Soft Constraints for Vector Field Design

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This note describes how to add "soft constraints" to the algorithm described in Crane et al, *Trivial Connections on Discrete Surfaces* (SGP 2010). By soft constraints we mean that the user provides an additional guidance vector field $\tilde{\beta} \in \mathbb{R}^{|F|}$ and a weight $\lambda > 0$ that determines the influence of this vector field in the final solution; $\lambda = 0$ means that the guidance vector field will have no influence.

Let $\tilde{x} \in \mathbb{R}^{|E|}$ be the discrete connection corresponding to $\tilde{\beta}$, i.e., the smallest difference between angles β_i and β_j along each oriented dual edge e^*_{ij}. The optimization problem originally presented in Equation 2 now becomes

$$\begin{array}{ll} \min\limits_x & ||x||^2 + \lambda||x - x_0||^2 \\ \text{s.t.} & Ax = -b. \end{array}$$

In other words, we penalize the deviation of x from the guidance vector field x_0; λ controls the strength of the penalty term. The task now is to transform this problem into one that can be solved more easily, i.e., by just a standard linear solve. To do so, we expand the objective function and then complete the square as follows:

$$\begin{array}{rcl} ||x||^2 + \lambda||x - x_0||^2 & = & (1+\lambda)||x||^2 - 2\lambda x^T x_0 + \lambda||x_0||^2 \\ & = & ||(1+\lambda)^{1/2}x - \frac{\lambda x_0}{(1+\lambda)^{1/2}}||^2 + (\lambda - \frac{\lambda^2}{1+\lambda})||x_0||^2. \end{array}$$

The second term is independent of the choice of x and can therefore be omitted. To simplify the remaining term, we define

$$a := \sqrt{1+\lambda}$$

and

$$y := \frac{\lambda}{\sqrt{1+\lambda}} x_0$$

at which point our optimization problem becomes

$$\begin{array}{ll} \min\limits_x & ||ax - y||^2 \\ \text{s.t.} & Ax = -b. \end{array}$$

This problem can be further simplified by making the change of variables

$$z := ax - y,$$

which means that

$$x = \frac{1}{a}(y + z)$$

and hence the linear constraint $Ax = b$ becomes $Az = -ab - Ay$. Finally, letting $c := -ab - Ay$ we get the optimization problem

$$\begin{array}{ll} \min\limits_{z} & ||z||^2 \\ \text{s.t.} & Az = c, \end{array}$$

whose form is identical to the original problem, i.e., we need only find the minimum 2-norm solution to a sparse linear system. To summarize, the overall procedure is:

I. Compute $a \leftarrow \sqrt{1 + \lambda}$.

II. Compute $y \leftarrow \frac{\lambda}{\sqrt{1+\lambda}} x_0$.

III. Compute $c \leftarrow -ab - Ay$.

IV. Find the minimum 2-norm solution z^* to $Az = c$ (using sparse QR).

V. Recover the final solution $x \leftarrow \frac{1}{a}(y + z)$.

Note that computationally each of these steps is quite cheap (the most expensive one being the linear solve, as usual).