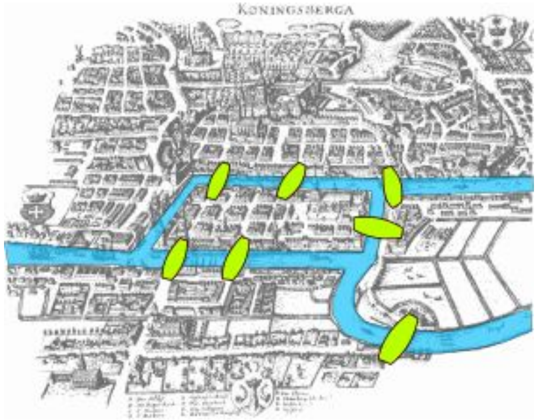
Graph Theory

Katherine and Noam

The Seven Bridges of Königsberg

The Seven Bridges of Königsberg

- Königsberg is a city in Russia
- It is separated by several rivers, and there are bridges over these rivers



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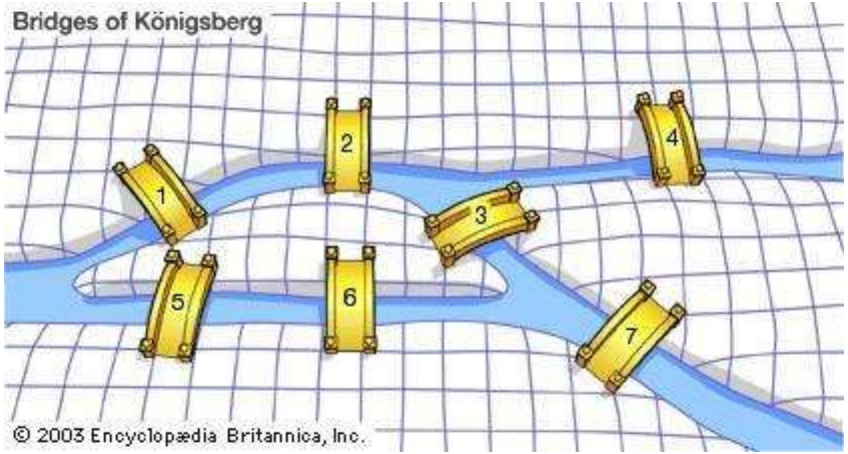
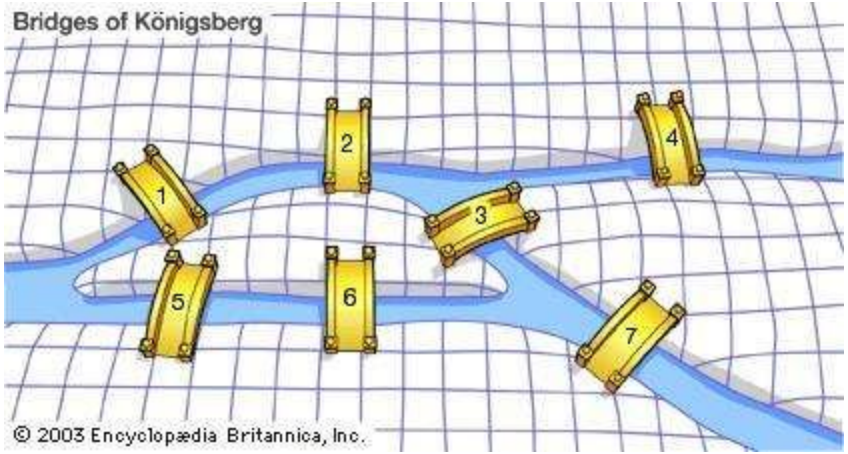
The Seven Bridges of Königsberg

- Is there a way to cross each of these bridges exactly once?



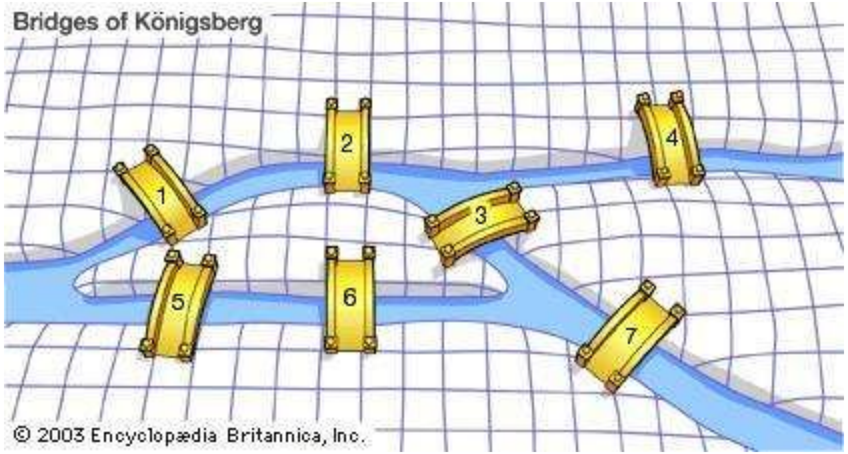
The Seven Bridges of Königsberg: Activity!

- Is there a way to cross each of these bridges exactly once?
- Try it!



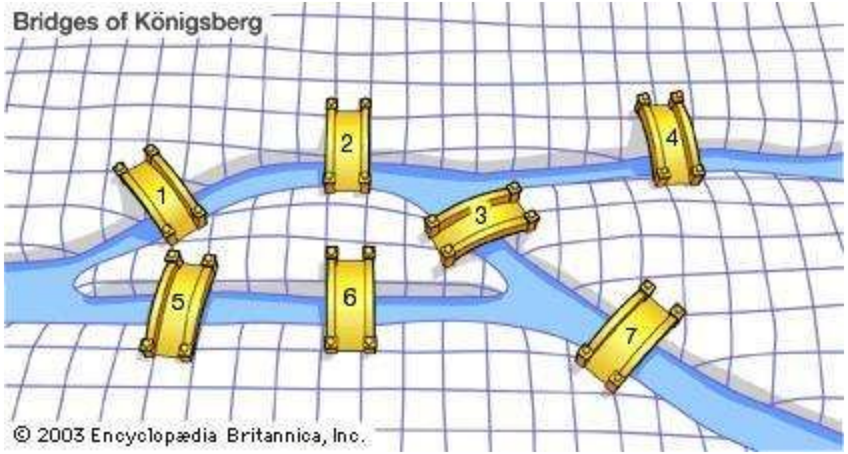
The Seven Bridges of Königsberg

- Is there a way to cross each of these bridges exactly once? NO!



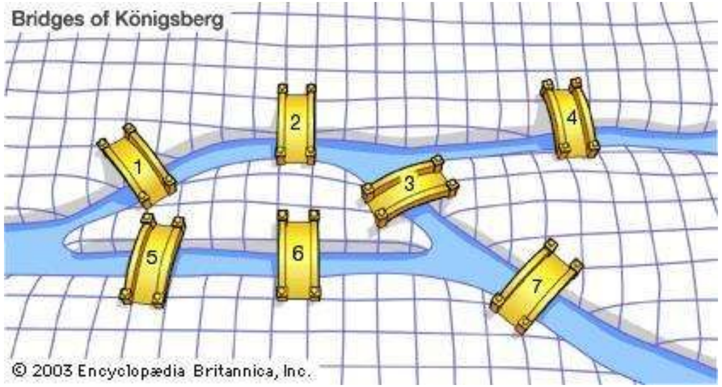
The Seven Bridges of Königsberg

- Is there a way to cross each of these bridges exactly once? NO!
- How can we understand this mathematically?



Rephrase the problem!

- Let's think of each land mass as a point
- Think of the bridges as edges between the points



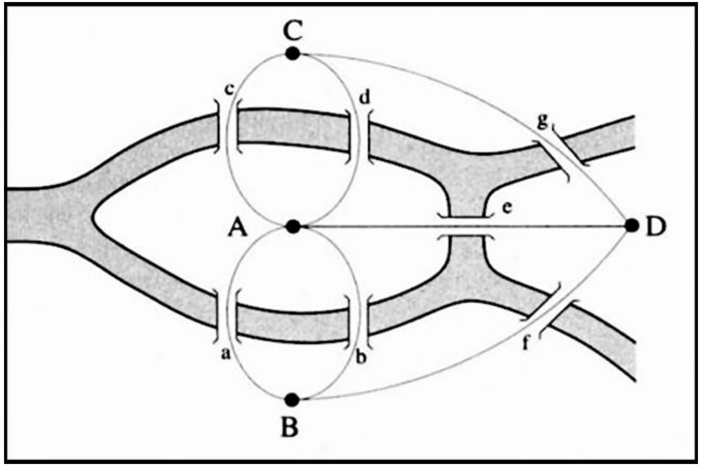
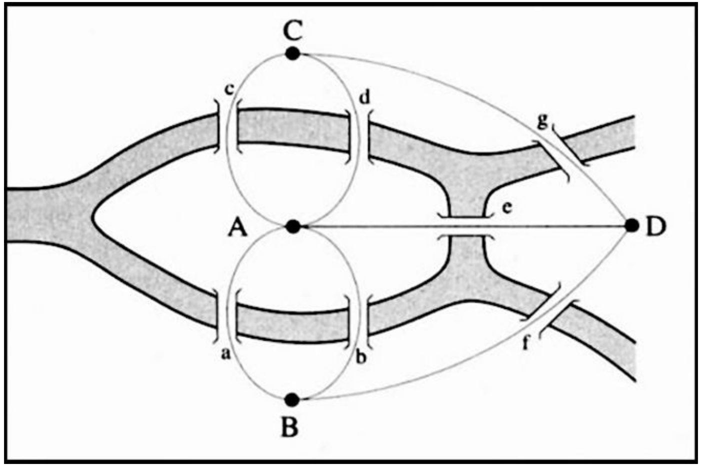


Image credit: https://physics.weber.edu/carroll/honors/konigsberg.htm

Graph theory!

- This kind of representation of our problem is a *graph*.



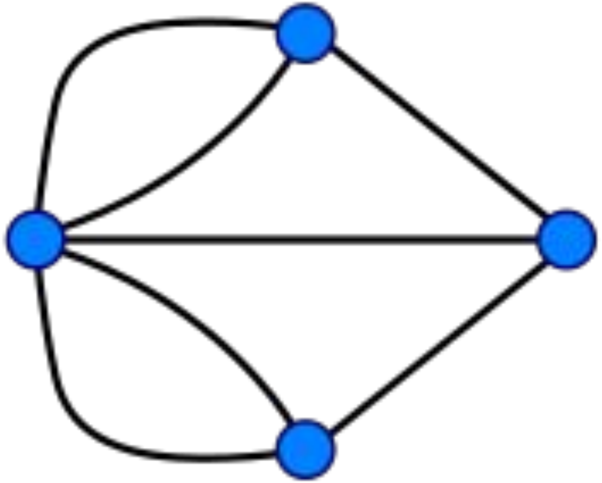
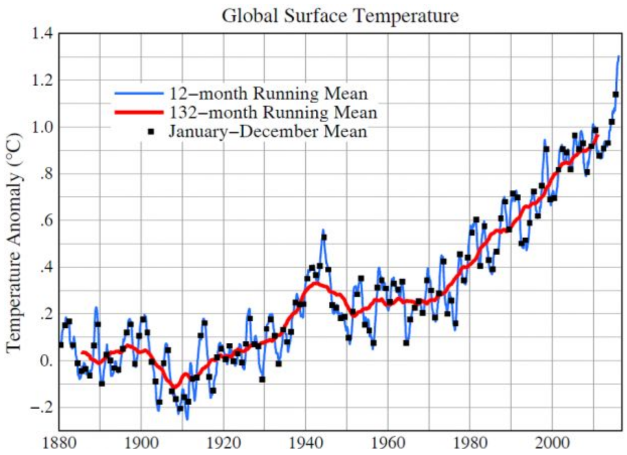


Image credit: Wikipedia

Wait, what’s a graph?

What is a graph?

- Sometimes when people say “graph”, they mean something like this or this:



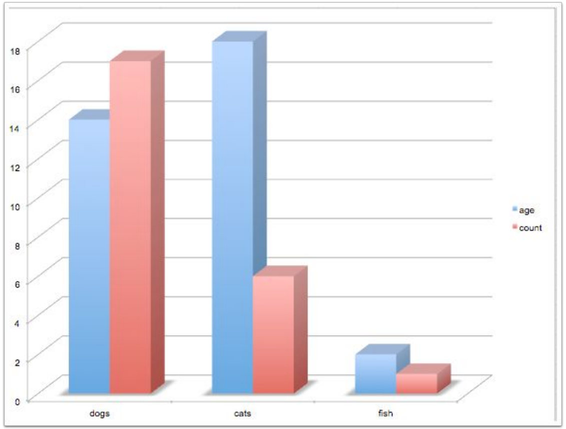
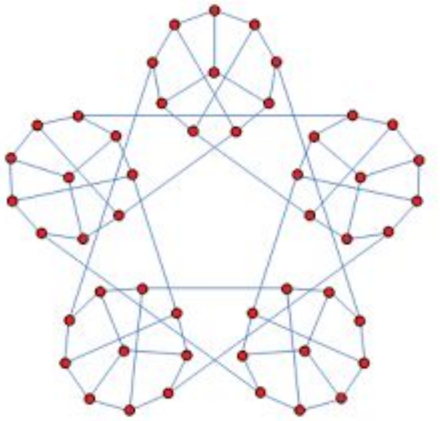


Image credit: https://www.podfeet.com/blog/tutorials-5/how-to-make-2d-excel-graphs-look-3d/, http://csas.ei.columbia.edu/2016/09/26/a-better-graph/

What is a graph?

- But often, mathematicians mean something like this or this:



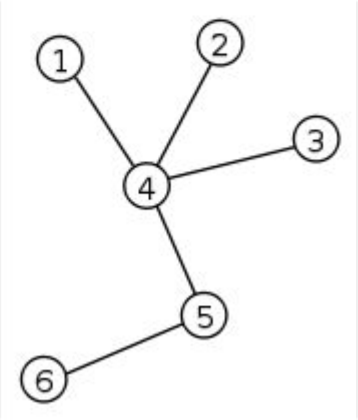
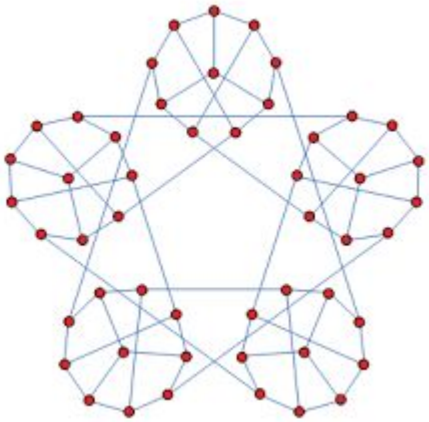
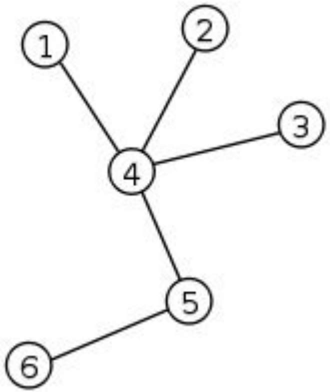


Image credit: https://www.maths.unsw.edu.au/about/graph-theory-unsw-past-and-present, Wikipedia

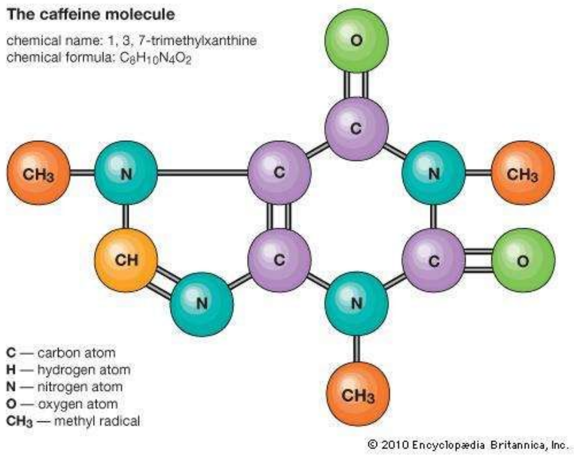
What is a graph?

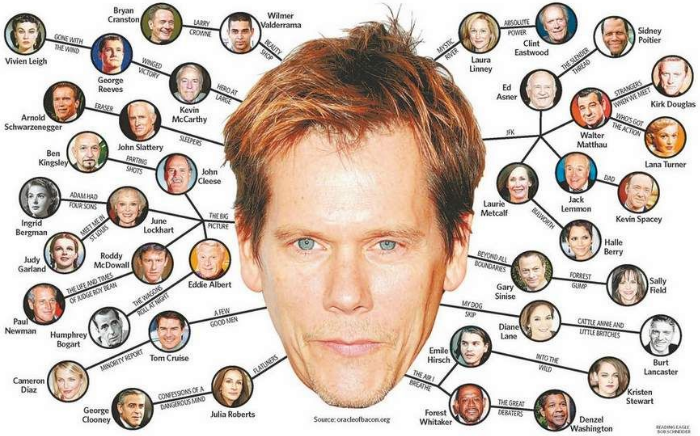
- More formally, a graph is just a set of points (vertices) and edges connecting those points.
- The points can be labeled (but don't have to be)

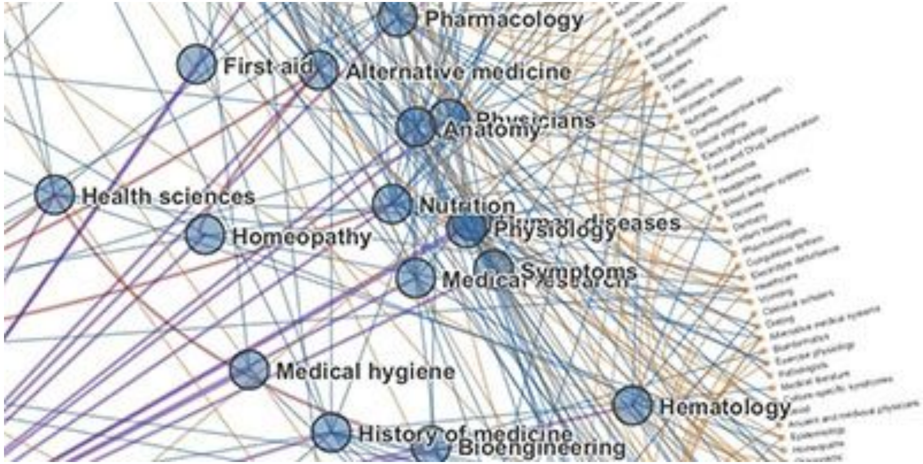




Graphs in the real world







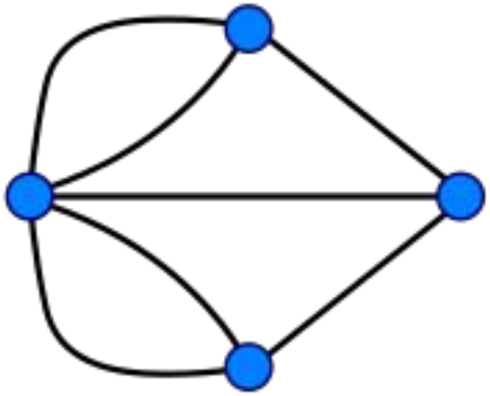
Eulerian Paths

Eulerian Path

- An *Eulerian path* is a path through a graph that travels through each edge exactly once.

Eulerian Path

- An *Eulerian path* is a path through a graph that travels through each edge exactly once.
- The Bridges of Königsberg is really asking whether we can find an Eulerian path through this graph.

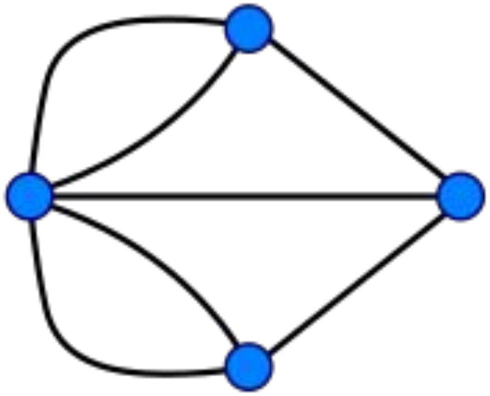


Eulerian Path: Activity!

- An *Eulerian path* is a path through a graph that travels through each edge exactly once.
- Let’s understand when we can and can’t find an Eulerian path through a graph.

Eulerian Path Recap

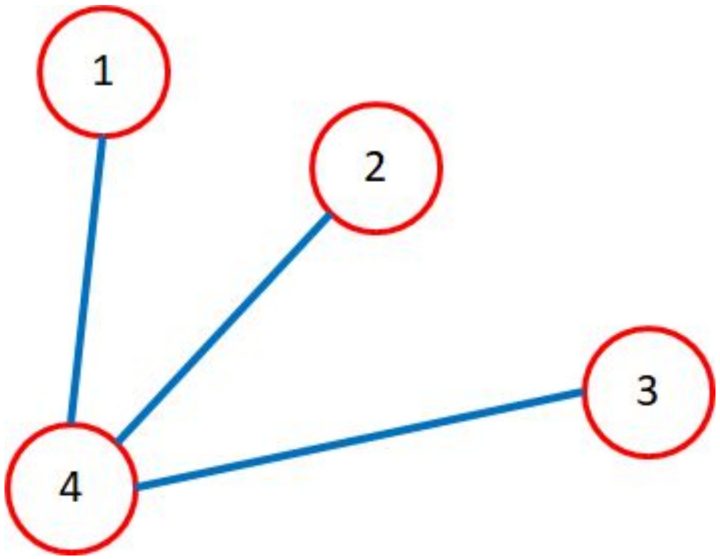
- If we have an Eulerian path, what vertices can have an odd number of edges coming from them?
  - The start vertex
  - The end vertex



Planar Graphs

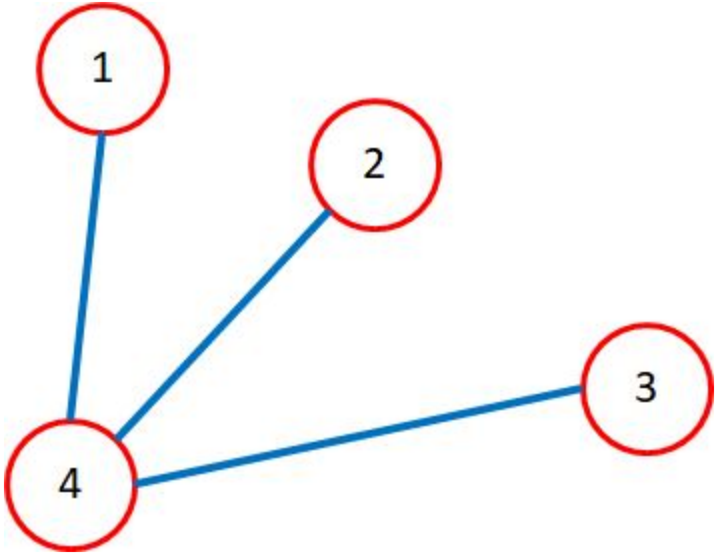
Graph theory

- Question: What information determines a graph?



Graph theory

- Question: What information determines a graph?
- Answer: Just the vertices (points) and the edges (connections between points). Nothing else matters!



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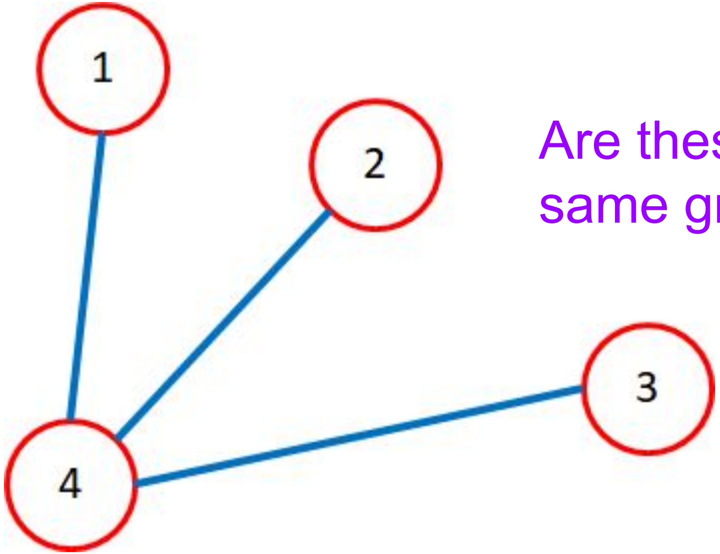
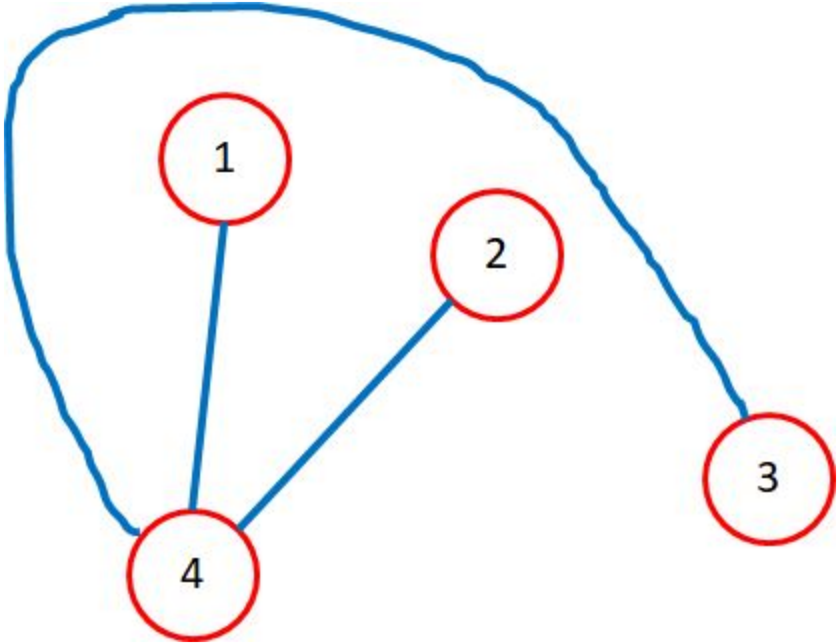


Are these the same graph?



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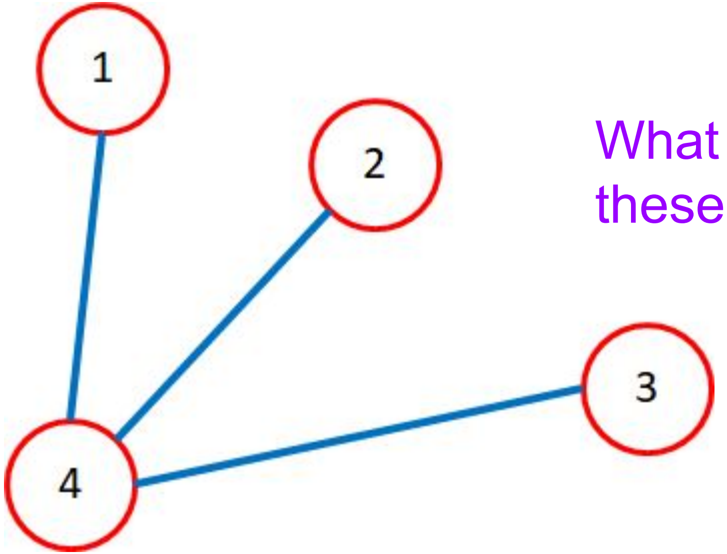
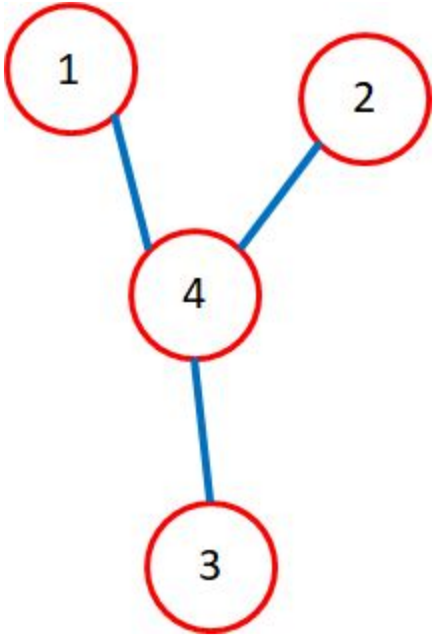


What about these?



Graph theory

- These look really different, but are all the same graph

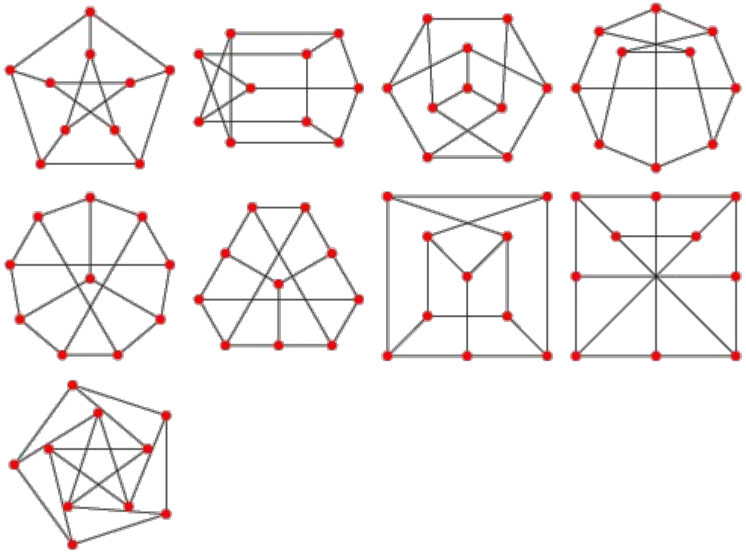


Image credit: http://mathworld.wolfram.com/PetersenGraph.html

Planar graphs

- If we can draw a graph without its edges crossing, then the graph is *planar*

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- Example:

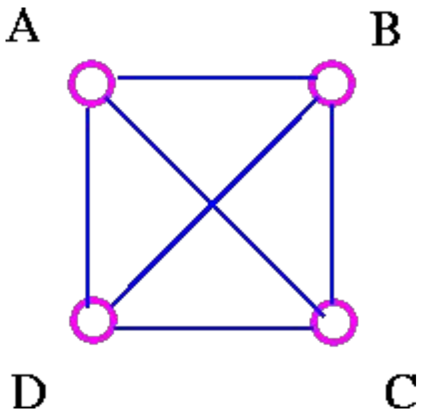

Image credit: http://www.personal.kent.edu/~rmuhamma/GraphTheory/MyGraphTheory/planarity.htm

Planar graphs

- If we can draw a graph without its edges crossing, then the graph is *planar*
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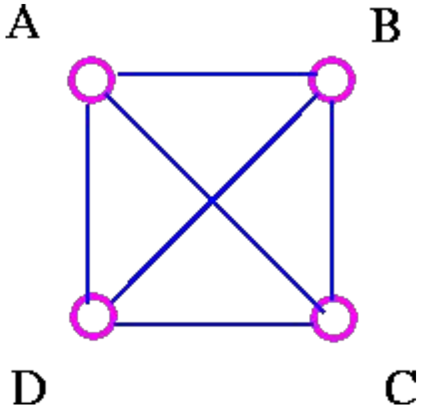

The edges are crossing right now, but can we redraw it?

Image credit: http://www.personal.kent.edu/~rmuhamma/GraphTheory/MyGraphTheory/planarity.htm

Planar graphs

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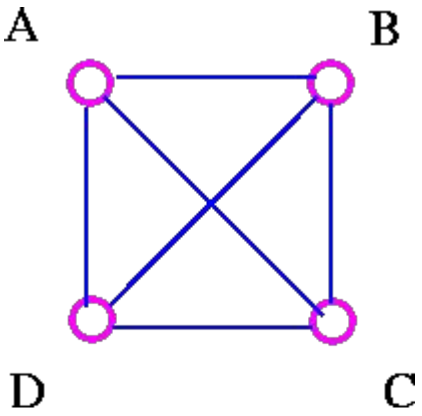
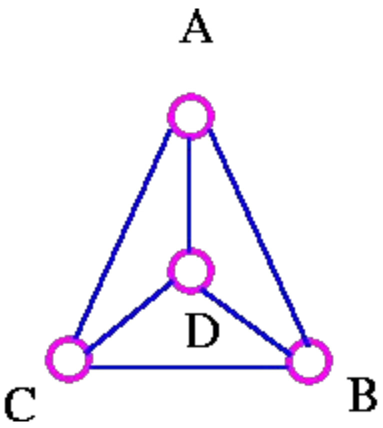
- Example:





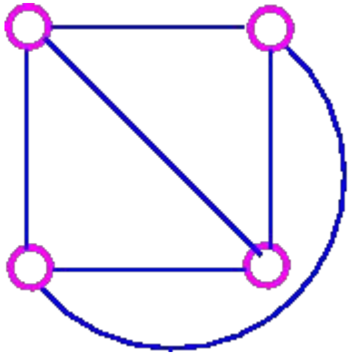
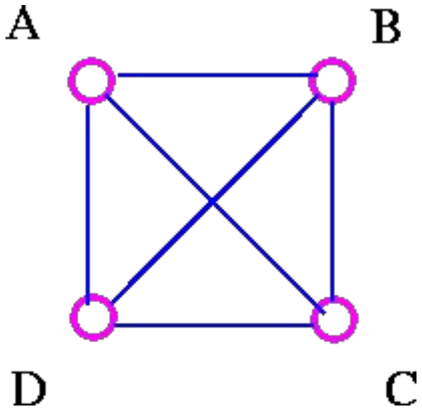
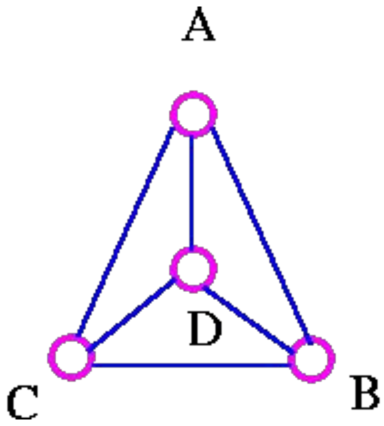


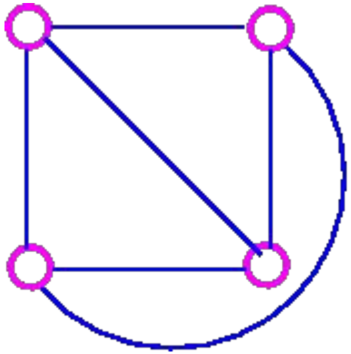
Image credit: http://www.personal.kent.edu/~rmuhamma/GraphTheory/MyGraphTheory/planarity.htm

Planar graphs

- If we can draw a graph without its edges crossing, then the graph is *planar*
- Now you try!







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Graph Theory and Coloring

Map Coloring

Fill in every region so that no two adjacent regions have the same color.

How many colors would it take to color this map?

You can do it with four!

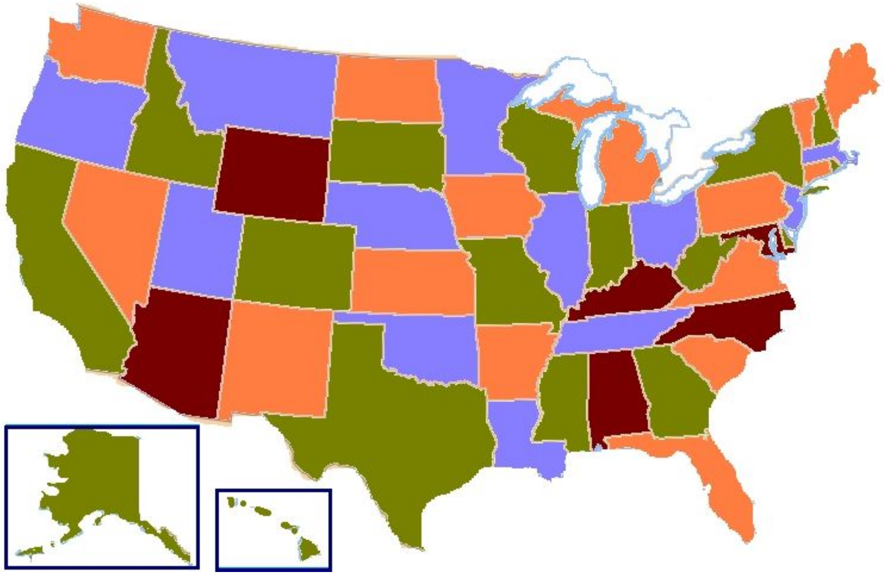


Image credit: Wikipedia

Now, challenge each other…

Pair up, and make a challenge map for your partner that you think should require **many** colors.

Then switch maps and try to color your partner’s challenge map with **as few colors** as possible!

The Four Color Theorem

Any map can be colored with just four colors.

Very hard to prove!

The Five Color Theorem

Any map can be colored with five colors.

From maps to graphs

- Do you think we could represent this map as a graph?

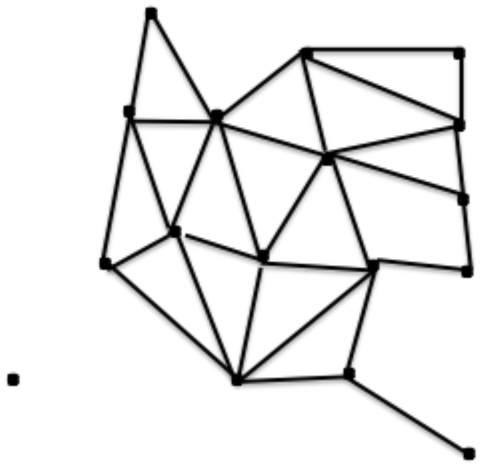
From maps to graphs

- Regions -> points, edges connect adjacent regions

Like this!

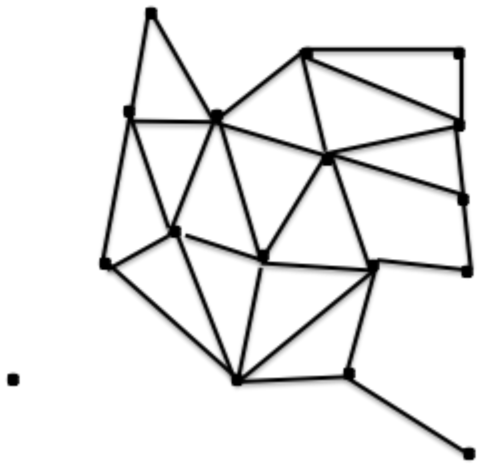
From maps to graphs

If we represent a map as a graph, is it planar?



From maps to graphs

- If we represent a map as a graph, is it planar?
- YES!



The Five Color Theorem

Using graph theory language!

We can restate the result:

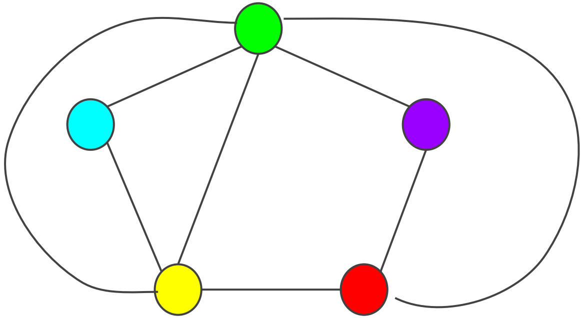
Every planar graph can be colored with five colors so that any two vertices connected by an edge have different colors.

Why is this true?

- What about for planar graphs with at most 5 vertices?

Why is this true?

- What about for planar graphs with at most 5 vertices?
- Just make each vertex a different color!



Why is this true?

- Now, let’s consider more general planar graphs.
- A fact from math: Every planar graph has a vertex that’s connected to at most 5 edges.

Why is this true?

- Now, let’s consider more general planar graphs.
- A fact from math: Every planar graph has a vertex that’s connected to at most 5 edges.
- What if we’ve colored our graph except for that vertex?

Why is this true?

- In other words, we’ve colored our graph except for one vertex that’s connected to at most 5 edges.

Why is this true?

- We’ve colored our graph except for one vertex that’s connected to at most 5 edges.
  - What if it’s only connected to 4 edges?

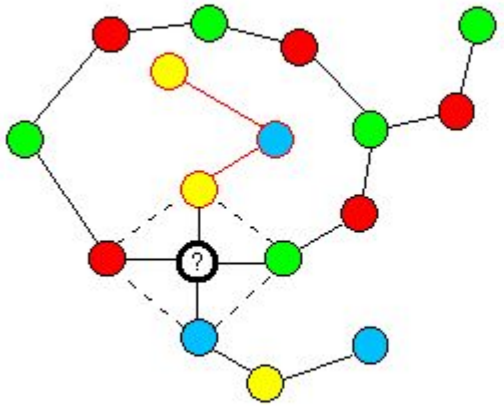


Image credit:
https://astarmathsandphysics.com/university-maths-notes/topology/2389-the-four-colour-mapping-theorem.html

Why is this true?

- We’ve colored our graph except for one vertex that’s connected to at most 5 edges.
  - What if it’s only connected to exactly 5 edges?

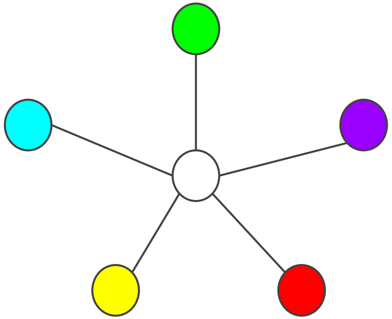


Image credit: http://cgm.cs.mcgill.ca/~athens/cs507/Projects/2003/MatthewWahab/5color.html

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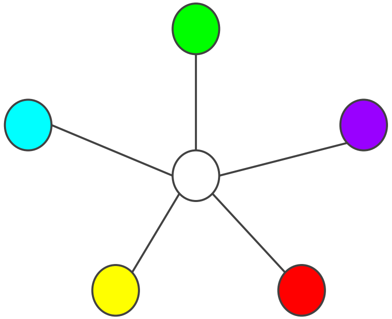

Can we make the red vertex yellow?

Image credit: http://cgm.cs.mcgill.ca/~athens/cs507/Projects/2003/MatthewWahab/5color.html

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- Let's say we've colored our graph except for one vertex that's connected to at most 5 edges.
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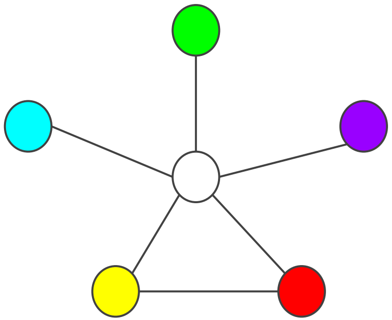

What if they're connected?

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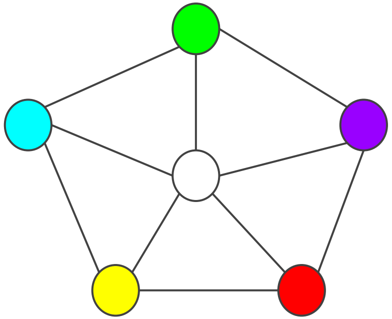

Can we make the green vertex yellow?

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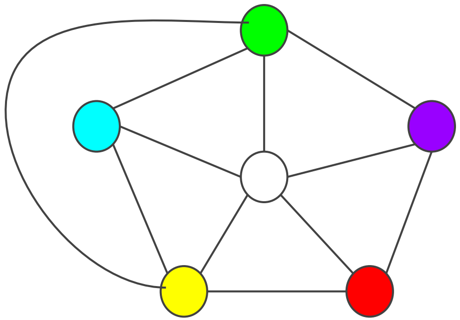

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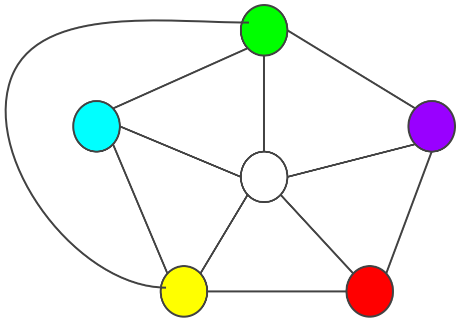

Can we make the blue vertex red?

Image credit: http://cgm.cs.mcgill.ca/~athens/cs507/Projects/2003/MatthewWahab/5color.html

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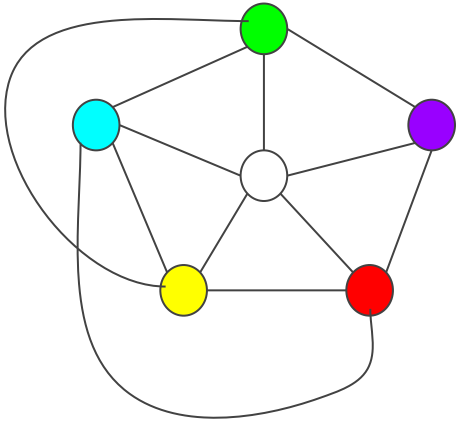

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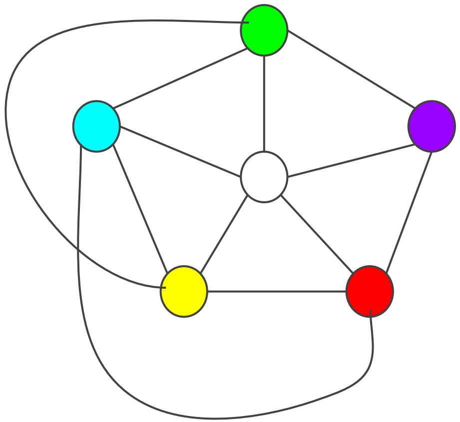


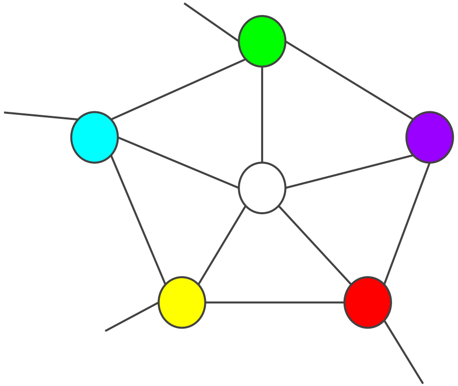
What if they’re connected?

THEY CAN’T BE--our graph is planar.

Image credit: http://cgm.cs.mcgill.ca/~athens/cs507/Projects/2003/MatthewWahab/5color.html

Why is this true?

- Can we always either make the green vertex yellow OR make the red vertex blue?

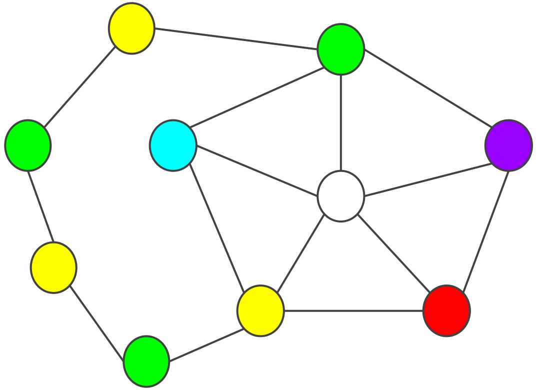


We don’t know what the rest of our graph looks like!

Image credit: http://cgm.cs.mcgill.ca/~athens/cs507/Projects/2003/MatthewWahab/5color.html

Why is this true?

- The only time we can't make the green vertex yellow is when something like this happens...



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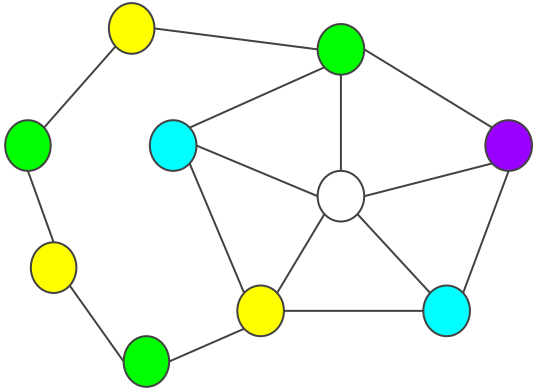
- And then, since our graph is planar, we will be able to make the red vertex blue!



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Why is this true?

- We’ve colored our graph except for one vertex that’s connected to exactly 5 edges.
- We can color the neighboring vertices with just 4 colors...

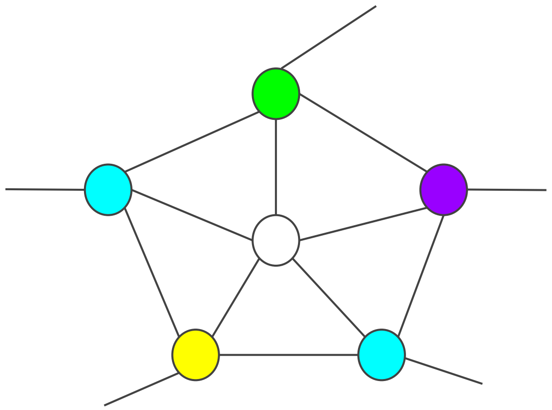
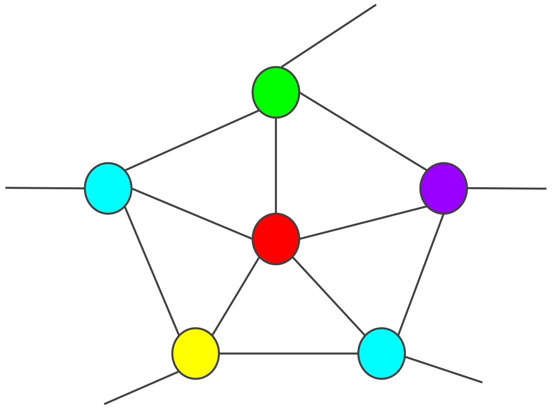


Image credit: http://cgm.cs.mcgill.ca/~athens/cs507/Projects/2003/MatthewWahab/5color.html

Why is this true?

- We’ve colored our graph except for one vertex that’s connected to exactly 5 edges.
- We can color the neighboring vertices with just 4 colors... and the middle vertex with the fifth.



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Mathematical induction

- We know that a graph with 1 vertex can be 5-colored
- We know that if we can 5-color a graph with n vertices, then we can 5-color a graph with n+1 vertices (why?)

Mathematical induction

- We know that a graph with 1 vertex can be 5-colored
- We know that if we can 5-color a graph with n vertices, then we can 5-color a graph with n+1 vertices (why?)
- By *induction*, this proves the 5-color theorem!

Context and Applications

You can do lots of other things with graphs…

- We’ve focused on *undirected*, *unweighted* graphs
- But you could have graphs with directed edges

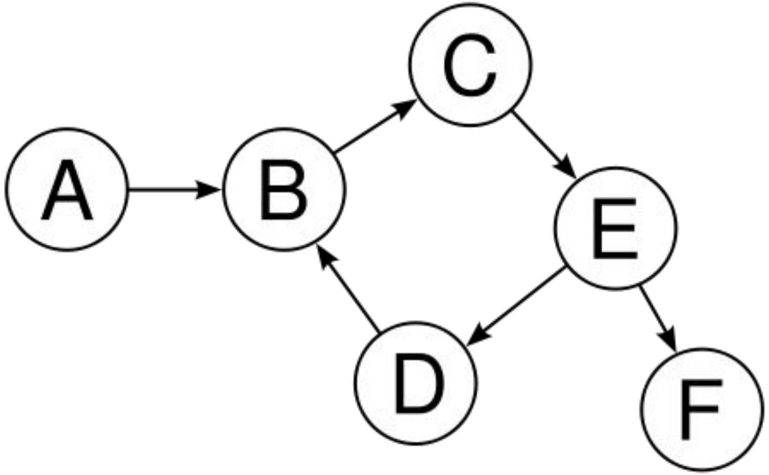

Image credit: Wikimedia

You can do lots of other things with graphs...

- We’ve focused on *undirected*, *unweighted* graphs
- But you could have graphs with weighted edges

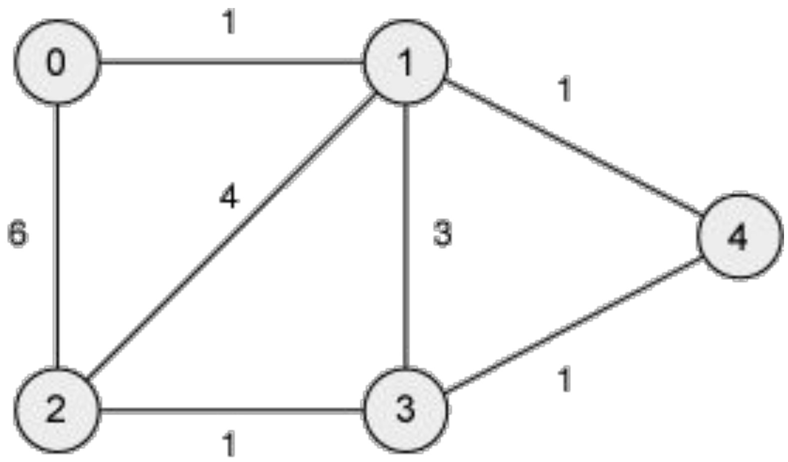

Image credit:
http://pages.cpsc.ucalgary.ca/~jacobs/Courses/cpsc331/F08/tutorials/tutorial14.html

Graph Theory Uses

- Modeling
  - Social networks
  - Rumor spreading
  - Disease transmission
  - Molecules
  - Atomic structures
- Advanced math
  - Knot theory
  - Group theory