15-453

FORMAL LANGUAGES, AUTOMATA AND COMPUTABILITY

THE ARITHMETIC HIERARCHY

THURSDAY, MAR 6

ORACLE MACHINES

An ORACLE is a set B to which the TM may pose membership questions "Is w in B?"
(formally: TM enters state $q_?$)
and the TM always receives a correct answer in one step
(formally: if the string on the "oracle tape" is in B, state $q_?$ is changed to q_{YES}, otherwise q_{NO})

This makes sense even if B is not decidable!
(We do not assume that the oracle B is a computable set!)

We say A is semi-decidable in B
if there is an oracle TM M with oracle B that semi-decides A

We say A is decidable in B
if there is an oracle TM M with oracle B that decides A

THE ARITHMETIC HIERARCHY

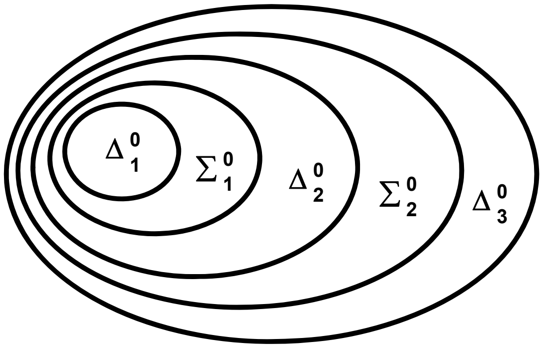
Δ^0_1 **= { decidable sets }   (sets = languages)**

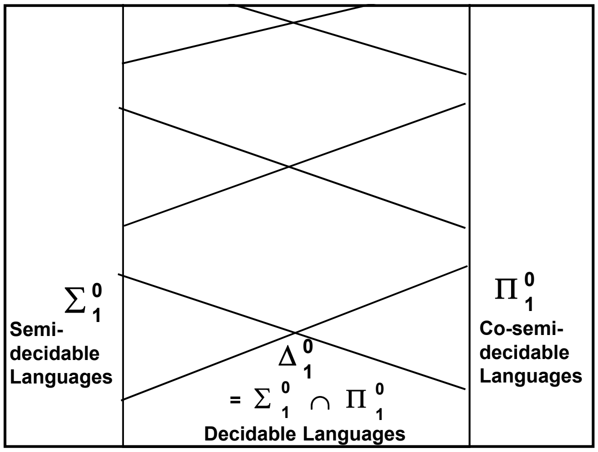
Σ^0_1 **= { semi-decidable sets }**

Σ^0_{n+1} **= { sets semi-decidable in some B $\in \Sigma^0_n$ }**

Δ^0_{n+1} **= { sets decidable in some B $\in \Sigma^0_n$ }**

Π^0_n **= { complements of sets in Σ^0_n }**





Σ^0_1

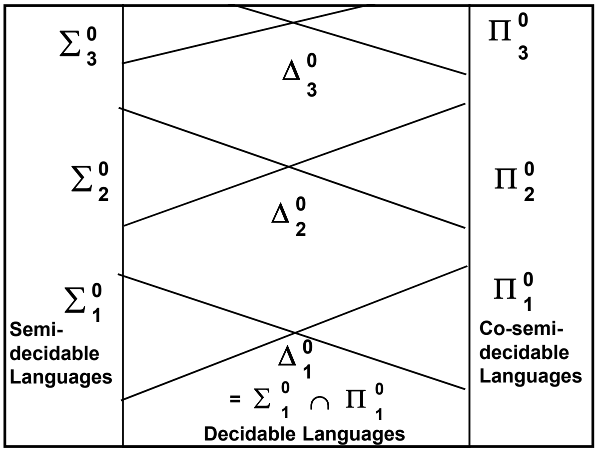
Semi-decidable Languages

Π^0_1

Co-semi-decidable Languages

$\Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1$

Decidable Languages



Σ^0_3 Δ^0_3 Π^0_3

Σ^0_2 Δ^0_2 Π^0_2

Σ^0_1

Semi-decidable Languages

Π^0_1

Co-semi-decidable Languages

$\Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1$

Decidable Languages

Definition: A decidable predicate R(x,y) is some proposition about x and y[1], where there is a TM M such that

for all x, y, R(x,y) is TRUE ⇒ M(x,y) accepts
R(x,y) is FALSE ⇒ M(x,y) rejects

We say M "decides" the predicate R.

EXAMPLES:
R(x,y) = "x + y is less than 100"
R(<N>,y) = "N halts on y in at most 100 steps"
Kleene's T predicate, T(<M>, x, y): M accepts x in y steps.

[1] x, y are positive integers or elements of Σ*

Definition: A decidable predicate R(x,y) is some proposition about x and y[1], where there is a TM M such that

for all x, y, R(x,y) is TRUE ⇒ M(x,y) accepts
R(x,y) is FALSE ⇒ M(x,y) rejects

We say M "decides" the predicate R.

EXAMPLES:
R(x,y) = "x + y is less than 100"
R(<N>,y) = "N halts on y in at most 100 steps"
Kleene's T predicate, T(<M>, x, y): M accepts x in y steps.

[1] x, y are positive integers or elements of Σ*

Definition: A decidable predicate R(x,y) is some proposition about x and y[1], where there is a TM M such that

for all x, y, R(x,y) is TRUE ⇒ M(x,y) accepts
R(x,y) is FALSE ⇒ M(x,y) rejects

We say M "decides" the predicate R.

EXAMPLES:
R(x,y) = "x + y is less than 100"
R(<N>,y) = "N halts on y in at most 100 steps"
Kleene's T predicate, T(<M>, x, y): M accepts x in y steps.

Note: A is decidable ⇔ A = {x | R(x,ε)},
for some decidable predicate R.

Theorem: A language A is semi-decidable if and only if there is a decidable predicate R(x, y) such that A = { x | ∃y R(x,y) }

Proof:

(1) If A = { x | ∃y R(x,y) } then A is semi-decidable

(2) If A is semi-decidable, then A = { x | ∃y R(x,y) }

Theorem: A language A is semi-decidable if and only if there is a decidable predicate R(x, y) such that: $A = \{ x \mid \exists y\, R(x,y) \}$

Proof:

(1) If $A = \{ x \mid \exists y\, R(x,y) \}$ then A is semi-decidable

Because we can enumerate over all y's

(2) If A is semi-decidable, then $A = \{ x \mid \exists y\, R(x,y) \}$

Theorem: A language A is semi-decidable if and only if there is a decidable predicate R(x, y) such that: $A = \{ x \mid \exists y\, R(x,y) \}$

Proof:

(1) If $A = \{ x \mid \exists y\, R(x,y) \}$ then A is semi-decidable

Because we can enumerate over all y's

(2) If A is semi-decidable, then $A = \{ x \mid \exists y\, R(x,y) \}$

Let M semi-decide A

Then, $A = \{ x \mid \exists y\, T(\langle M\rangle, x, y) \}$ **(Here M is fixed.)**

where

Kleene's T predicate, $T(\langle M\rangle, x, y)$: M accepts x in y steps.

Theorem

Σ^0_1 **= { semi-decidable sets }**

= languages of the form $\{ x \mid \exists y\, R(x,y) \}$

Π^0_1 **= { complements of semi-decidable sets }**

= languages of the form $\{ x \mid \forall y\, R(x,y) \}$

Δ^0_1 **= { decidable sets }**

$= \Sigma^0_1 \cap \Pi^0_1$

Where R is a decidable predicate

Theorem

Σ^0_2 **= { sets semi-decidable in some semi-dec. B }**

= languages of the form $\{ x \mid \exists y_1 \forall y_2\, R(x,y_1,y_2) \}$

Π^0_2 **= { complements of Σ^0_2 sets}**

= languages of the form $\{ x \mid \forall y_1 \exists y_2\, R(x,y_1,y_2) \}$

$\Delta^0_2 = \Sigma^0_2 \cap \Pi^0_2$

Where R is a decidable predicate

Theorem

Σ^0_n **= languages $\{ x \mid \exists y_1 \forall y_2 \exists y_3 \ldots Q y_n\, R(x,y_1,\ldots,y_n) \}$**

Π^0_n **= languages $\{ x \mid \forall y_1 \exists y_2 \forall y_3 \ldots Q y_n\, R(x,y_1,\ldots,y_n) \}$**

$\Delta^0_n = \Sigma^0_n \cap \Pi^0_n$

Where R is a decidable predicate

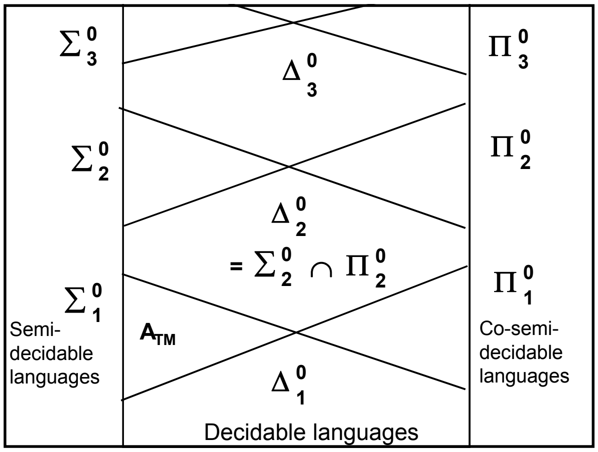
Example

Decidable predicate

Σ^0_1 **= languages of the form $\{ x \mid \exists y \boxed{R(x,y)} \}$**

We know that A_{TM} is in Σ^0_1 Why?

Show it can be described in this form:



Π^0_1 **= languages of the form { x | ∀y R(x,y) }**

Show that EMPTY (ie, E_{TM}) = { M | L(M) = ∅ } is in Π^0_1

Π^0_1 **= languages of the form { x | ∀y R(x,y) }**

Show that EMPTY (ie, E_{TM}) = { M | L(M) = ∅ } is in Π^0_1

$$\textbf{EMPTY} = \{ M \mid \underbrace{\forall w \forall t}_{\text{two quantifiers??}} [\, \underbrace{\neg T(\langle M\rangle, w, t)}_{\text{decidable predicate}} \,] \}$$

THE PAIRING FUNCTION

Theorem. There is a 1-1 and onto computable function < , >: Σ* x Σ* → Σ* and computable functions π_1 and π_2 : Σ* → Σ* such that

$z = \langle w, t\rangle \Rightarrow \pi_1(z) = w$ and $\pi_2(z) = t$

THE PAIRING FUNCTION

Theorem. There is a 1-1 and onto computable function < , >: Σ* x Σ* → Σ* and computable functions π_1 and π_2 : Σ* → Σ* such that

$z = \langle w, t\rangle \Rightarrow \pi_1(z) = w$ and $\pi_2(z) = t$

EMPTY = { M | ∀w∀t [M doesn't accept w in t steps] }

EMPTY = { M | ∀z[M doesn't accept π_1 (z) in π_2(z) steps]}

EMPTY = { M | ∀z[¬T(<M>, π_1 (z) , π_2(z))] }

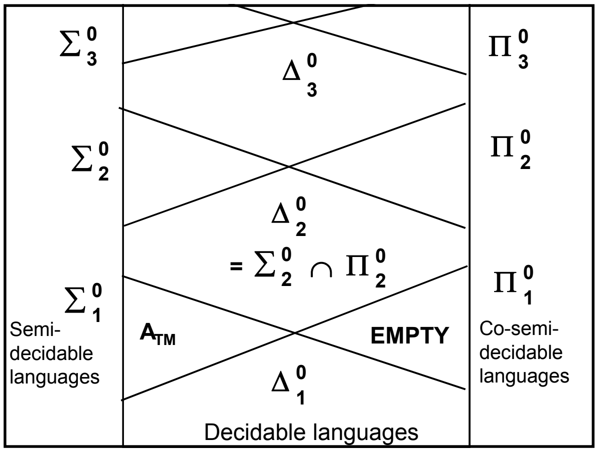
THE PAIRING FUNCTION

Theorem. There is a 1-1 and onto computable function < , >: Σ* x Σ* → Σ* and computable functions π_1 and π_2 : Σ* → Σ* such that

$z = \langle w, t\rangle \Rightarrow \pi_1(z) = w$ and $\pi_2(z) = t$

Proof: Let $w = w_1 \ldots w_n \in \Sigma^*$, $t \in \Sigma^*$.
Let a, b ∈ Σ, a ≠ b.
$\langle w, t\rangle := a\, w_1 \ldots a\, w_n\, b\, t$

π_1 (z) := "if z has the form $a\, w_1 \ldots a\, w_n\, b\, t$, then output $w_1 \ldots w_n$, else output ε"
π_2(z) := "if z has the form $a\, w_1 \ldots a\, w_n\, b\, t$, then output t, else output ε"



Σ^0_3

Δ^0_3

Π^0_3

Σ^0_2

Π^0_2

Δ^0_2

$= \Sigma^0_2 \cap \Pi^0_2$

Σ^0_1

Semi-decidable languages

A_{TM}

EMPTY

Π^0_1

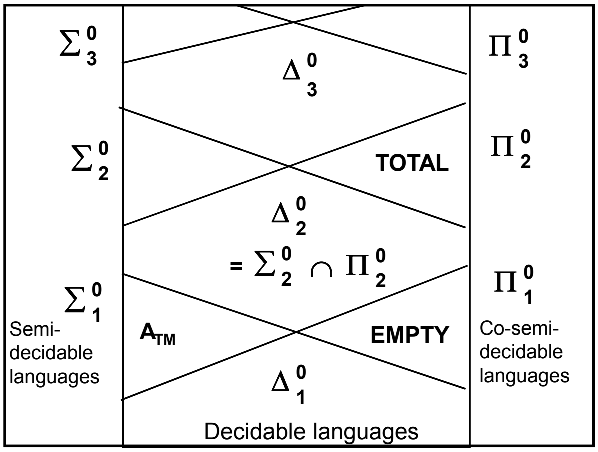
Co-semi-decidable languages

Δ^0_1

Decidable languages

Π^0_2 = languages of the form { x | ∀y∃z R(x,y,z) }

Show that TOTAL = { M | M halts on all inputs } is in Π^0_2



Σ^0_3

Δ^0_3

Π^0_3

Σ^0_2

TOTAL

Π^0_2

Δ^0_2

$= \Sigma^0_2 \cap \Pi^0_2$

Σ^0_1

Semi-decidable languages

A_{TM}

EMPTY

Π^0_1

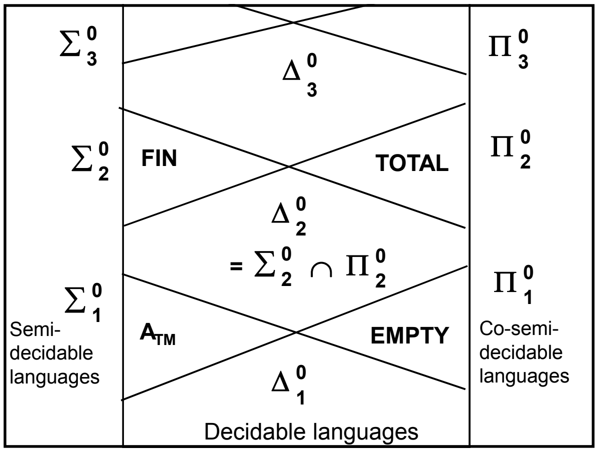
Co-semi-decidable languages

Δ^0_1

Decidable languages

Σ^0_2 = languages of the form { x | ∃y∀z R(x,y,z) }

Show that FIN = { M | L(M) is finite } is in Σ^0_2



Σ^0_3

Δ^0_3

Π^0_3

Σ^0_2

FIN

TOTAL

Π^0_2

Δ^0_2

$= \Sigma^0_2 \cap \Pi^0_2$

Σ^0_1

Semi-decidable languages

A_{TM}

EMPTY

Π^0_1

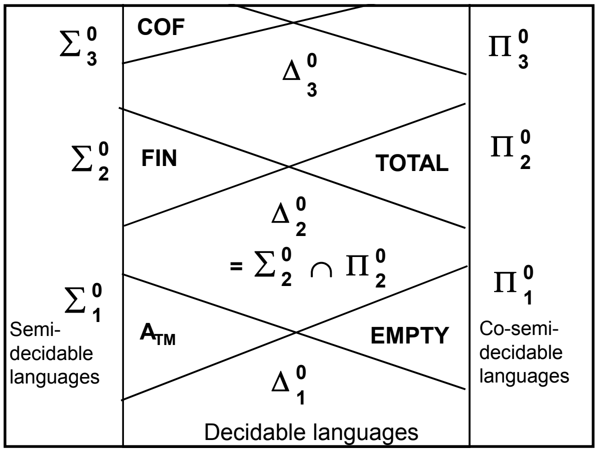
Co-semi-decidable languages

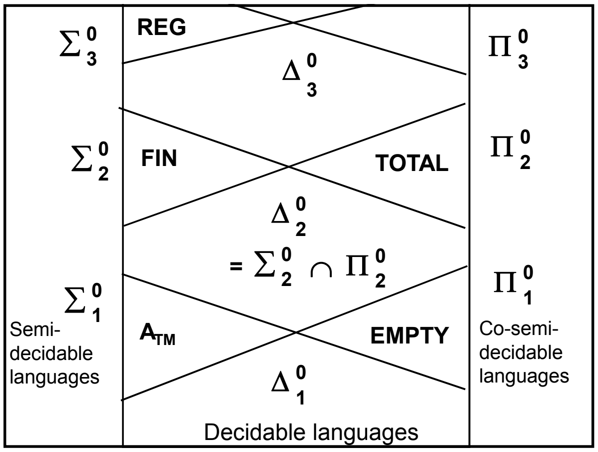
Δ^0_1

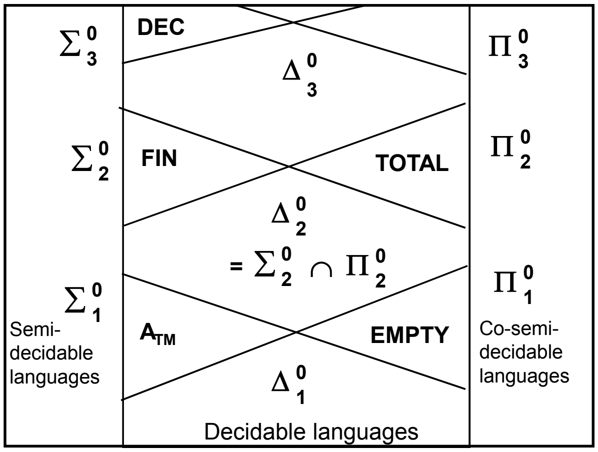
Decidable languages

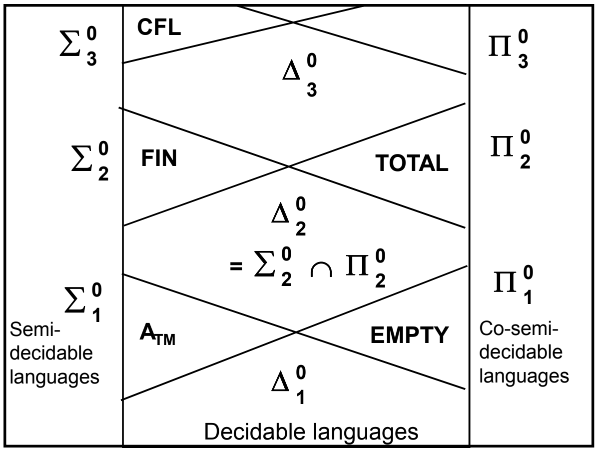
Σ^0_3 = languages of the form { x | ∃y∀z∃u R(x,y,z,u)

Show that COF = { M | L(M) is cofinite } is in Σ^0_2









Each is m-complete for its level in hierachy and cannot go lower (by the SuperHalting Theorem, which shows the hierarchy does not collapse).

L is m-complete for class C if
i) L $\in$ C and
ii) L is m-hard for C,

ie, for all L' $\in$ C , L' $\leq_m$ L

A_{TM} is m-complete for class C = Σ_1^0

i) $A_{TM} \in$ C

ii) A_{TM} is m-hard for C,

Suppose L $\in$ C . Show: L $\leq_m A_{TM}$

Let M semi-decide L. Then Map
$\Sigma^* \rightarrow \Sigma^*$

where w → (M, w).

Then, w $\in$ L $\Leftrightarrow$ (M,w) $\in A_{TM}$ QED

FIN is m-complete for class C = Σ^0_2

i) FIN ∈ C
ii) FIN is m-hard for C:

Suppose L ∈ C . Show: L $\leq_m$ FIN

So suppose L= { w | ∃y∀z R(w,y,z) }
where R is decided by some TM D

Map $\Sigma^* \to \Sigma^*$
where w → $N_{D,w}$

Supose L ∈ Σ^0_2 ie L= { w | ∃y∀z R(w,y,z) }
where R is decided by some TM D

Show: L $\leq_m$ FIN

Map $\Sigma^* \to \Sigma^*$
where w → $N_{D,w}$

Define $N_{D,w}$ On input s:

1. Write down all strings y of length |s|
2. For each y, try to find a z such that ¬ R(w, y, z) and accept if all are successful (here use D and w)

So, w ∈ L ⇔ $N_{D,w}$ ∈ FIN

ORACLES not all powerful

The following problem cannot be decided, **even by a TM with an oracle for the Halting Problem:**

SUPERHALT = { (M,x) | M, with an oracle for the Halting Problem, halts on x}

Can use diagonalization here!

Suppose H decides SUPERHALT (with oracle)

Define **D(X) = "if H(X,X) accepts (with oracle) then LOOP, else ACCEPT."**

D(D) halts ⇔ H(D,D) accepts ⇔ D(D) loops…

ORACLES not all powerful

Theorem: The arithmetic hierarchy is strict. That is, the nth level contains a language that isn't in any of the levels below n.

Proof IDEA: Same idea as the previous slide.

SUPERHALT^0 = HALT = { (M,x) | M halts on x}.

SUPERHALT^1 = { (M,x) | M, with an oracle for the Halting Problem, halts on x}

SUPERHALT^n = { (M,x) | M, with an oracle for SUPERHALT^{n-1}, halts on x}

WWW.FLAC.WS

Read Chapter 6.4 for next time