Covering Spaces in Homotopy Type Theory

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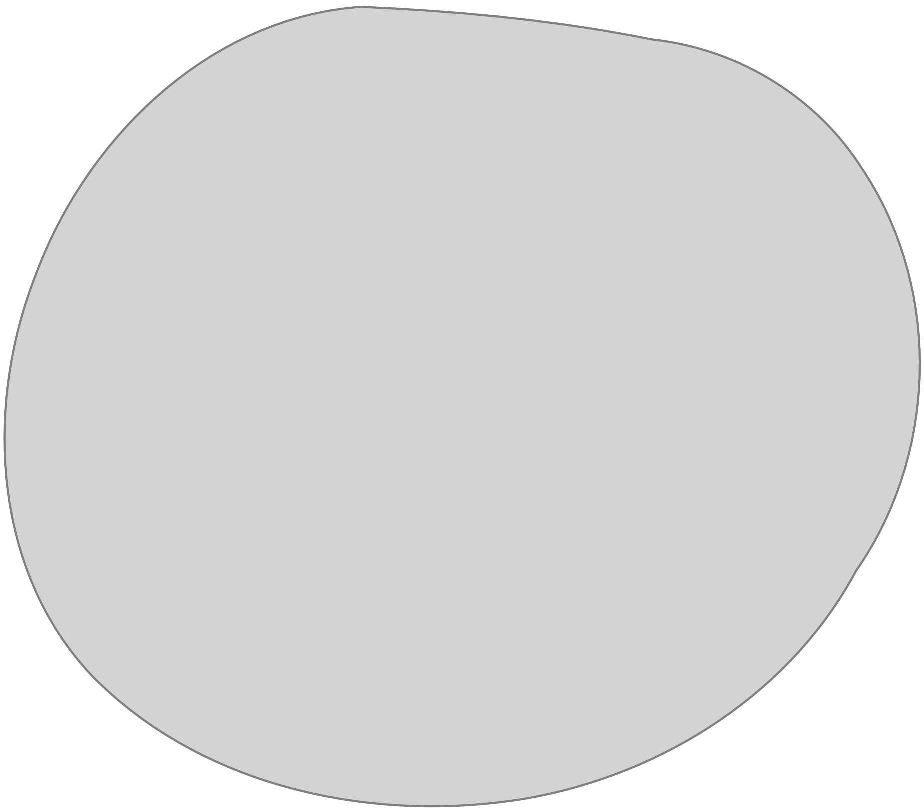
Homotopy Type Theory (HoTT)

A	Type	Space
$a : A$	Term	Point
$f : A \longrightarrow B$	Function	Continuous Mapping
$C : A \longrightarrow Type$	Dependent Type	Fibration
$C(a)$		Fiber
$a =_A b$	Identity	Path

Every type is an ∞-groupoid



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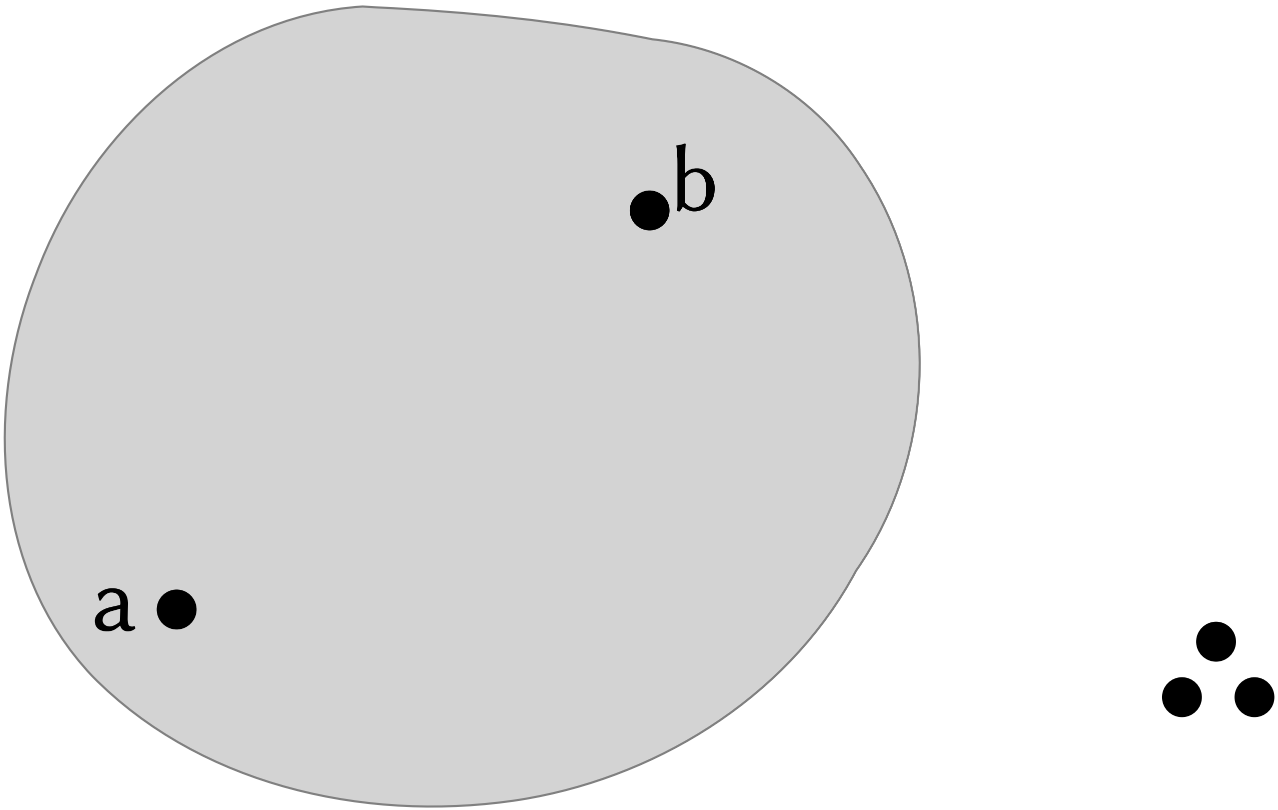

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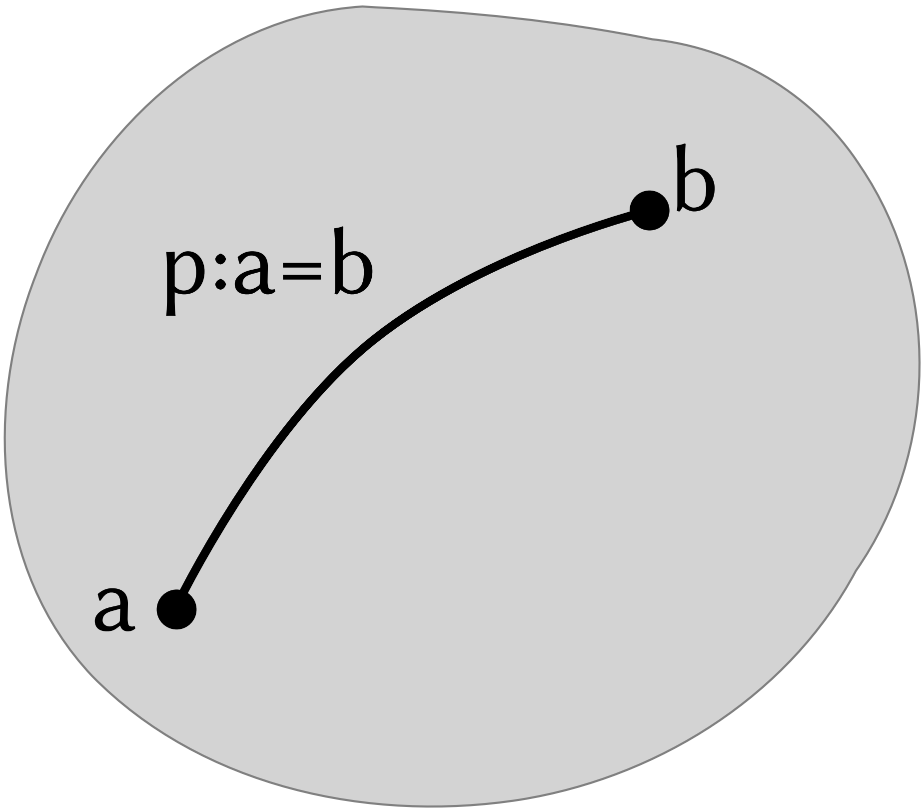

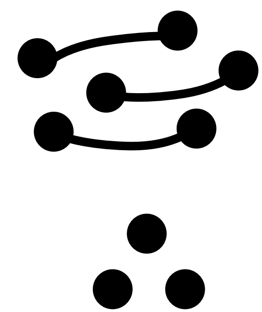

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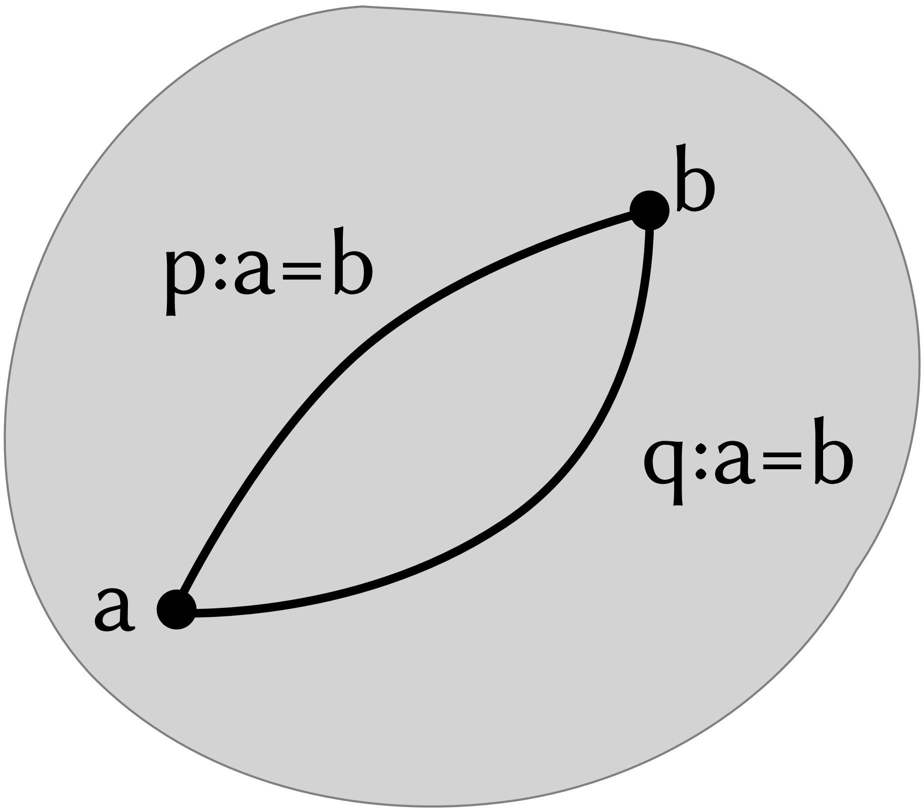

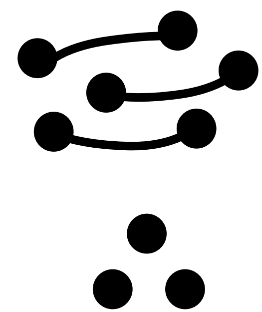

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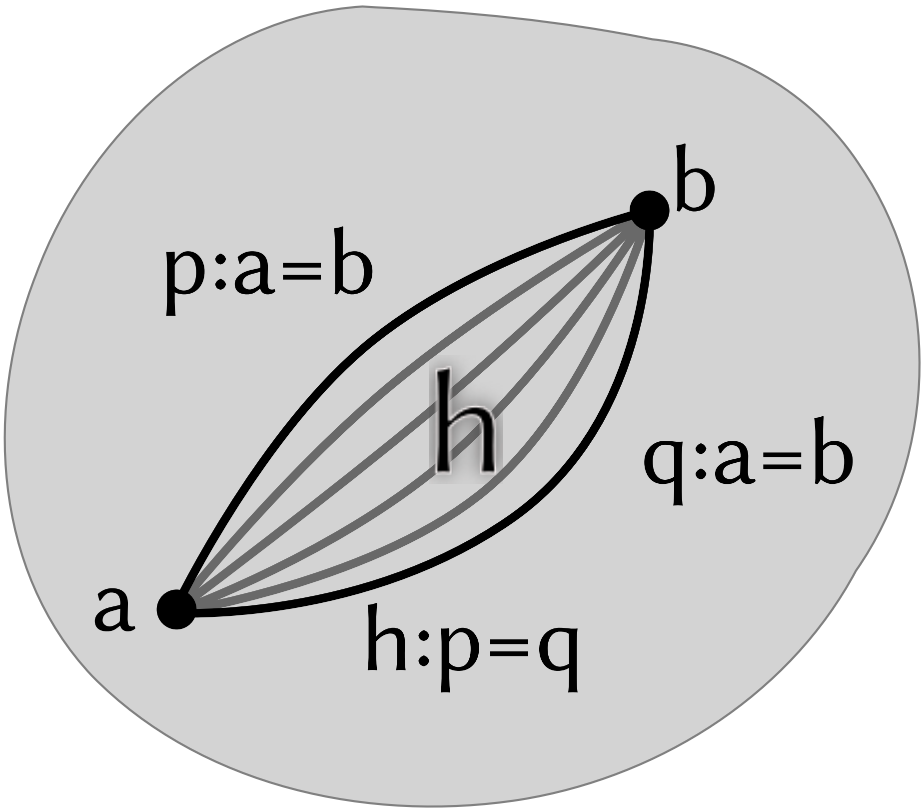
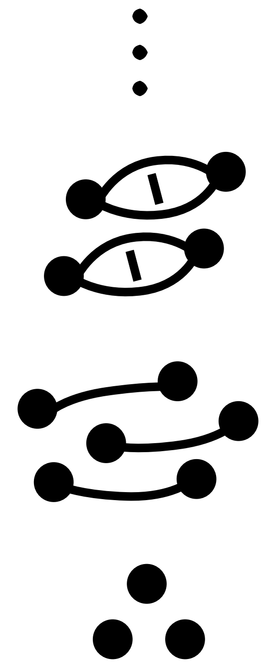


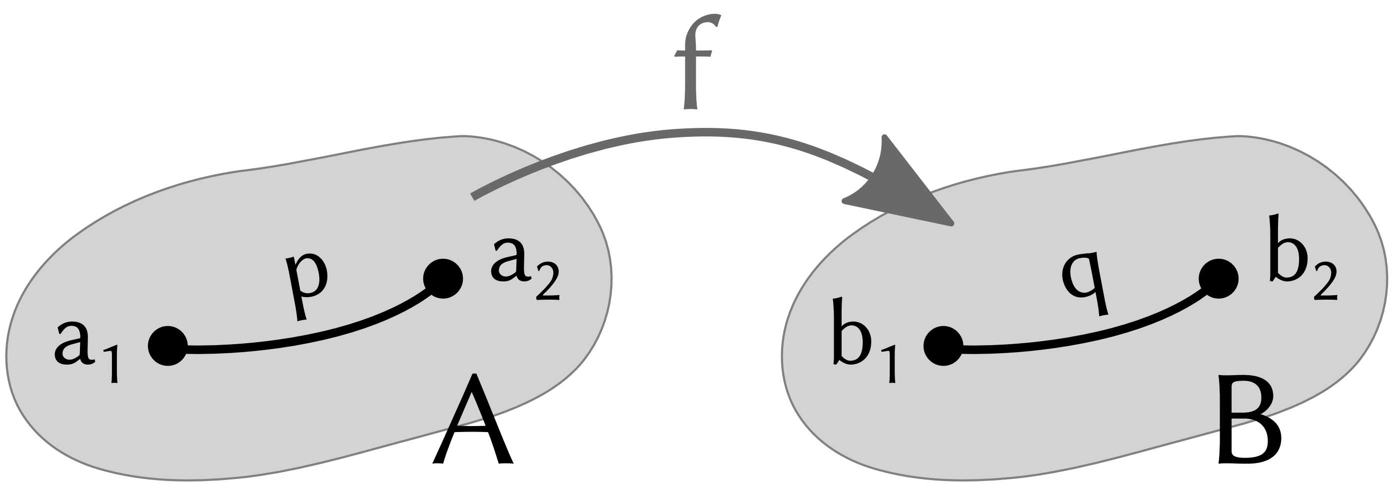


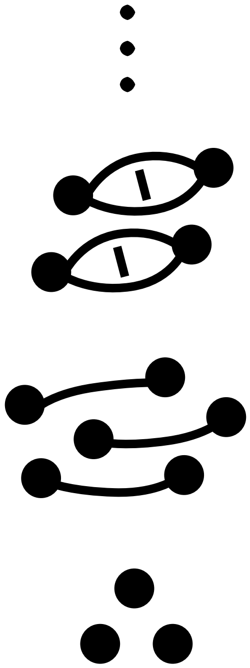
$$f : A \longrightarrow B$$

$$a : A \quad \longmapsto \quad b : B$$

$$p : a_1 = a_2 \quad \longmapsto \quad q : b_1 = b_2$$

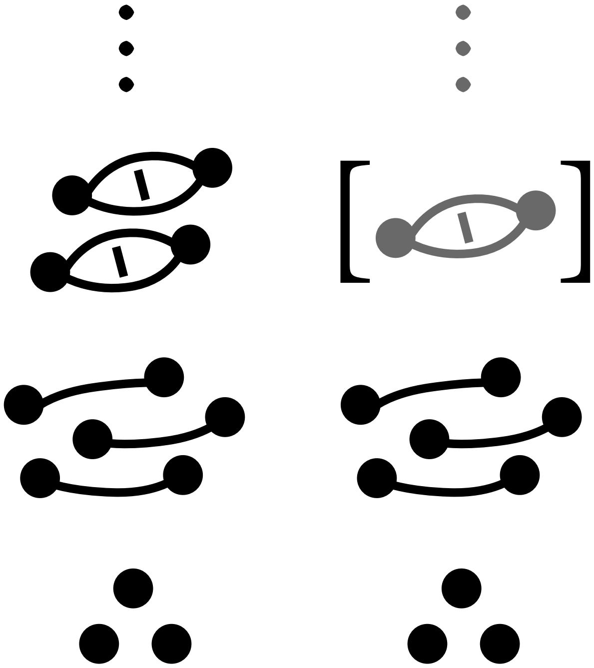
$$\vdots$$





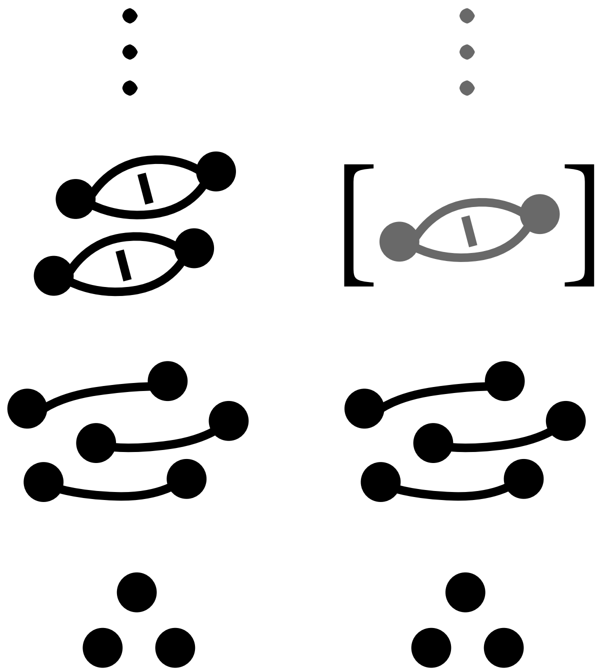
type

A

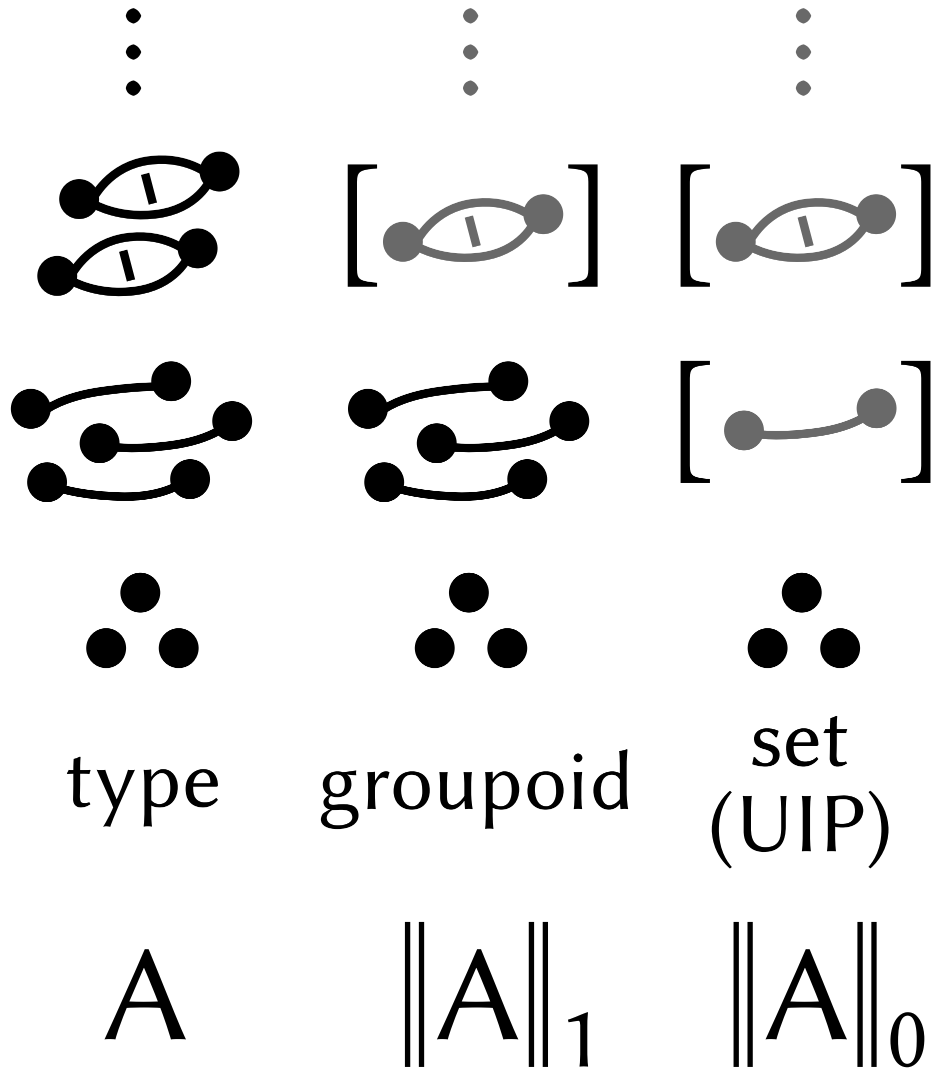


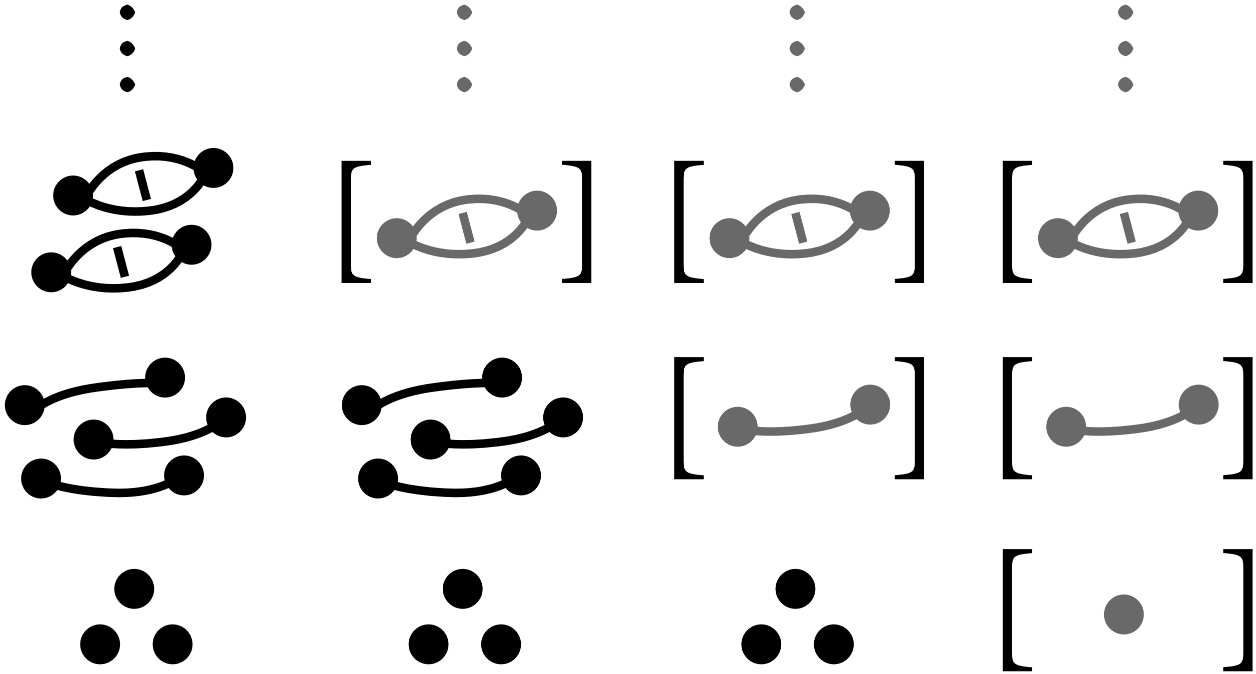
type groupoid

A



A $\qquad$ $\|A\|_1$


type
groupoid
set
(UIP)
A
$\|A\|_1$
$\|A\|_0$



type	groupoid	set (UIP)	prop. (squash)
A	$\|A\|_1$	$\|A\|_0$	$\|A\|_{-1}$

Covering Spaces

Continuously changing families of sets

Classical definition:

A covering space of A is a space C together with a continuous surjective map $p : C \longrightarrow A$, such that for every $a \in A$, there exists an open neighborhood U of a, such that $p^{-1}(U)$ is a union of disjoint open sets in A, each of which is mapped homeomorphically onto U by p.

HoTT definition:

$$F : A \longrightarrow \mathrm{Set}$$

Question: Is it correct (up to homotopy)?

Covering Spaces

$$F : A \longrightarrow \mathbf{Set}$$

$$\begin{array}{rcl} a : A & \longmapsto & F(a) : \mathrm{Set} \\ p : a_1 = a_2 & \longmapsto & \mathrm{iso} : F(a_1) = F(a_2) \\ q : p_1 = p_2 & \longmapsto & \text{(trivial)} \\ & \vdots & \end{array}$$

Classification Theorem

Suppose A is pointed (a_0) and connected.

$$F : A \to \mathbf{Set}$$

$$\wr\!\wr$$

$$a_0 : A \longmapsto F(a_0) : \mathbf{Set}$$

$$\mathrm{loop} : a_0 = a_0 \longmapsto \mathrm{auto} : F(a_0) = F(a_0)$$

This is an action of $\|a_0=a_0\|_0$ on $F(a_0)$.
$\|a_0=a_0\|_0$ is the fundamental group $\pi_1(A, a_0)$.

Classification Theorem

Suppose A is pointed (a_0) and connected.

$$(A \longrightarrow \mathrm{Set}) \simeq \pi_1(A, a_0)\text{-Set}$$

Pointed (a_0) and connected:

$$(a_0 : A) \times ((x : A) \longrightarrow (y : A) \longrightarrow \|x = y\|_{-1})$$

Fundamental group $\pi_1(A, a_0)$: $\|a_0 = a_0\|_0$

G-Set: $(X : \mathrm{Set}) \times (\alpha : G \longrightarrow (X \longrightarrow X)) \times$
$(\alpha\ \mathrm{unit} = \mathrm{id}) \times (\alpha\ (g_1 \cdot g_2) = \alpha\ g_1 \circ \alpha\ g_2)$

Suppose $a_0 : A$ and $(x : A) \longrightarrow (y : A) \longrightarrow \|x = y\|_{-1}$.

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$$(A \to \mathrm{Set}) \simeq \pi_1(A, a_0)\text{-Set}$$

$$F \longmapsto (F(a_0), \star_0, \ldots)$$

Suppose $a_0 : A$ and $(x : A) \to (y : A) \to \|x = y\|_{-1}$.

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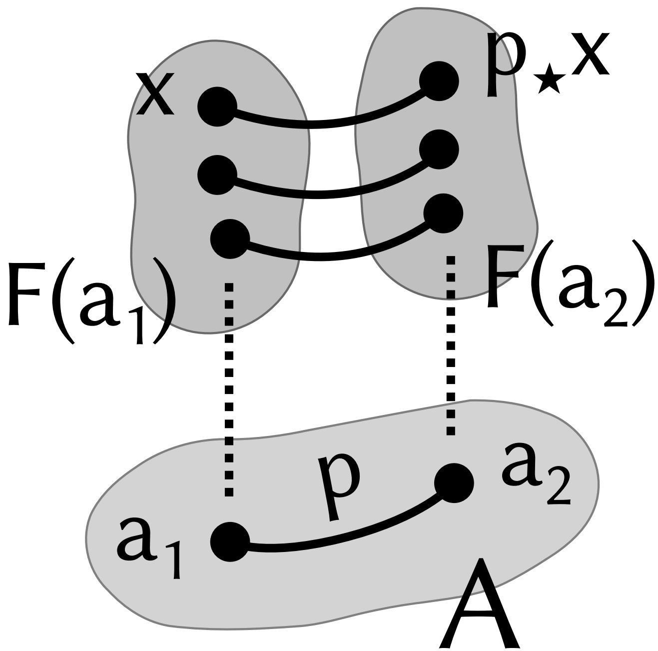


transport x along p ($p_\star x$)

$\star : a_1 = a_2 \to F(a_1) \to F(a_2)$

$\star_0 : \|a_1 = a_2\|_0 \to F(a_1) \to F(a_2)$

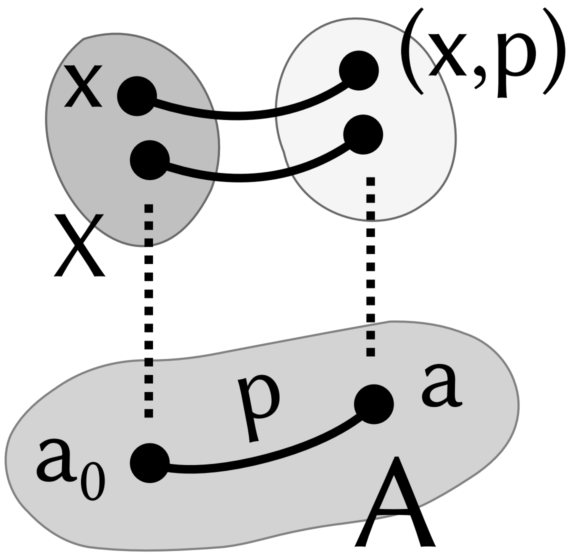
($\star$ for set-truncated paths)

Suppose $a_0 : A$ and $(x : A) \to (y : A) \to \|x = y\|_{-1}$.

$(A \to \mathrm{Set}) \simeq \pi_1(A, a_0)\text{-Set}$

$$F \longmapsto (F(a_0), \star_0, \ldots)$$

$$? \longleftarrow (X, \alpha, -, -)$$



Idea: formal transports

Suppose $a_0 : A$ and $(x : A) \longrightarrow (y : A) \longrightarrow \|x = y\|_{-1}$.

$(A \rightarrow \mathrm{Set}) \simeq \pi_1(A, a_0)\text{-Set}$

$$F \longmapsto (F(a_0), \star_0, \ldots)$$

$$R_{X,\alpha} \longleftarrow\!\!\!\dashv (X, \alpha, -, -)$$

$R_{X,\alpha}(a) :\equiv X \times \|a_0 = a\|_0$ quotiented by *some relation ~.*

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Goal: $F = R_{F(a0),\star 0}$

$F(a) \simeq F(a_0) \times \|a_0 = a\|_0$ quotiented by *some relation ~.*

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$$p_{\star 0}x \quad \longleftarrow\!\!\dashv \quad (x, p)$$

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$$p_{\star 0}x \longleftarrow\!\!\dashv (x, p)$$

$$x \longmapsto (q^{-1}{}_{\star 0}x, q)?$$

We only have $\|a_0 = a\|_{-1}$ but need $q : \|a_0 = a\|_0$.

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We only have $\|a_0 = a\|_{-1}$ but need $q : \|a_0 = a\|_0$.

Lemma: If $(q_1^{-1}{}_{\star 0} x, q_1) = (q_2^{-1}{}_{\star 0} x, q_2)$ then $\|a_0 = a\|_{-1}$ is fine.

Suppose $a_0 : A$ and $(x : A) \longrightarrow (y : A) \longrightarrow \|x = y\|_{-1}$.

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Wants $(q_1{}^{-1}{}_{\star 0}x, q_1) = (q_2{}^{-1}{}_{\star 0}x, q_2)$.

$$(\alpha \text{ loop } x , p) \sim (x , \text{loop} \blacksquare p)$$

Intuition: $p_{\star 0}(\text{loop}_{\star 0}x) = (\text{loop} \blacksquare p)_{\star 0}x$

$$\begin{aligned}(q_1{}^{-1}{}_{\star 0}x, q_1) &= (q_1{}^{-1}{}_{\star 0}x, (q_1 \blacksquare q_2{}^{-1}) \blacksquare q_2)\\ &= ((q_1 \blacksquare q_2{}^{-1})_{\star 0}(q_{1\star 0}x), q_2) = (q_2{}^{-1}{}_{\star 0}x, q_2)\end{aligned}$$

Suppose $a_0 : A$ and $(x : A) \longrightarrow (y : A) \longrightarrow \|x = y\|_{-1}$.

$(A \longrightarrow \text{Set}) \simeq \pi_1(A, a_0)\text{-Set}$

$$F \longmapsto (F(a_0), \star_0, \ldots)$$

$$R_{X,\alpha} \longleftarrow\!\!| \ (X, \alpha, \text{—}, \text{—})$$

$R_{X,\alpha}(a) :\equiv X \times \|a_0 = a\|_0$ quotiented by

$(\alpha \text{ loop } x \text{ , path}) \sim (x \text{ , loop} \cdot \text{path})$

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The other round trip is easy.
(G-sets $\longrightarrow$ covering spaces $\longrightarrow$ G-sets)

Summary

- A simple formulation: $A \longrightarrow \mathrm{Set}$.
- Type equivalence of $A \longrightarrow \mathrm{Set}$ and $\pi_1(A)$-Set.

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Notes

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- Fibers need not to be decidable types.
  - ☞ "path-constant" spaces, not just discrete ones?
- $A \longrightarrow \mathrm{Groupoid}$?

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Thank you

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