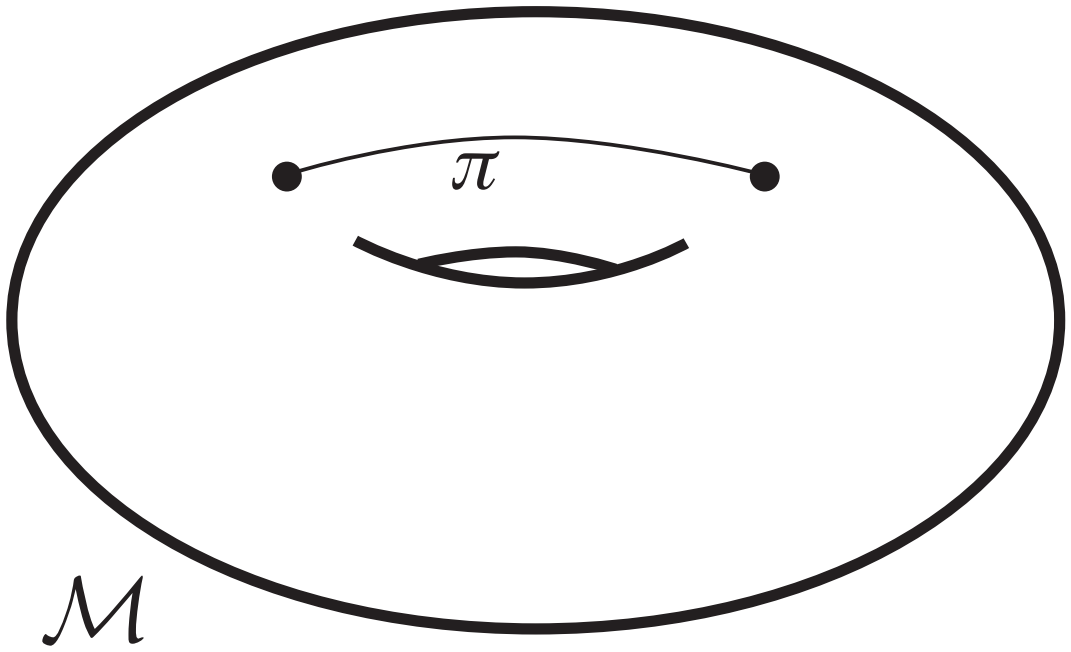
The Story of Lemma 5.6

from

"Optimally Cutting a Surface into a Disk"

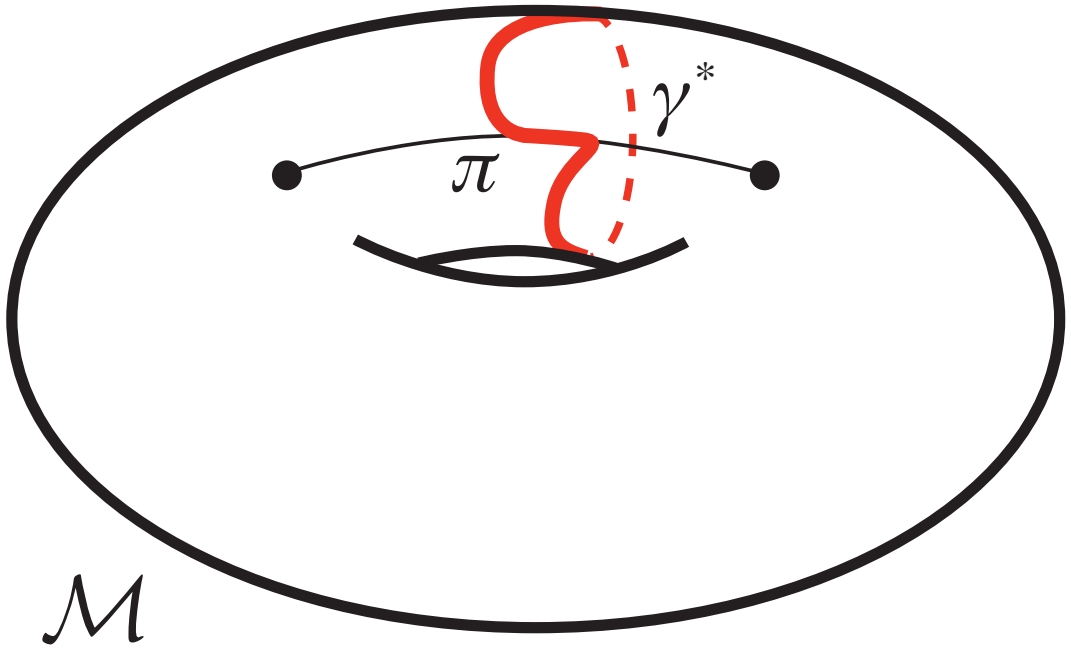
(*An Experiment in Effective Proof Writing)*

Original proof by Jeff Erickson and Sariel Har-Peled.
Pictures and prose by Keenan Crane.



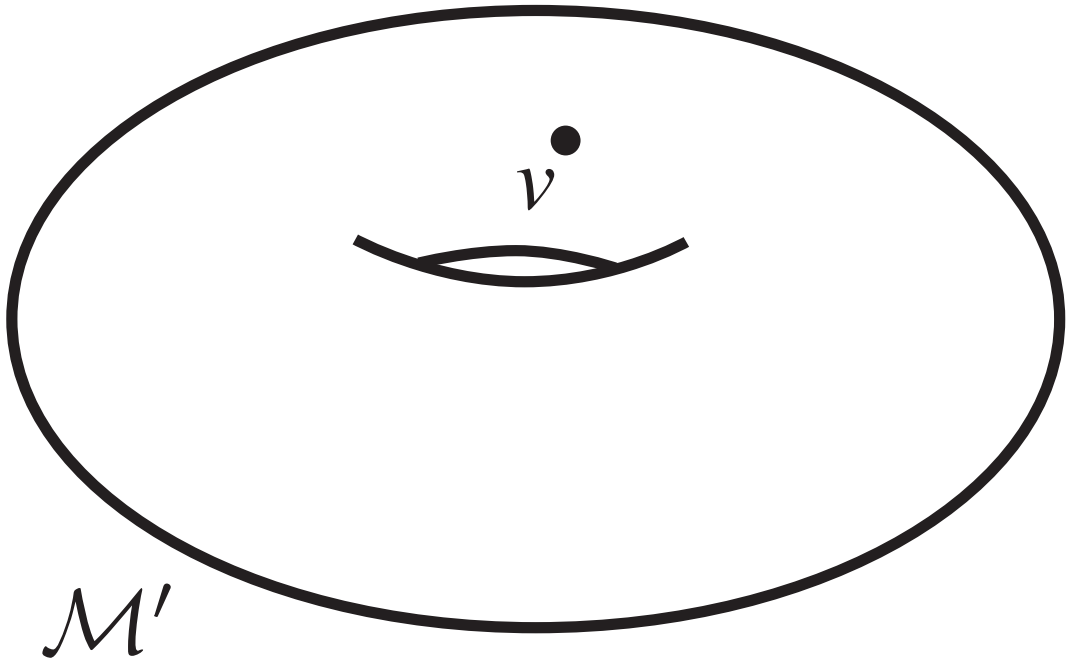
Meet π. He is the shortest path between the two marked vertices on $\mathcal{M}$.

π: a shortest path on $\mathcal{M}$



Meet γ^*, the red cycle. He is the shortest essential cycle in $\mathcal{M}$ which intersects π.

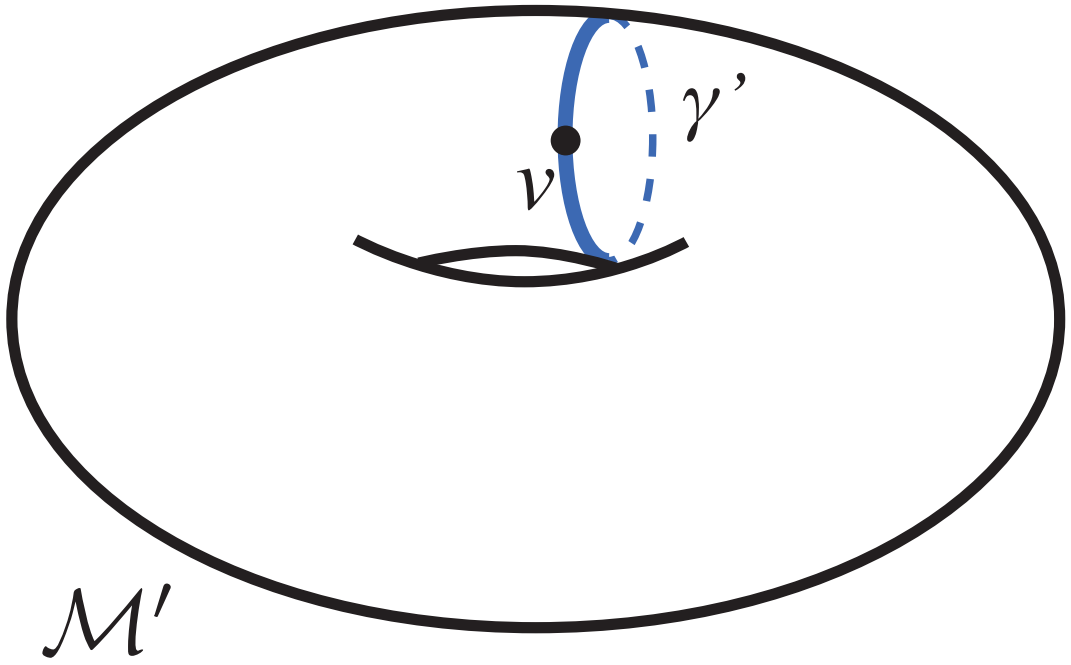
π: a shortest path on $\mathcal{M}$



One day, π decided to move to a manifold called $\mathcal{M}'$ where he changed his name to v and contracted himself into a point!

(But all of the edges which were incident on π are still incident on v, and their length didn't change.)

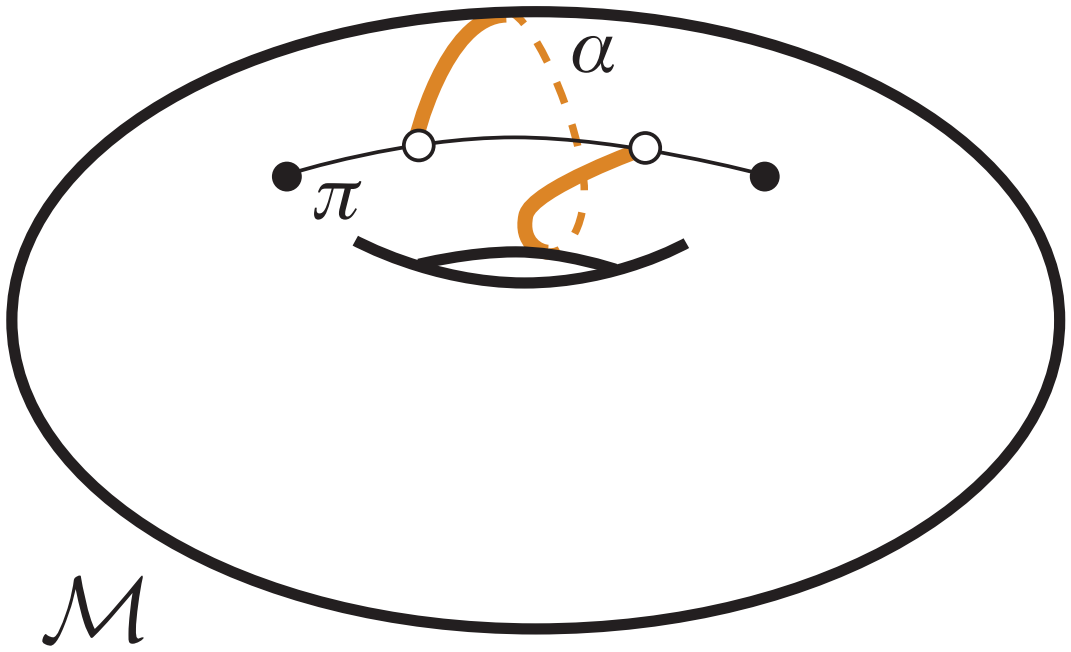
γ^*: the shortest essential cycle through the path π on $\mathcal{M}$



Meet γ', a blue cycle who lives in $\mathcal{M}'$ with v. He is the shortest essential cycle through v. He is no longer than γ^*, because he contains all the same edges as γ^*, except for the ones which were contracted.

$$\boxed{|\gamma'| \leq |\gamma^*|}$$

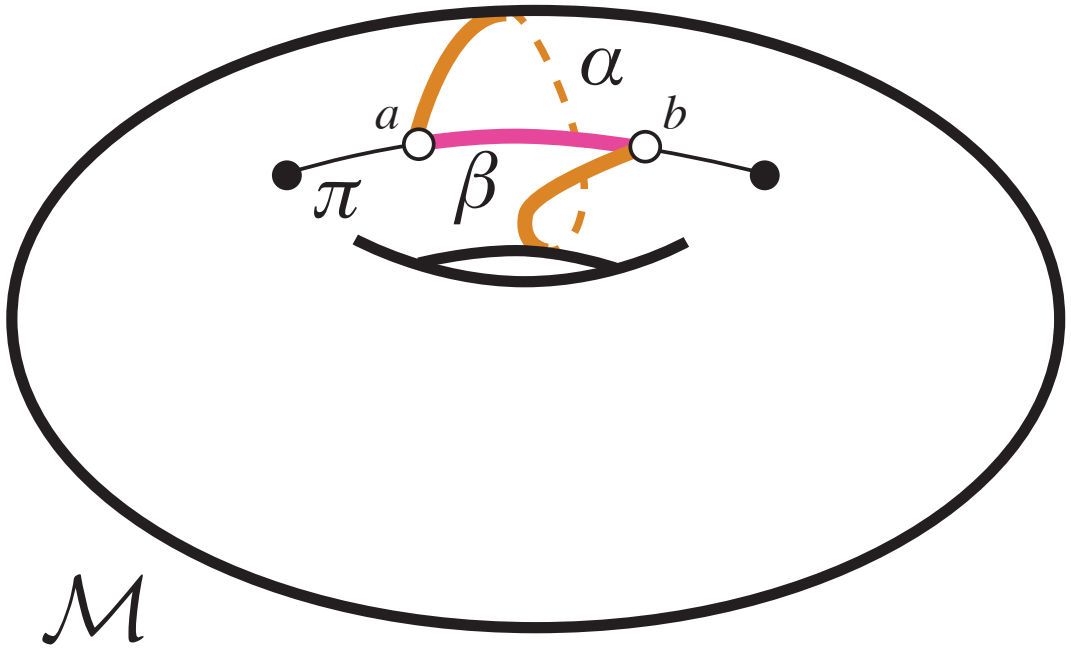
γ': the shortest essential cycle through the point v on $\mathcal{M}'$



Eventually π (or v, as they had come to call him) decided to show γ' where he used to live in $\mathcal{M}$. When γ' got to $\mathcal{M}$, though, he found that his edges no longer formed a cycle, but an orange path called α!

$$\boxed{|\alpha| = |\gamma'|}$$

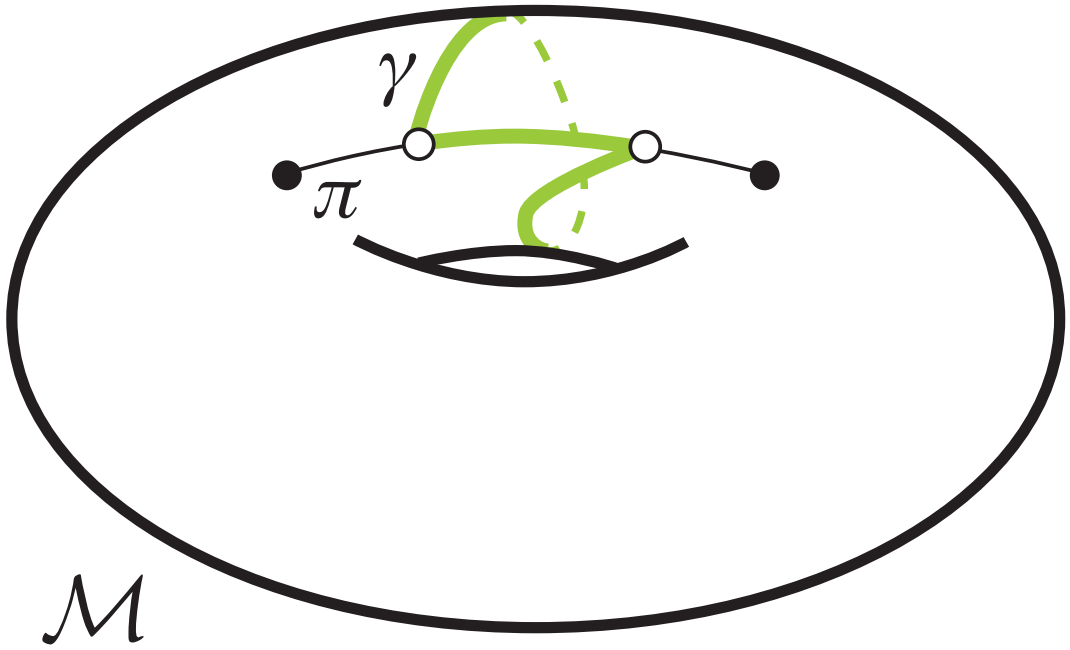
π: a shortest path on $\mathcal{M}$



Sitting along π between the endpoints of α was a pink path called β, who unexpectedly found herself holding hands (a and b) with α. β was a *shortest path*, and didn't like the idea of holding hands with a longer path like α.

$$\boxed{|\beta| \leq |\alpha|}$$

α: edges from the shortest essential cycle through the point v on $\mathcal{M}'$



Over time, though, α and β grew to love each-other, and formed a new green path called γ.

$$\boxed{|\gamma| = |\alpha| + |\beta|}$$

The moral of the story:

$$|\gamma| = |\alpha| + |\beta| \leq 2|\alpha| = 2|\gamma'| \leq 2|\gamma^*|.$$