15-453

FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY

NP-COMPLETENESS: THE COOK-LEVIN THEOREM

TUESDAY March 25

Theorem (Cook-Levin): SAT is NP-complete

Corollary: SAT $\in$ P if and only if P = NP

SAT = { ϕ | ϕ is a satisfiable boolean formula }

3-SAT = { ϕ | ϕ is a satisfiable 3cnf-formula }

A 3cnf-formula is of the form:

$$\underbrace{(x_1 \vee \neg x_2 \vee x_3)}_{\text{clauses}} \wedge \underbrace{(x_4 \vee x_2 \vee x_5)}_{\text{clauses}} \wedge \underbrace{(x_3 \vee \neg x_2 \vee \neg x_1)}_{\text{clauses}}$$

Theorem (Cook-Levin): SAT is NP-complete

Proof:

(1) SAT $\in$ NP (also, 3SAT $\in$ NP)

(2) Every language A in NP is polynomial time reducible to SAT

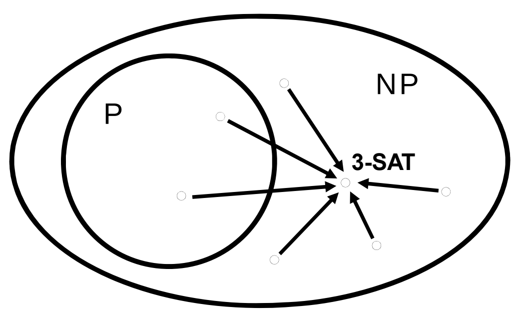
We build a poly-time reduction from A to SAT

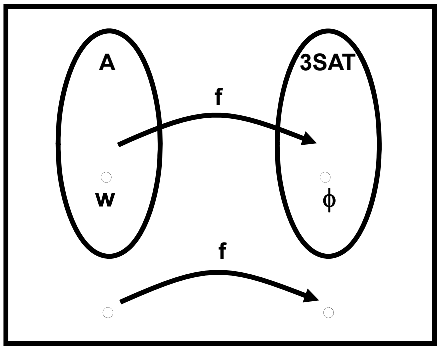
The reduction turns a string w into a 3-cnf formula ϕ such that w $\in$ A iff ϕ $\in$ 3-SAT.

ϕ will *simulate* the NP machine N for A on w.

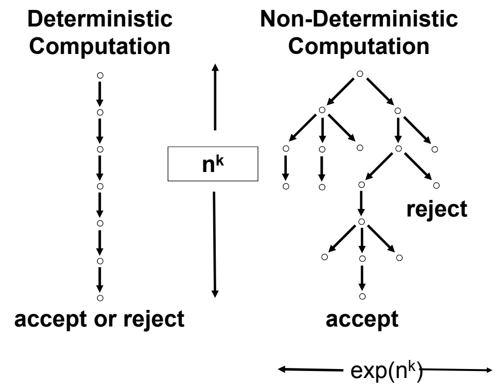
Here N is a non-deterministic TM that decides A in time n^k

So proof will also show:
3-SAT is NP-Complete

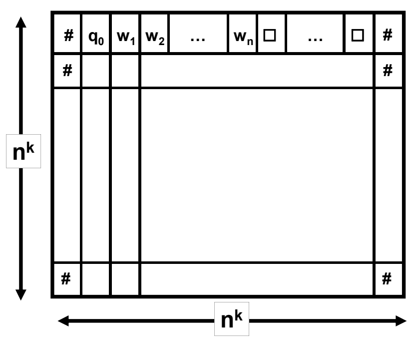




The reduction f turns a string w into a 3-cnf formula ϕ such that: $w \in A \Leftrightarrow \phi \in 3SAT$.
ϕ will simulate the NP machine N for A on w.



A tableau for N on w is an $n^k \times n^k$ table whose rows are the configurations of *some* possible computation of N on input w ($A \in NTIME(n^k)$)



A tableau is accepting if any row of the tableau is an accepting configuration

Determining whether N accepts w is equivalent to determining whether there is an accepting tableau for N on w

Given w, our 3cnf-formula ϕ will describe a *generic* tableau for N on w (in fact, essentially *generic* for N on any string w of length n).

The 3cnf formula ϕ will be satisfiable *if and only if there is* an accepting tableau for N on w.

VARIABLES of ϕ

Let $C = Q \cup \Gamma \cup \{ \# \}$

Each of the $(n^k)^2$ entries of a tableau is a cell

cell[i,j] = the cell at row i and column j

For each i and j ($1 \le i, j \le n^k$) and for each $s \in C$ we have a variable $x_{i,j,s}$

 # variables = $|C|n^{2k}$, ie $O(n^{2k})$, since |C| only depends on N

These are the variables of ϕ and represent the contents of the cells

We will have: $x_{i,j,s} = 1 \Leftrightarrow cell[i,j] = s$

$x_{i,j,s} = 1$

means

cell[i, j] = s

We now design ϕ so that a satisfying assignment to the variables $x_{i,j,s}$ corresponds to an accepting tableau for N on w

The formula ϕ will be the AND of four parts:
$\phi = \phi_{cell} \wedge \phi_{start} \wedge \phi_{accept} \wedge \phi_{move}$

ϕ_{cell} ensures that for each i,j, exactly one $x_{i,j,s} = 1$

ϕ_{start} ensures that the first row of the table is the *starting (initial)* configuration of N on w

ϕ_{accept} ensures* that an accepting configuration occurs somewhere in the table

ϕ_{move} ensures* that every row is a configuration that legally follows from the previous config

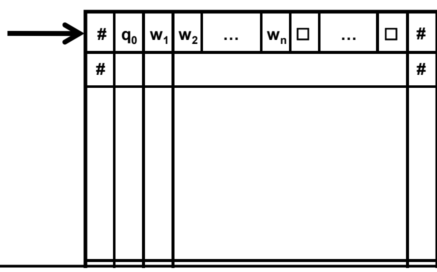
*if the other components of ϕ **hold**

ϕ_{cell} ensures that for each i,j, exactly one $x_{i,j,s}$ = 1

$$\phi_{cell} = \bigwedge_{1 \le i,\, j \le n^k} \left[\underbrace{\left(\bigvee_{s \in C} x_{i,j,s} \right)}_{\text{at least one variable is turned on}} \wedge \underbrace{\left(\bigwedge_{\substack{s,t \in C \\ s \neq t}} (\neg x_{i,j,s} \vee \neg x_{i,j,t}) \right)}_{\text{at most one variable is turned on}} \right]$$

Thus, ϕ_{cell} is satisfiable
(ie, there exist assignment to the variables s.t. ϕ_{cell} evaluates to 1)
$\Leftrightarrow$
each cell in the **tableau** has exactly one symbol (from C.)

$$\phi_{start} = x_{1,1,\#} \wedge x_{1,2,q_0} \wedge x_{1,3,w_1} \wedge x_{1,4,w_2} \wedge \cdots \wedge x_{1,n+2,w_n} \wedge x_{1,n+3,\square} \wedge \cdots \wedge x_{1,n^k-1,\square} \wedge x_{1,n^k,\#}$$



$$\phi_{start} = x_{1,1,\#} \wedge x_{1,2,q_0} \wedge x_{1,3,w_1} \wedge x_{1,4,w_2} \wedge \cdots \wedge x_{1,n+2,w_n} \wedge x_{1,n+3,\square} \wedge \cdots \wedge x_{1,n^k-1,\square} \wedge x_{1,n^k,\#}$$

Thus, ϕ_{start} is satisfiable $\Leftrightarrow$ the first row of the tableau represents the start configuration for N on input w

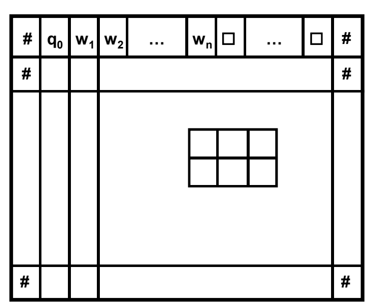
ϕ_{accept} ensures that an accepting configuration occurs somewhere in the table

$$\phi_{accept} = \bigvee_{1 \le i,\, j \le n^k} x_{i,j,q_{accept}}$$

Thus, ϕ_{accept} is satisfiable $\Leftrightarrow$
at least one cell in the tableau has the symbol q_{accept}.

ϕ_{move} ensures that every row is a configuration that legally follows from the previous

It works by ensuring that each 2 $\times$ 3 "window" of cells is legal (does not violate N's rules)



If $\delta(q_1,a) = \{(q_1,b,R)\}$ and $\delta(q_1,b) = \{(q_2,c,L), (q_2,a,R)\}$
Which of the following windows are legal:

a	q_1	b		a	q_1	b		a	a	q_1
q_2	a	c		q_1	a	a		a	a	b
#	b	a		a	b	a		b	q_1	b
#	b	a		a	b	q_2		q_2	b	2
a	b	a		a	q_1	b		b	b	b
a	a	a		a	a	q_2		c	b	b

If $\delta(q_1,a) = \{(q_1,b,R)\}$ and $\delta(q_1,b) = \{(q_2,c,L), (q_2,a,R)\}$
Which of the following windows are legal:

a	q_1	b		a	q_1	b		a	a	q_1
q_2	a	c		q_1	a	a		a	a	b
#	b	a		a	b	a		b	q_1	b
#	b	a		a	b	q_2		q_2	b	q_2
a	b	a		a	q_1	b		b	b	b
a	a	a		a	a	q_2		c	b	b

CLAIM:
If
- **the top row of the tableau is the start configuration, and**
- **and every window is legal,**

Then
each row of the tableau is a configuration that legally follows the preceding one.

	a	
	a	

Proof:
In upper configuration, every cell that doesn't contain the boundary symbol #, is the center top cell of a window.

Case 1. center cell of window is a non-state symbol and not adjacent to a state symbol

CLAIM:
If
- **the top row of the tableau is the start configuration, and**
- **and every window is legal,**

Then
each row of the tableau is a configuration that legally follows the preceding one.

	a				q	
	a			ok	ok	ok

Proof:
In upper configuration, every cell that doesn't contain the boundary symbol #, is the center top cell of a window.

Case 1. center cell of window is a non-state symbol and not adjacent to a state symbol
Case 2. center cell of window is a state symbol

#	q_0	w_1	w_2	w_3	w_4	…	w_n	□	…	□	#
#	ok	ok	w_2	w_3	w_4						#

#	q_0	w_1	w_2	w_3	w_4	…	w_n	□	…	□	#
#	ok	ok	w_2	w_3	w_4						#

#	q_0	w_1	w_2	w_3	w_4	…	w_n	□	…	□	#
#	ok	ok	w_2	w_3	w_4						#

#	a_1	q	a_2	a_3	a_4	a_5	…	a_n	□	…	□	#
#	ok	ok	ok	a_3	a_4	a_5						#

#	a_1	q	a_2	a_3	a_4	a_5	…	a_n	□	…	□	#
#	ok	ok	ok	a_3	a_4	a_5						#

#	a_1	q	a_2	a_3	a_4	a_5	…	a_n	□	…	□	#
#	ok	ok	ok	a_3	a_4	a_5						#

So the lower configuration follows from the upper!!!

	col. j-1	col. j	col. j+1
row i	**(i,j-1)** **a_1**	**(i,j)** **a_2**	**(i,j+1)** **a_3**
row i+1	**(i+1,j-1)** **a_4**	**(i+1,j)** **a_5**	**(i+1,j+ 1)** **a_6**

The (i,j) Window

$$\phi_{move} = \bigwedge_{1 \le i, j \le n^k} (\text{ the } (i, j) \text{ window is legal })$$

the (i, j) window is legal =

$$\bigvee_{\substack{a_1, \dots, a_6 \\ \text{is a legal window}}} (x_{i,j-1,a_1} \wedge x_{i,j,a_2} \wedge x_{i,j+1,a_3} \wedge x_{i+1,j-1,a_4} \wedge x_{i+1,j,a_5} \wedge x_{i+1,j+1,a_6})$$

This is a disjunct over all ($\le |C|^6$) legal sequences (a_1, …, a_6).

This disjunct is satisfiable
⇔
There is *some* assignment to the cells (ie variables) in the window (i,j) that makes the window legal

$$\phi_{move} = \bigwedge_{1 \le i, j \le n^k} (\text{ the (i, j) window is legal })$$

the (i, j) window is legal =

$$\bigvee_{\substack{a_1, \ldots, a_6 \\ \text{is a legal window}}} (x_{i,j-1,a_1} \wedge x_{i,j,a_2} \wedge x_{i,j,+1,a_3} \wedge x_{i+1,j-1,a_4} \wedge x_{i+1,j,a_5} \wedge x_{i+1,j+1,a_6})$$

This is a disjunct over all ($\le |C|^6$) legal sequences ($a_1, \ldots, a_6$).

So ϕ_{move} is satisfiable
⇔
There is *some* assignment to each of the variables that makes *every* window legal.

$$\phi_{move} = \bigwedge_{1 \le i, j \le n^k} (\text{ the (i, j) window is legal })$$

the (i, j) window is legal =

$$\bigvee_{\substack{a_1, \ldots, a_6 \\ \text{is a legal window}}} (x_{i,j-1,a_1} \wedge x_{i,j,a_2} \wedge x_{i,j,+1,a_3} \wedge x_{i+1,j-1,a_4} \wedge x_{i+1,j,a_5} \wedge x_{i+1,j+1,a_6})$$

This is a disjunct over all ($\le |C|^6$) legal sequences ($a_1, \ldots, a_6$).

Can re-write as equivalent conjunct:

$$\equiv \bigwedge_{\substack{a_1, \ldots, a_6 \\ \text{ISN'T a legal window}}} (\bar{x}_{i,j-1,a_1} \vee \bar{x}_{i,j,a_2} \vee \bar{x}_{i,j,+1,a_3} \vee \bar{x}_{i+1,j-1,a_4} \vee \bar{x}_{i+1,j,a_5} \vee \bar{x}_{+1,j+1,a_6})$$

This is a conjunct over all ($\le |C|^6$) illegal sequences ($a_1, \ldots, a_6$).

$$\phi = \phi_{cell} \wedge \phi_{start} \wedge \phi_{accept} \wedge \phi_{move}$$

ϕ is satisfiable (ie, there is some assignment to each of the varialbes s.t. ϕ evaluates to 1)
⇔
there is some assignment to each of the variables s.t. ϕ_{cell} and ϕ_{start} and ϕ_{accept} and ϕ_{move} each evaluates to 1
⇔
There is some assignment of symbols to cells in the tableau such that:
- **The first row of the tableau is a start configuration and**
- **Every row of the tableau is a configuration that follows from the preceding by the rules of N and**
- **One row is an accepting configuration**

,⇔
There is some accepting computation for N with input w

$$\phi = \phi_{cell} \wedge \phi_{start} \wedge \phi_{accept} \wedge \phi_{move}$$

WHAT'S THE LENGTH OF ϕ?

$$\phi = \phi_{cell} \wedge \phi_{start} \wedge \phi_{accept} \wedge \phi_{move}$$

$$\phi_{cell} = \bigwedge_{1 \le i, j \le n^k} \left[\left(\bigvee_{s \in C} x_{i,j,s} \right) \wedge \left(\bigwedge_{\substack{s,t \in C \\ s \neq t}} (\neg x_{i,j,s} \vee \neg x_{i,j,t}) \right) \right]$$

$O(n^{2k})$ clauses

$$\text{Length}(\phi_{cell}) = O(n^{2k}) \underbrace{O(\log n^k)}_{\text{length(indices)}} = O(n^{2k} \log n)$$

$$\phi = \phi_{cell} \wedge \phi_{start} \wedge \phi_{accept} \wedge \phi_{move}$$

$$\begin{aligned}\phi_{start} = \; & x_{1,1,\#} \wedge x_{1,2,q_0} \wedge \\ & x_{1,3,w_1} \wedge x_{1,4,w_2} \wedge \cdots \wedge x_{1,n+2,w_n} \wedge \\ & x_{1,n+3,\square} \wedge \cdots \wedge x_{1,n^k-1,\square} \wedge x_{1,n^k,\#}\end{aligned}$$

$O(n^k)$

$$\phi_{accept} = \bigvee_{1 \le i,\, j \le n^k} x_{i,j,q_{accept}}$$

$$O(n^{2k})$$

$$\phi_{move} = \bigwedge_{1 \le i,\, j \le n^k} (\text{ the (i, j) window is legal })$$

the (i, j) window is legal =

$$\bigwedge_{\substack{a_1, \ldots, a_6 \\ \text{ISN'T a legal window}}} (\overline{x}_{i,j-1,a_1} \vee \overline{x}_{i,j,a_2} \vee \overline{x}_{i,j,+1,a_3} \vee \overline{x}_{i+1,j-1,a_4} \vee \overline{x}_{i+1,j,a_5} \vee \overline{x}_{i+1,j+1,a_6})$$

This is a conjunct over all ($\le |C|^6$) illegal sequences ($a_1, \ldots, a_6$).

$$O(n^{2k})$$

Theorem (Cook-Levin): SAT is NP-complete

Corollary: SAT $\in$ P if and only if P = NP

Theorem (Cook-Levin): 3SAT is NP-complete

Corollary: 3SAT $\in$ P if and only if P = NP

3-SAT?

How do we convert the whole thing into a 3-cnf formula?

Everything was an AND of ORs
We just need to make those ORs with 3 literals

If a clause has less than three variables:

$a \equiv (a \vee a \vee a)$, $(a \vee b) \equiv (a \vee b \vee b)$

If a clause has more than three variables:

$(a \vee b \vee c \vee d) \equiv (a \vee b \vee z) \wedge (\neg z \vee c \vee d)$

$(a_1 \vee a_2 \vee \ldots \vee a_t) \equiv$

$(a_1 \vee a_2 \vee z_1) \wedge (\neg z_1 \vee a_3 \vee z_2) \wedge \ldots (\neg z_{t-3} \vee a_{t-1} \vee z_t)$

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