be analyzed by data-driven analysis. The rest of the program is analyzed by static analysis.

Hybrid AARA is based on a formal typing system that extends AARA with a new type judgment:

$$\Gamma; p_0 \vdash_{\mathcal{D}} \mathsf{stat}_{\ell}\ e : \langle a, q_0 \rangle. \tag{2.21}$$

The judgment (2.21) extends the judgment (2.6) of Conventional AARA by including a dataset $\mathcal{D}$ of runtime cost measurements. A key technical challenge is the design of the interface between (i) data-driven resource analysis using sampling-based Bayesian inference algorithms and (ii) Conventional AARA using static inference and linear programming.

2.4.1 Hybrid BayesWC and Opt

Typing rules Opt and BayesWC are integrated into the AARA type system by adding the following rules for stat_ℓ subexpressions:

$$\frac{\begin{array}{c}p_0 + \Phi(V_i^\ell : \Gamma) \geq q_0 + \Phi(v_i^\ell : a) + c_i^\ell \\ (i = 1, \ldots, |\mathcal{D}_\ell|)\end{array}}{\Gamma; p_0 \vdash_{\mathcal{D}} \mathsf{stat}_\ell\ e : \langle a, q_0 \rangle}\ \text{H:Opt} \qquad \frac{\begin{array}{c}p_0 + \Phi(V_i^\ell : \Gamma) \geq q_0 + \Phi(v_i^\ell : a) + c'^\ell_{|V_i^\ell|} \\ c'^\ell_n \sim \pi^\ell_{\mathbf{v}^\ell}(\cdot | \mathbf{c}^\ell) \qquad (i = 1, \ldots, |\mathcal{D}_\ell|; n \in N_{\mathcal{D}_\ell})\end{array}}{\Gamma; p_0 \vdash_{\mathcal{D}} \mathsf{stat}_\ell\ e : \langle a, q_0 \rangle}\ \text{H:BayesWC} . \tag{2.22}$$

H:Opt states that the consequent holds whenever the input potential $p_0 + \Phi(V : \Gamma)$ is large enough to cover both cost c and leftover potential $q_0 + \Phi(v : a)$ for every measurement $(\ell, V, v, c) \in \mathcal{D}_\ell$. For H:BayesWC, given a dataset $\mathcal{D}_\ell = \{(V_i^\ell, v_i^\ell, c_i^\ell) \mid i = 1, \ldots, M_\ell\}$, I define

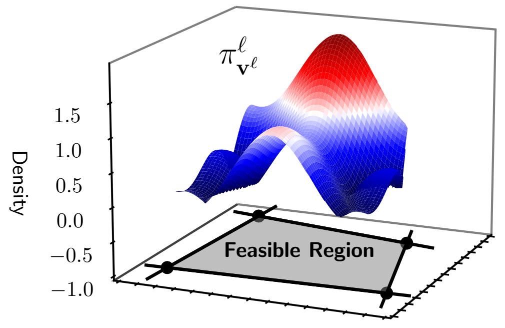
$$\mathbf{v}^\ell := ((V_i^\ell, v_i^\ell), i = 1, \ldots, M_\ell) \qquad \mathbf{c}^\ell := (c_i^\ell, i = 1, \ldots, M_\ell). \tag{2.23}$$

The corresponding probabilistic model used within stat_ℓ is denoted π^ℓ. H:BayesWC is similar to H:Opt, except that each observed cost c within a measurement (ℓ, V, v, c) is replaced with a posterior sample $c'^\ell_{|V|}$ for worst-case costs drawn from BayesWC (2.18).

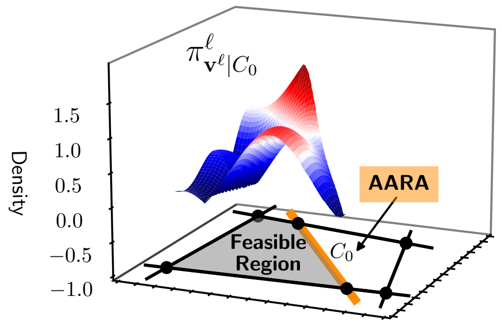
Type inference Because the premises of H:Opt and H:BayesWC are linear constraints over the resource coefficients in e, type inference operates similarly to conventional AARA. The type inference for Hybrid BayesWC operates as follows. Given the runtime data $\mathcal{D}$, we first perform data-driven BayesWC inference to produce M batches of posterior samples of $\mathbf{c}'^\ell_j := (c'^\ell_{n,j}, n \in N_{\mathcal{D}_\ell})$ for $j = 1, \ldots, M$ and each label ℓ, which define M versions of H:BayesWC for each stat_ℓ subexpression. Next, for each $j = 1, \ldots, M$, we perform a static pass that constructs a template typing tree according to Conventional AARA's type system augmented with the inference rule H:BayesWC. This process produces M systems of linear constraints C_j ($j = 1, \ldots, M$), where the linear constraints within each C_j are derived from two provenances: those from the Conventional AARA type system (applied to non-annotated code fragments) and those from the H:BayesWC type rule (applied to code fragments annotated with stat). Each C_j is provided to an LP solver to provide a typing judgment J_j for the root node's typing context, which translates to an inferred cost bound.

2.4.2 Hybrid BayesPC

Key challenge The integration of BayesPC and Conventional AARA pose a challenge in the design of their interface. Resource annotations in the typing judgment $\Gamma; p_0 \vdash_{\mathcal{D}} \mathsf{stat}_\ell\ e : \langle a, q_0 \rangle$ are sampled using Bayesian inference in BayesPC, while they are optimized using LP solvers in Conventional AARA. Thus, their integration requires an interface between sampling-based probabilistic inference and linear programming. We cannot simply perform BayesPC to infer potential functions of the input and output of $\mathsf{stat}_\ell\ e$ and plug them into a linear program produced by AARA's type system, because we do not know in advance how much potential should be stored in the output. Unlike in fully data-driven resource analysis, in hybrid resource analysis, the output of $\mathsf{stat}_\ell\ e$ may be used in a subsequent computation that also consumes potential. Thus, the input potential (i.e., Γ and p_0) of an annotated expression $\mathsf{stat}_\ell\ e$ should be sufficient to pay for not only its own cost but also the cost of subsequent computation.



(a) BayesPC



(b) Hybrid BayesPC

Figure 2: Posterior distributions over resource coefficients restricted to convex polytopes using BayesPC. (a) In pure BayesPC, the feasible region of the distribution $\pi^{\ell}_{\mathbf{v}^{\ell}}$ (and its posterior distribution) is defined by the linear constraints from a dataset $\mathcal{D}_{\ell}$ of cost measurements. (b) In Hybrid BayesPC, the feasible region of the distribution $\pi^{\ell}_{\mathbf{v}^{\ell}|C_0}$ (and its posterior distribution) is defined by the linear constraints C_0 produced by Conventional AARA's type system, as well as the linear constraints induced by the dataset $\mathcal{D}_{\ell}$.

Type inference I address this challenge by adding linear constraints to the BayesPC probabilistic models that encode feasible regions of linear programs computed by Conventional AARA. This approach guarantees that judgments from BayesPC cannot impose new constraints that cause the linear programs to fail to have a solution.

Hybrid BayesPC operates as follows. Suppose we have an annotated code fragment with a resource-annotated typing judgment

$$\Gamma; p_0 \vdash_{\mathcal{D}} \mathsf{stat}_{\ell}\ e : \langle a, q_0 \rangle. \tag{2.24}$$

We start by performing a static-analysis pass through the program using the conventional AARA type system to obtain a set of linear constraints C_0, treating any stat_{ℓ} using the rule H:Opt to ensure consistency with runtime data $\mathcal{D}_{\ell}$. Next, for each subexpression stat_{ℓ} encountered in the first pass, we apply a variant of BayesPC where a Bayesian model $\pi^{\ell}_{\mathbf{v}^{\ell}}$ (2.20) is extended by including all LP variables appearing in the constraints C_0, as well as the original LP variables in the typing judgment (2.24). The additional LP variables added to the Bayesian model are assigned with uninformative priors (e.g., uniform if C_0 is bounded).

We then run reflective Hamiltonian Monte Carlo (reflective HMC), which is a recently developed sampling algorithm that allows HMC sampling to be restricted to a convex polytope defined by linear constraints [15–17, 53, 64]. Specifically, we run reflective HMC on the extended Bayesian model, subject to (i) the linear constraints induced by the dataset $\mathcal{D}_{\ell}$ and (ii) the linear constraints C_0 imposed by AARA from the first pass. Let $\pi^{\ell}_{\mathbf{v}^{\ell}|C_0}$ denote the extended Bayesian model restricted to the convex polytope defined by the linear constraints C_0 from the first pass. The linear constraints of $\mathcal{D}$ are already included in the original Bayesian model $\pi^{\ell}_{\mathbf{v}^{\ell}}$ (Remark 2.3), and hence are also included in the extended Bayesian model $\pi^{\ell}_{\mathbf{v}^{\ell}|C_0}$. Fig. 2 illustrates the difference between the distribution $\pi^{\ell}_{\mathbf{v}^{\ell}}$ for pure BayesPC and the distribution $\pi^{\ell}_{\mathbf{v}^{\ell}|C_0}$ in Hybrid BayesPC. Thanks to reflective HMC, any posterior sample drawn from the posterior distribution of $\pi^{\ell}_{\mathbf{v}^{\ell}|C_0}$ is guaranteed to satisfy these linear constraints.

```ocaml
let rec merge_sort xs =
  let _ = Raml.tick 1.0 in
  match xs with
  | [] → []
  | [ x ] → [ x ]
  | _ →
    let lo, hi = split xs in
    let lo_sorted = merge_sort lo in
    let hi_sorted = merge_sort hi in
    merge lo_sorted hi_sorted
```

(a) MergeSort.

```ocaml
let rec bubble_sort xs =
  let _ = Raml.tick 1.0 in
  let is_xs_sorted, xs_swapped =
    traverse_and_swap xs in
  if is_xs_sorted then
    xs_swapped
  else
    bubble_sort xs_swapped
```

(b) BubbleSort.

Lst. 1: MergeSort and BubbleSort in OCaml. (a) The construct `Raml.tick 1.0` (§2.5.1) indicates the cost of 1.0 for every function call to `merge_sort`. This construct comes from RaML [42], an implementation of AARA for OCaml. (b) The function `traverse_and_swap` (§2.5.1) traverses the input `xs` and returns two outputs: (i) whether `xs` is sorted; and (ii) the result of swapping all out-of-order pairs of consecutive elements in `xs`.

Finally, we draw M many samples from the posterior distribution:

$$(\Gamma_j^\ell, p_{0,j}^\ell, a_j^\ell, q_{0,j}^\ell) \sim \pi_{\mathbf{v}^\ell|C_0}^\ell(\Gamma, p_0, q_0, a|\mathbf{c}^\ell) \qquad (j = 1, \ldots, M; \ell \in L'). \tag{2.25}$$

Substituting these posterior samples for the corresponding LP variables in C_0, we obtain M new constraints:

$$C_j := C_0 \oplus \{(\Gamma_j^\ell, p_{0,j}^\ell, a_j^\ell, q_{0,j}^\ell) \mid \ell \in L'\} \qquad (j = 1, \ldots, M) \tag{2.26}$$

Each C_j is then fed to an LP solver to obtain M posterior samples $(B_1, \ldots, B_M)$ of cost bounds.

2.5 Resource Decomposition (Completed)

This section introduces the second hybrid-analysis framework, dubbed *resource decomposition* [71], which integrates two complementary analysis techniques using a different interface design from Hybrid AARA. I first describe two limitations of AARA using concrete examples. I then motivate vertical integration of resource analyses, which cannot be achieved by Hybrid AARA. Finally, I illustrate resource decomposition on the examples.

2.5.1 Limitations of Hybrid AARA

Running examples I use MergeSort (Listing 1a) and BubbleSort (Listing 1b) as running examples. In BubbleSort, the input list `xs` is iteratively traversed, and every out-of-order pair of consecutive list elements is swapped until the list is ordered. As a concrete resource metric, I consider the number of function calls performed during evaluation, including all recursive calls and helper functions, as an illustrative metric. This metric is represented with functions `Raml.tick` in the code. With this metric, MergeSort and BubbleSort have cost bounds $f(x) = 1 + 3.5|x| + 3.5|x|\lceil\log_2(|x|)\rceil$ and $f(x) = 1 + 2|x| + |x|^2$, respectively, where $|x|$ is the length of the input list x.

AARA on running examples MergeSort and BubbleSort are both challenging for traditional static resource analyses, including AARA. To infer a resource bound for MergeSort, the static analysis must infer that the input list is always split in half. This property can then be used to conclude that (i) the recursion depth scales logarithmically; (ii) the cost across all recursive calls at the same depth scales linearly. AARA automatically infers a bound $f(x) = 1 - 2.5|x| + 3.5|x|^2$ for MergeSort, which has incorrect asymptotics.

For BubbleSort, the analysis must infer that the number of out-of-order pairs of consecutive list elements decreases over time. If the analysis only considers changes in the input size, it cannot infer the termination of BubbleSort, let alone infer its quadratic resource bound. AARA fails to infer any bound for BubbleSort altogether.

Limitations of AARA AARA, as implemented in RaML, has two major limitations. The first limitation is that AARA can only express polynomial bounds, but not logarithmic bounds. Even though AARA infers a sound quadratic bound for MergeSort, it cannot infer an asymptotically tight bound of the form $f(x) = c|x|\log(|x|)$. The second limitation of AARA is that it cannot infer a polynomial cost bound of BubbleSort, because the size of the input list does not decrease at each recursive step. The potential function assigned to the input list `xs` by AARA can only be parametric in the size of `xs`, but not in its content. In BubbleSort, however, the length of the input list `xs` is the same at each recursive call. Instead, what decreases over time is the number of out-of-order pairs of elements in the list. But this is beyond the expressive and reasoning power of AARA.

These two limitations are not unique to AARA—they are also representative for other static analysis techniques [47, 57]. It is difficult to design an automatic analysis that can derive arbitrary cost functions. If we want, for instance, compositionality, then it is nontrivial to statically infer cost bounds involving logarithm (e.g., $c|x|\log(|x|)$ for MergeSort) because logarithm does not compose well with other functions such as polynomials. For BubbleSort, the resource analysis requires reasoning about the input list's content as well as its size, which goes beyond the expressive power of all existing static analysis techniques that I am aware of.

Vertical integration of resource analyses To analyze MergeSort and BubbleSort by hybrid analysis (without resorting to fully data-driven analysis), we seek to *vertically integrate* two analysis techniques. Vertical integration means, given a program P, two analysis techniques separately analyze two quantities of different resource metrics, and then their inferred symbolic bounds are combined into an overall cost bound of the program P. For instance, in MergeSort, the logarithmic factor in the asymptotically tight bound $cn \log n$ stems from the logarithmic recursion depth of the function `merge_sort`. It is this logarithmic recursion depth that poses a challenge to AARA. So we wish to perform two analysis techniques (e.g., data-driven analysis and AARA) to infer, respectively, the recursion depth and the combined cost of all recursive calls at the same recursion depth, which are two quantities of different resource metrics. Likewise, in BubbleSort, although AARA cannot reason about the linear recursion depth, AARA is still capable of inferring a linear cost bound of each recursive call. Hence, we would like to separately infer the linear recursion depth and the linear cost of a single recursive call, then composing (i.e., multiplying) their inferred symbolic bounds.

```
let rec merge_sort xs =
  let _ = Raml.mark0 1.0 in
  let _ = Raml.tick 1.0 in
  let result =
    match xs with
    | [] → []
    | [ x ] → [ x ]
    | _ →
      let lo, hi = split xs in
      let lo_sorted = merge_sort lo in
      let hi_sorted = merge_sort hi in
      merge lo_sorted hi_sorted
  in let _ = Raml.mark0 (-1.0)
  in result
```

(a) Resource-decomposed MergeSort.

```
let rec merge_sort xs r =
  let _ = Raml.tick 1.0 in
  let r1 = decrement_r r in
  let result, r_final =
    match xs with
    | [] → ([], r1)
    | [ x ] → ([ x ], r1)
    | _ →
      let (lo, hi) = split xs in
      let lo_sorted, r2 = merge_sort lo r1 in
      let hi_sorted, r3 = merge_sort hi r2 in
      (merge lo_sorted hi_sorted, r3)
  in (result, increment_r r_final)
```

(b) Resource-guarded MergeSort.

Lst. 2: Resource-decomposed and resource-guarded MergeSort. A resource component is the recursion depth of the function `merge_sort`. (a) The annotations `Raml.mark0 1.0` (§2.5.2) and `Raml.mark0 (-1.0)` (§2.5.2) increment and decrement a resource-component counter by one, respectively. Hence, the recursion depth of `merge_sort` is equal to the high-water mark of this counter. The suffix `0` in the annotation `Raml.mark0` identifies a resource component (i.e., the first resource component). (b) The function `merge_sort` is extended with a resource guard `r`. The function `decrement_r` (§2.5.2) decrements the resource guard by one (if it is positive, else raises an exception). The function `increment_r` (§2.5.2) increments the resource guard by one.

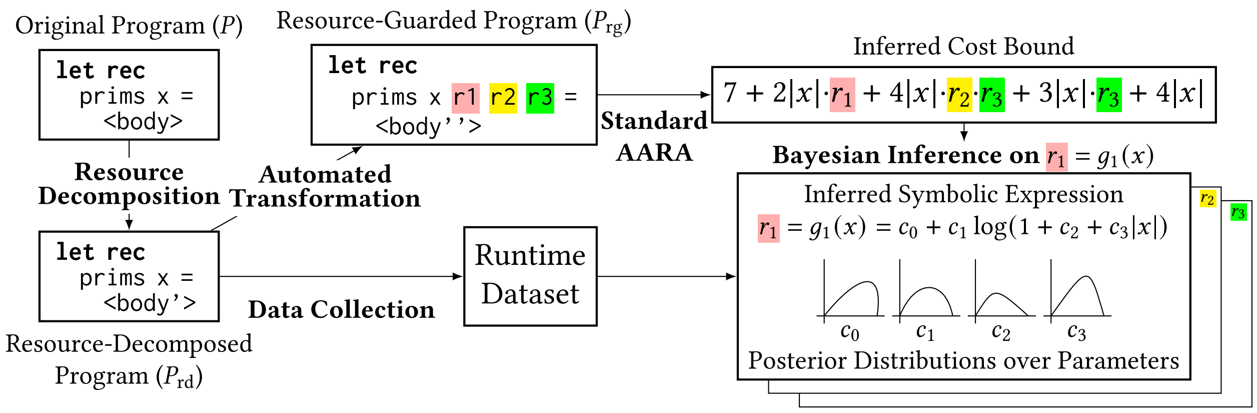
Unfortunately, Hybrid AARA does not support vertical integration of analysis techniques. Hybrid AARA only supports *horizontal integration*, where two resource analyses are performed on quantities of the same resource metric (e.g., running time) of different code fragments.

The root cause is that Hybrid AARA uses resource-annotated types in an interface between AARA and data-driven analysis. In Hybrid AARA, given an annotated code fragment $\mathsf{stat}_\ell\ e$, the inference result of a data-driven analyses is expressed using a resource-annotated typing judgment $\Gamma; p_0 \vdash_{\mathcal{D}} \mathsf{stat}_\ell\ e : \langle a, q_0 \rangle$, which contains polynomial potential functions assigned to the input (i.e., Γ and p_0) and output (i.e., a and q_0). These statistically inferred resource-annotated types are then plugged into a typing tree inferred by AARA. As all resource-annotated types for an input program P must concern the same resource metric (e.g., running time of code fragments inside P), we cannot use them to capture quantities of different resource metrics inside the same program (e.g., recursion depth and the running time of a single recursive call).

2.5.2 Overview

I introduce a general hybrid-resource-analysis framework, resource decomposition, to enable vertical integration of two complementary analysis techniques.

Key insight To illustrate a key insight of the new resource analysis framework, consider a modified version of MergeSort in Listing 2b. This version is obtained by augmenting the original program (Listing 1a) with a nonnegative numeric variable `r`, called a *resource guard*. This variable is intended to track the recursion depth of MergeSort. At the start of the function



Original Program (P)
let rec prims x = <body>
Resource-Guarded Program (P_{rg})
let rec prims x r1 r2 r3 = <body''>
Inferred Cost Bound
$7 + 2|x| \cdot r_1 + 4|x| \cdot r_2 \cdot r_3 + 3|x| \cdot r_3 + 4|x|$
Standard AARA
Bayesian Inference on $r_1 = g_1(x)$
Resource Decomposition
Automated Transformation
Inferred Symbolic Expression
$r_1 = g_1(x) = c_0 + c_1 \log(1 + c_2 + c_3|x|)$
r_2
r_3
let rec prims x = <body'>
Runtime Dataset
Data Collection
c_0
c_1
c_2
c_3
Posterior Distributions over Parameters
Resource-Decomposed Program (P_{rd})

Figure 3: Overview of resource decomposition instantiated with AARA and Bayesian data-driven analysis.

body, r is decremented (§2.5.2). If an attempt to decrement r is made when its value is zero, then an exception is raised. The decremented resource guard r is then successively passed on to the next recursive calls (§2.5.2), if there are any. Finally, once the last recursive call terminates, its output value of the resource guard r is incremented (§2.5.2) before it is returned as the second output of the current function call, alongside the original output of MergeSort.

For this modified MergeSort, AARA infers a symbolic resource bound $f(x, r) = 1+3.5 \cdot r \cdot |x|$, where r denotes the resource guard variable r (which is always a nonnegative number) and $|x|$ is the size of the input list xs. The modified program is related to the original version as follows: as long as the resource guard r is initialized to at least the recursion depth of MergeSort (i.e., $1 + \lceil \log_2(|x|) \rceil$), the modified code successfully terminates (i.e., it does not raise an exception), returns the same output as the original code, and incurs the same cost.

Resource decomposition My collaborators and I have developed *resource decomposition* as a way to systematically exploit the insight illustrated with the modified code of MergeSort. To derive a bound for the original function, we first derive the bound $f(x, r) = 1 + 3.5 \cdot r \cdot |x|$ for the modified function. Next, we use another analysis method to infer a cost bound (say $g(x)$) for r in terms of the original input x, and substitute the result in f to obtain an overall bound $f(x) = 1 + 3.5 \cdot g(x) \cdot |x|$.

Workflow Fig. 3 shows the overall workflow of resource decomposition. Given a program $P(x)$, the user (or an automatic tool) first decides on a set of $m \geq 1$ *resource components*, i.e., the quantities that the resource guards should track. Examples of resource components include the recursion depth, the number of recursive calls, and the cost of a code fragment. The program $P(x)$ is then (manually or automatically) instrumented with code annotations $\text{mark}_\ell\, q$, where ℓ is a label that uniquely identifies a resource component and $q \in \mathbb{Q}$ is a rational number, such that their high-water marks (i.e., the highest values reached so far) are equal to the user-specified resource components. Let $P_{rd}(\mathbf{x})$ denote the resulting *resource-decomposed program*. By way of example, a resource-decomposed program of MergeSort is displayed in Listing 2a, where a user-specified resource component is the recursion depth of the function merge_sort.

Next, the decomposed program $P_{\text{rd}}(x)$ is automatically translated to a *resource-guarded program* $P_{\text{rg}}(x, \mathbf{r})$ by augmenting the former with resource guards $\mathbf{r} = (r_1, \ldots, r_m)$ as extra input variables, one for each user-specified resource component. In contrast to the annotations $\text{mark}_\ell\ q$ for resource components, the resource guards $\mathbf{r}$ count down, i.e., they are decremented whenever the corresponding resource components are incremented. If the resource-guarded program P_{rg} attempts to decrement a resource guard that is zero, the program raises an exception. We then conduct resource analyses (possibly using different techniques) on (i) the resource-guarded program P_{rg} to derive a symbolic cost bound $f(x, \mathbf{r})$ for the cost of program P; (ii) the resource components of P_{rd} to derive their symbolic bounds $r_i = g_i(x)$ ($i = 1, \ldots, m$). Finally, we substitute the resource components' symbolic bounds $g_i(x)$ for the resource guards r_i ($i = 1, \ldots, m$) in the bound $f(x, \mathbf{r})$, obtaining an overall cost bound $f(x, g_1(x), \ldots, g_m(x))$ for the original program P.

Comparison with Hybrid AARA Thanks to the use of resource guards in the interface between constituent resource analyses, resource decomposition enables vertical integration without any restriction on symbolic bounds of resource components. Resource components can represent any user-specified quantities as long as they can be expressed as high-water-mark costs. Hence, resource components are allowed to have different resource metrics from each other and also from the overall cost bound of the program. This enables vertical integration, as well as horizontal integration, of resource analyses. Additionally, resource guards, which are numeric variables tracking resource components, can be any symbolic bounds, including non-polynomial symbolic bounds that cannot be captured by resource-annotated types.

Furthermore, resource decomposition is more general than Hybrid AARA in term of supported analysis techniques. Hybrid AARA is specific to the integration of AARA (and its variants) and data-driven analysis, where cost bounds from both analyses are expressed as resource-annotated types. By contrast, resource decomposition can be applied to any combination of analysis techniques: static, data-driven, and interactive resource analyses.

Although resource decomposition is better at vertical integration than Hybrid AARA, the latter also some advantages over the former in terms of expressive power. In Hybrid AARA, cost bounds are expressed by resource-annotated types assigned to the input and output of an expression. Hence, Hybrid AARA can express cost bounds parametric in output sizes as well as input sizes. By contrast, in resource decomposition, symbolic bounds of resource components can only be parametric in input sizes, but not output sizes.

2.5.3 Formalization

This section first introduces a programming language with constructs for resource-decomposed programs. I then describe the program transformation from resource-decomposed programs to resource-decomposed ones. Finally, I state the soundness theorem of the program transformation, which has been proved using a logical relation in Pham et al. [71].

Code annotations Fix a countable set $\mathcal{L}$ of labels to identify resource components. For resource-decomposed programs, I define a call-by-value higher-order programming language $\text{R}\textsc{pcf}_n$, where $n \in \mathbb{N}$ is the (statically known) number of distinct labels $\ell \in \mathcal{L}$ that appear in the source code of an input program. The parameter $n \in \mathbb{N}$ determines the number of resource guards that are tracked in the denotational semantics. In addition to $\mathsf{tick}\ q$ ($q \in \mathbb{Q}$) for resource consumption, Rpcf$_n$ contains the following constructs for resource decomposition:

(C1) $\mathsf{mark}_\ell\, q : \mathtt{unit}$ increments a counter of the resource component $\ell \in \mathcal{L}$ by $q \in \mathbb{Q}$.

(C2) $\mathsf{reset}_\ell : \mathtt{unit}$ records the current high-water mark of the resource component $\ell \in \mathcal{L}$ in a dataset $\mathcal{D}_\ell$ and resets the counter to zero.

These constructs have the following semantics. During program execution, each resource component has a "current" value c_ℓ and a "high-water mark" value h_ℓ (i.e., the maximum value reached so far), which are both initialized to zero. The expression $\mathsf{mark}_\ell\, q$ sets $c_\ell \leftarrow c_\ell + q$ and $h_\ell \leftarrow \max(h_\ell, c_\ell)$. The expression reset_ℓ first saves the measurement (ℓ, h_ℓ) in a dataset $\mathcal{D}_\ell$, then resets $c_\ell \leftarrow 0$ and $h_\ell \leftarrow 0$. These expressions must be inserted in such a way that the high-water mark h_ℓ of the resource component ℓ is equal to the quantity it is intended to represent.

In many settings, it is possible to automate the insertion of expressions $\mathsf{mark}_\ell\, q$ and reset_ℓ to produce desirable resource-decomposed programs. For example, consider a program $P(x)$ that contains a subexpression $H(y)$, where H is a recursive function with available source code. Suppose the user would like a resource component ℓ to represent the recursion depth of an evaluation $H(y)$. First, we insert $\mathsf{mark}_\ell\, 1$ at the beginning and $\mathsf{mark}_\ell\, (-1)$ at the end of the function body of H. Next, we insert reset_ℓ right after the function call $H(y)$ from $P(x)$, so that it records the high-water mark of the resource component ℓ while evaluating $H(y)$.

Types and Semantics The type system of Rpcf$_n$ makes computational effects (e.g., divergence and costs) explicit by distinguishing between computation types and value types. A computation type is marked with a modality F. For example, an expression $e : \mathsf{F}1$, where 1 is the unit (value) type, is a computation that returns the unit element (if terminates) and possibly incurs costs.

I work with domain-theoretic denotational semantics of the language Rpcf$_n$. To track resource usage during program execution, I use a bicyclic monoid $(\mathbb{N}^2, (0,0), \oplus)$ to represent a *resource state*, which is a pair (h, r) of a high-water mark cost h so far and the leftover resource r [39]. A resource effect $\sigma : (\mathbb{N}^2)^n \to (\mathbb{N}^2)^n$ of a computation $e : \mathsf{F}A$ is a mapping from the initial resource states (for all n resource components) before running e to updated resource states.

Program Transformation An expression $e : \mathsf{F}A$ in a resource-annotated program is transformed to an expression $\ulcorner e \urcorner_n : (\mathtt{nat}^2)^n \to \mathsf{F}\left((\mathtt{nat}^2)^n \times \ulcorner A \urcorner_n\right)$ in the corresponding resource-guarded program, where $\ulcorner A \urcorner_n$ is the result of recursively translating the type A. In this transformation, the expression e is extended with n resource guards, each of which is a pair of numeric non-negative variables. We need two numeric variables for each resource guard to maintain both the current counter value and its high-water mark. The latter is used when we encounter the expression reset_ℓ to record and reset the counter.

Thm. 2.4 formally states the soundness of the program transformation of resource-decomposed programs to resource-guarded ones. The theorem is proved by a logical relation, which is a type-indexed binary relation between the source and target programs that makes the inductive proof go through.

Theorem 2.4 (Soundness of program translation)**.** *Given a closed computation $e : \mathsf{F1}$ of* $\textsc{Rpcf}_n$*, suppose e terminates with resource effect σ and cost c, where*

$$((h_1, r_1), \ldots, (h_n, r_n)) \coloneqq \sigma(\underbrace{(0,0), \cdots, (0,0)}_{n \text{ times}}). \tag{2.27}$$

Then the corresponding resource-guarded expression $\ulcorner e \urcorner_n((h_1, h_1), \cdots, (h_n, h_n))$ terminates with cost c as well.

Thm. 2.4 states that, if we run a resource-decomposed expression e and record high-water marks h_i ($i = 1, \ldots, n$) of the n resource components, then we can safely (i.e., without raising an exception) run the corresponding resource-guarded expression $\ulcorner e \urcorner_n$ with the arguments $h_1, \ldots, h_n$ for the n resource guards. Furthermore, the resource-guarded expression $\ulcorner e \urcorner_n$ has the same cost as the resource-decomposed expression e. Therefore, if we obtain a sound cost bound for the resource-guarded program, it also constitutes a sound bound for the resource-decomposed program (and hence the original unannotated program).

2.6 Swiftlet: Instantiation of Resource Decomposition (Completed)

This section presents *Swiftlet* [71], a concrete instantiation of resource decomposition that runs (i) AARA to infer an overall cost bound of the resource-guarded program $P_{\text{rg}}(x, \mathbf{r})$ and (ii) Bayesian analysis to infer (linear or logarithmic) symbolic bounds for resource components that are recursion depths of recursive functions.

2.6.1 Data Collection

A dataset for Bayesian inference is constructed as follows. Let P_{rd} be a resource-decomposed program obtained by annotating an original program with mark_ℓ and reset_ℓ. For some $M > 0$, let $\{\ell_1, \ldots, \ell_M\} \subset \mathcal{L}$ be the set of resource-component labels that appear in the source code of P_{rd}. When executing $P_{\text{rd}}(v)$ on a concrete input v, we obtain, for each label ℓ_i ($i = 1, \ldots, M$), a set $\{(\ell_i, h_{\ell_i,1}), \ldots, (\ell_i, h_{\ell_i,k_i})\}$ of $k_i \geq 0$ measurements of high-water marks. As our goal is to infer worst-case cost bounds and all the high-water marks from the execution $P_{\text{rd}}(v)$ are associated with the *same* input v, it is sufficient to save only the maximum measurement $h_{\ell_i,v} \coloneqq \max_{1 \leq j \leq k_i} h_{\ell_i,j}$ for each label ℓ_i (whenever $k_i > 0$). The resulting dataset associated with an input v is then $\{(\ell_i, v, h_{\ell_i,v}) \mid 1 \leq i \leq M, k_i > 0\}$. Repeating this pattern, we execute $P_{\text{rd}}(v_i)$ on $N \geq 1$ concrete inputs $\{v_1, \ldots, v_N\}$ to obtain a collection of M datasets $\mathcal{D}_1, \ldots, \mathcal{D}_M$, where $\mathcal{D}_i \coloneqq \{(\ell_i, v_j, h_{\ell_i,v_j}) \mid 1 \leq j \leq N, k_i > 0\}$ contains the measurements of high-water marks for resource component ℓ_i across all N inputs.

2.6.2 Bayesian Inference for Recursion Depths

Overview The goal of Bayesian inference is to infer, for each resource component ℓ in the resource-decomposed program P_{rd}, a cost bound $p_\tau(v)$ that relates an input $v : \tau$ with its maximum high-water mark $h_{\ell,v}$ obtained while executing $P_{\text{rd}}(v)$. Each bound is learned from the data $\mathcal{D}_i$.

In general, a resource component $\ell \in \mathcal{L}$ is allowed to be any user-specified quantity, as long as the quantity can be expressed as the high-water mark of expressions $\mathsf{mark}_\ell\, q$. In Swiftlet, however, I focus specifically on recursion depths. This motivation stems from the fact that static resource analysis methods such as conventional AARA often fail on recursive programs: while they succeed in finding the cost of a *single* recursive step, they fail to compute bounds on the *number of* recursive steps or *depth of* recursion, which causes the overall analysis to fail. Swiftlet lets us decompose the analysis so that static analysis using AARA computes a bound on the cost of a single recursion and data-driven analysis using Bayesian inference computes a bound on the recursion depth.

Symbolic recursion bounds To enable data-driven inference of cost-bounds from the dataset $\mathcal{D}$, our approach associates each program input $v : \tau$ with a numeric value $m_\tau(v) \in \mathbb{N}$ that denotes its "size". My collaborators and I develop a domain-specific language (DSL) over size measures m_τ and corresponding cost bounds $p_\tau(v)$ that admit linear and logarithmic cost expressions:

$$m_{\texttt{unit}} = m_{\texttt{int}} ::= \lambda v.1 \tag{2.28}$$

$$m_{L(\tau)} ::= \lambda[v_1, \dots, v_k].k \mid \lambda[v_1, \dots, v_k].\max\{m_\tau(v_1), \dots, m_\tau(v_k)\} \tag{2.29}$$

$$m_{\tau_1 \times \tau_2} ::= \lambda\langle v_1, v_2\rangle.m_{\tau_1}(v_1) \mid \lambda\langle v_1, v_2\rangle.m_{\tau_2}(v_2) \tag{2.30}$$

$$p_\tau(v) ::= c_0 + c_1 m_\tau(v) \mid c_0 + c_1 \log(1 + c_2 + c_3 m_\tau(v)); \qquad c \in \mathbb{R}_{\geq 0}. \tag{2.31}$$

Eqs. (2.28)–(2.30) show the language of size measures for base and composite types. The base types Eq. (2.28) have a trivial size measure of 1. The composite types Eqs. (2.29) and (2.30) are associated with multiple size measures. Consider, for example, a nested list type $\tau := L(L(\texttt{int}))$. In the DSL, one size measure m_τ is the outer list length and another size measure is the maximum inner list length. For a graph algorithm where the input is a nested adjacency list, the first size measure corresponds to the number of vertices and the second corresponds to the maximum degree.

Eq. (2.31) shows the language of cost bounds that contains linear and logarithmic terms, each with their own unknown coefficients $c \in \mathbb{R}_{\geq 0}$. The DSL is grounded in two main assumptions. First, each symbolic bound p_τ contains exactly one size measure, even if m_τ contains multiple nontrivial size measures. Second, only degree-one (i.e., affine) polynomials of size measures are admitted. These assumptions are generally sufficient for expressing bounds that relate to recursion depths in the benchmark programs in Pham et al. [71].

Bayesian model Let τ be the input datatype of a functional program under analysis and assume momentarily that we have already selected a size measure m_τ from the DSL (2.28)–(2.30), with $|v| := m_\tau(v)$. Let $\mathbf{v} := (v_1, \dots, v_N)$ be a vector of input values and $\mathbf{h}_\ell := (h_{\ell,1}, \dots, h_{\ell,N})$ the corresponding high-water marks of the resource component ℓ. Rather than pre-specify either a linear or logarithmic symbolic form of the cost bound p_τ from the DSL Eq. (2.31), I use Bayesian model averaging [38] to infer the appropriate symbolic bound from the data.

Selecting the size measure via mutual information Recall from Eqs. (2.29) and (2.30) that a composite type τ may be associated with many size measures m_τ. To select the size measure

m_τ to appear in symbolic bounds p_τ of resource components in the Bayesian model, we select the one with the highest statistical dependence with the observed high-water marks $\mathbf{h}_\ell$. The quantitative measure of dependence we use is *mutual information*, which characterizes all types of possible dependencies (e.g., linear, nonlinear, etc.) between a pair of random variables. The method of Kraskov et al. [54] is used to estimate mutual information from finitely many samples $\{(h_{\ell,i}, m_\tau(v_i)) \mid 1 \leq i \leq N\}$.

2.7 Inference of Program-Input Generators (Proposed Work)

To improve the accuracy of data-driven resource analysis, my collaborators and I will develop a data-driven analysis method that statistically infers not only worst-case cost bounds but also *worst-case input generators*. An *input generator* is a program that can generate program inputs of varying sizes that all conform to a certain pattern. For instance, consider a program $P : L(\texttt{int}) \to L(\texttt{int})$ takes integer lists as input. A trivial input generator for the program P is a random generator that fills in the content of an input list by sampling integers from a distribution (e.g., uniform distribution over some interval of integers). The goal is to infer a worst-case input generator, which, for each input size, generates a worst-case input (with a high probability).

2.7.1 Motivation

In the evaluation of data-driven and hybrid resource analyses in Hybrid AARA [72], I run random input generators to generate program inputs, which are then fed to the programs to record cost measurements. However, when the average-case complexity of a program is significantly lower than its worst-case complexity, statistical analysis struggles to infer correct worst-case bounds from cost measurements of randomly generated inputs. For example, in QuickSort, the worst-case complexity is $O(n^2)$, while the average-case complexity over randomly generated inputs is $O(n \log n)$. Furthermore, the complexity of QuickSort concentrates tightly around $O(n \log n)$ [60–62, 84]. Unless the user explicitly embeds this knowledge into a statistical model, it is difficult for statistical analysis to correctly infer the $O(n^2)$ worst-case complexity from a dataset that highly concentrates around $O(n \log n)$.

To outperform data-driven analysis that uses randomly generated inputs, I enable data-driven analysis to test an input program P with different input generators, instead of using a fixed dataset of cost measurements. The data-driven analysis then infers a worst-case input generator in addition to a worst-case symbolic bound, where the worst-case generator serves as a witness of the worst-case bound. Thanks to the ability to run the input program P, the new data-driven analysis is capable of inferring more accurate worst-case bounds than a random input generator. For instance, for QuickSort, if the data-driven analysis successfully notices that sorted lists accrue higher costs than randomly generated lists, we use sorted lists, instead of randomly generated lists, to record cost measurements. We then perform statistical analysis on these cost measurements, correctly inferring a $O(n^2)$ worst-case bound.

2.7.2 Domain-Specific Language of Input Generators

I will first develop a domain-specific language (DSL) for program input generators. Generators in the DSL can generate program inputs of inductive types. Fix a set $\mathcal{T}$ of type names. Inductive types are formed by this grammar:

$$\tau ::= T \in \mathcal{T} \mid \texttt{unit} \mid \texttt{bool} \mid \texttt{int} \mid \tau_1 + \tau_2 \mid \tau_1 \times \tau_2. \tag{2.32}$$

Inductive types are defined by definitions of the form $T := \tau_T$, where $T \in \mathcal{T}$ is a type name being defined and τ_X is the type definition that may mention the type name T.

Probabilistic generators Unlike in existing works (e.g., Singularity [86]), I would like input generators to be probabilistic: they generate program inputs probabilistically. Probabilistic generators offer two benefits. The first benefit is that probabilistic generators always have a positive probability/density of generating any input. Consequently, as we generate more program inputs using a probabilistic generator, we will eventually find a worst-case input (for various input sizes). This property is essential for statistical soundness guarantees of data-driven and hybrid resource analyses, including Hybrid AARA. The second benefit of probabilistic generators is that they offer a more accurate characterization of worst-case inputs than deterministic generators. For instance, in QuickSort, its set of worst-case inputs contains all sorted lists. However, this diversity of worst-case inputs is never captured by deterministic generators (e.g., Singularity [86]), because they can only generate a single program input of a specific size.

Generators indexed by types For primitive types, the DSL admits the following probabilistic generators g taking $d \geq 0$ arguments:

$$g(x_1, \cdots, x_d) := () \qquad \text{if } \tau = \texttt{unit} \tag{2.33}$$

$$g(x_1, \cdots, x_d) := \textsc{Bern}(\sigma(e)) \qquad \text{if } \tau = \texttt{bool} \tag{2.34}$$

$$g(x_1, \cdots, x_d) := \textsc{Norm}(e_\mu, e_\sigma, e_\ell, e_u, e_{\text{BlackList}}) \qquad \text{if } \tau = \texttt{int}. \tag{2.35}$$

In Eq. (2.34), Bern denotes a Bernoulli distribution, and $\sigma : \mathbb{R} \to [0, 1]$ is the sigmoid function whose output is in the unit interval $[0, 1]$. In Eq. (2.35), Norm denotes a normal distribution. The expressions e_μ, e_σ, e_ℓ, and e_u specify the mean, standard deviation, lower bound, and upper bound, respectively, of a normal distribution to draw samples from. The expression e_{BlacList} is a list of numbers that should be excluded from sampling. It is useful when we would like to generate, for instance, an integer list containing distinct elements, which is a worst-case input to some functional programs.

For compound types (i.e., sum and product types), the corresponding generators are defined inductively, having code structures similar to the type structures. For a sum type $\tau_1 + \tau_2$, a probabilistic generator g has the form

$$\begin{aligned} g(x_1, \cdots, x_d) := {} & \text{let } b = g_{\texttt{bool}}(x_1, \cdots, x_d) \text{ in} \\ & \text{if } b \text{ then left} \cdot g_1(x_1, \cdots, x_d) \\ & \text{else right} \cdot g_2(x_1, \cdots, x_d), \end{aligned} \tag{2.36}$$

where g_{bool} is generator of the boolean type and g_i ($i = 1, 2$) is a generator of type τ_i. For a product types $\tau_1 \times \tau_2$, a probabilistic generator g has the form

$$\begin{aligned} g(x_1, \cdots, x_d) \coloneqq {} & \mathsf{let}\ v_1 = g_1(x_1, \cdots, x_d)\ \mathsf{in} \\ & \mathsf{let}\ v_2 = g_2(x_1, \cdots, x_d, v_1)\ \mathsf{in} \\ & \langle v_1, v_2 \rangle, \end{aligned} \tag{2.37}$$

where g_i ($i = 1, 2$) is a generator of type τ_i.

Challenges The design of a DSL for probabilistic generators poses two challenges:

1. The trade-off between expressiveness and ease of generator inference. The expressions e inside normal distributions Norm contain operators such as succ (for incrementing integers) and hd (for returning the head element of a non-empty list). The more operators the DSL supports, the more expressive it becomes. However, it also grows the state space of input generators, making the inference of worst-case generators more challenging.
2. Probabilistic generation of varying input sizes. In lists, for instance, if we probabilistically choose between two constructors (i.e., nil and cons) using the same Bernoulli distribution in all list cells, then the lengths of generated lists follow a geometric distribution. However, the user may want a different distribution for list lengths. To support such distributions, we need to select the size and shape of a data structure globally, instead of locally deciding what data constructor to use.

2.7.3 Statistical Inference of Worst-Case Input Generators

Once I design a DSL of probabilistic generators, the next step is to develop a methodology for statistically inferring a worst-case input generator (or a distribution thereof), in addition to a worst-case cost bound. Ideally, I would like to conduct Bayesian inference to infer a *posterior distribution* of worst-case generators, rather than a *single* generator. However, it is challenging to design a fully Bayesian methodology such that generators with worse program inputs (i.e., program inputs incurring higher computational costs) are more preferable.

In an optimization-based approach to generator inference (e.g., Singularity [86]), a generator g is assigned a numeric score based on the computational costs of program inputs generated by g. The goal is then to find an optimal generator with the highest score. A Bayesian approach to generator inference, however, cannot adopt the same scoring scheme as optimization. This is because, in Bayesian inference, the density of a prior distribution for generators, which is analogous to a score in optimization, cannot increase monotonically forever.

Example To illustrate this challenge, consider QuickSort that uses the head element of an input list as a pivot. Under a resource metric of running time, a worst-case input to QuickSort is a sorted list. Let g_{random} be a random input generator and g_{sorted} be a generator that generates sorted integer lists (with a sufficiently high probability). The generator g_{sorted} incurs an (asymptotically) higher cost on average than the random generator g_{random}: the former has a $O(n^2)$ cost bound, while the latter has a $O(n \log n)$ cost bond. Additionally, let f_{random} be a symbolic cost bound that fits well with a dataset $\mathcal{D}_{\text{random}}$ generated by the random generator g_{random}. Likewise, let f_{sorted} be a symbolic cost bound that fits well with a dataset $\mathcal{D}_{\text{sorted}}$ generated by the generator g_{sorted}.

Our goal is to correctly infer that the pair $(g_{\text{sorted}}, f_{\text{sorted}})$ is a worst-case choice of a generator and a cost bound for QuickSort.

Consider a probabilistic model $\pi(g, f, \mathcal{D})$, where a latent variable g is a worst-case generator, a latent variable f is a worst-case cost bound, and an observed variable $\mathcal{D}$ is a dataset of program inputs and their associated cost measurements. Typically, a probabilistic model assigns a higher density to a cost bound f if it is closer to a dataset $\mathcal{D}$ of cost measurements. This means the following two conditional densities are close to each other:

$$\pi(\mathcal{D}_{\text{random}} \mid g_{\text{random}}, f_{\text{random}}) \approx \pi(\mathcal{D}_{\text{sorted}} \mid g_{\text{sorted}}, f_{\text{sorted}}). \tag{2.38}$$

This is because the cost bounds f_{random} and f_{sorted} are both assumed to fit well with their respective datasets. Consequently, if we only concern how well cost bounds fit datasets, then the two pairs $(g_{\text{random}}, f_{\text{random}})$ and $(g_{\text{sorted}}, f_{\text{sorted}})$ are equally preferable by Eq. (2.38), even though the latter should be chosen over the former as the inference result of resource analysis.

To prefer the latter pair $(g_{\text{sorted}}, f_{\text{sorted}})$, one may attempt to design a prior distribution $\pi(f)$ such that

$$\pi(f_{\text{sorted}}) > \pi(f_{\text{random}}) > 0.$$

However, such a prior distribution $\pi(f)$ does not exist. The density of a probability distribution cannot strictly increase forever; otherwise, the integral $\int_f \pi(f)\, \mathrm{d}f$ would be infinite.

To overcome this challenge, one idea is to incorporate optimization into the Bayesian methodology of generator inference. Wei et al. [86], for instance, frame generator inference as an optimization problem. They develop a fuzzer for (deterministic) program-input generators, called Singularity [86], where the fuzzer runs a genetic algorithm on abstract syntax trees of generators to find a worst-case generator.

3 Timeline

Table 1: Thesis timeline.

Semester	Plan
Fall 2024	Submit the resource-decomposition paper to PLDI 2025
Spring 2025	Thesis proposal
	Complete the design of the DSL for input generators and implement a prototype
	Resubmit the resource-decomposition paper to ICFP or OOPSLA 2025 (if necessary)
	Complete the speaking-skill requirement
	Complete a thesis draft
Summer 2025	Thesis defense

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