Queue Logic: An Undisplayable Logic?

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Abstract

There is a variant of ordered logic where the ordered context is a queue, in the sense that left rules are applicable only on one end, and hypotheses are added on the other. We speculate that this means it lacks a display logic formulation, despite apparently being an otherwise sensible substructural logic.

1 Introduction

In display logic [Bel82], one requires the *display property*: that structural rules permit any proposition in a sequent to be fully exposed, and directly amenable to inference rules. Even in an calculus without the step-by-step structural display *rules* that make this property hold as such, it is common in calculi with structured contexts to formulate sequent left rules in such a way as to structurally permit inference deep within the context at any point. For instance, the bunched logic [OP99] left rule for multiplicative implication is

$$\frac{\Delta \vdash A \qquad \Gamma(B) \vdash C}{\Gamma(A -\!\!* B, \Delta) \vdash C}$$

where $\Gamma(\text{—})$ is understood as a context with a hole allowed to be filled by another context expression. Henceforth we take the phrase 'display property' and the adjective 'displayable' to mean informally the property that every proposition in the context is *accessible* in some sense, whether by small-step structural display rules or big-step notions of contexts-with-holes.

We assent that the question of where inference is allowed is of fundamental importance in defining a logic. The point of this note, however, is to dispute that the only answer to the question is 'everywhere'. Though plainly it is already refuted to some extent by the existence of logics that embody focusing proof search: it is essential that not every proposition can be decomposed at any time, for that is the mechanism by which they reduce the nondeterminism of ordinary proof search.

But perhaps one might still think that all *real* logics are displayable, and those that aren't are mere proof-search hacks. To dispose of that objection, we define a logic that appears natively undisplayable, which would have different provability characteristics if the display property were forced to hold in it.

2 Queue Logic

By queue logic we mean the logic whose syntax is

$$\begin{array}{rl} \text{Propositions } A & ::= A \twoheadrightarrow A \mid A \bullet A \mid 1 \mid a \\ \text{Contexts } \Omega & ::= \cdot \mid \Omega A \end{array}$$

where contexts are considered to be intrinsically associative (but not commutative) lists, and whose sequent rules are

$$\frac{}{a \vdash a}\, init \qquad \frac{}{\cdot \vdash 1}\, 1R \qquad \frac{\Omega \vdash C}{1\Omega \vdash C}\, 1L \qquad \frac{\Omega_1 \vdash A \qquad \Omega_2 \vdash B}{\Omega_1\Omega_2 \vdash A \bullet B}\, \bullet R$$

$$\frac{AB\Omega \vdash C}{(A \bullet B)\Omega \vdash C}\, \bullet L \qquad \frac{\Omega A \vdash B}{\Omega \vdash A \twoheadrightarrow B}\, \twoheadrightarrow R \qquad \frac{\Omega_1 \vdash A \qquad B\Omega_2 \vdash C}{(A \twoheadrightarrow B)\Omega_1\Omega_2 \vdash C}\, \twoheadrightarrow L$$

We could also add additive connectives without any difficulty.

The standard cut admissibility and identity expansion theorems are

Theorem 2.1

1. *If $\Omega \vdash A$ and $\Omega_L A \Omega_R \vdash C$, then $\Omega_L \Omega \Omega_R \vdash C$.*
2. *If $\Omega \vdash A$ and $A\Omega_R \vdash C$, then $\Omega\Omega_R \vdash C$.*

Proof By lexicographic induction on A and the derivations involved. When showing part 1, prioritize doing commutative cases in the derivation of $\Omega_L A \Omega_R \vdash C$. until Ω_L is empty, then pass to the special case that is part 2. At that stage, do commutative cases on the derivation of $\Omega \vdash A$ until the principal case is reached. ∎

Theorem 2.2 *$A \vdash A$ for all A.*

Proof By induction on A. ∎

3 Comparison to Ordered Logic

Note that during the cut admissibility proof we used the global property that every connective's left rule only allows decomposition on the left end of the context to insure that it is *possible* to keep going through commutative cases on Ω_L (dealing with commutative decompositions on C as necessary also, but they are not important here) until it is empty. For we could have defined the calculus to be simply a subset of ordered logic like so: (only the left rules have been changed by the addition of Ω')

$$\frac{}{a \vdash a}\, init \qquad \frac{}{\cdot \vdash 1}\, 1R \qquad \frac{\Omega'\Omega \vdash C}{\Omega' 1\Omega \vdash C}\, 1L \qquad \frac{\Omega_1 \vdash A \qquad \Omega_2 \vdash B}{\Omega_1\Omega_2 \vdash A \bullet B}\, \bullet R$$

$$\frac{\Omega' AB\Omega \vdash C}{\Omega'(A \bullet B)\Omega \vdash C}\, \bullet L \qquad \frac{\Omega A \vdash B}{\Omega \vdash A \twoheadrightarrow B}\, \twoheadrightarrow R \qquad \frac{\Omega_1 \vdash A \qquad \Omega' B\Omega_2 \vdash C}{\Omega'(A \twoheadrightarrow B)\Omega_1\Omega_2 \vdash C}\, \twoheadrightarrow L$$

This calculus actually satisfies the same two theorems as stated above, despite coinciding with the queue logic on right rules, but differing on left rules. We claim that this peculiar situation (which violates the expectation that once the right (resp. left) rules of a connective and the cut and identity principles are fixed, the left (resp. left) rules should be uniquely determined up to provability) is possible because in addition to the cut and identity principles, we ought to have specified slightly more information about the fundamental shape of sequents, namely their display characteristics: where inferences are *a priori* even possible.

To summarize: the second logic mentioned, a fragment of ordered logic, is fully displayable, while the first, queue logic, is not, and this is reflected in the global property required in the proof of cut admissibility for queue logic.

To show conclusively that the two are different, we give a simple litmus test counterexample, which is provable in ordered logic, but not queue logic: the sequent $(a \twoheadrightarrow b \bullet c \twoheadrightarrow d)(c \twoheadrightarrow a \bullet b)c \vdash d$, where $\twoheadrightarrow$ is right-associative and has lower precedence than $\bullet$, and a, b, c, d are distinct atomic propositions. In ordered logic we have the proof that applies $\twoheadrightarrow$L in the middle of the context:

$$\dfrac{\dfrac{}{c \vdash c} \qquad \dfrac{\dfrac{\dfrac{}{a \vdash a} \qquad \dfrac{\dfrac{}{b \vdash b} \qquad \dfrac{}{d \vdash d}}{(b \twoheadrightarrow d)b \vdash d}}{(a \twoheadrightarrow b \twoheadrightarrow d)ab \vdash d}}{(a \twoheadrightarrow b \twoheadrightarrow d)(a \bullet b) \vdash d}}{(a \twoheadrightarrow b \twoheadrightarrow d)(c \twoheadrightarrow a \bullet b)c \vdash d}$$

but in queue logic, we have only three possible ways to begin the proof, depending on how we split the context in $\twoheadrightarrow L$

$$\dfrac{\dfrac{?}{\cdot \vdash a} \qquad \dfrac{?}{(b \twoheadrightarrow d)(c \twoheadrightarrow a \bullet b)c \vdash d}}{(a \twoheadrightarrow b \twoheadrightarrow d)(c \twoheadrightarrow a \bullet b)c \vdash d}$$

$$\dfrac{\dfrac{?}{(c \twoheadrightarrow a \bullet b) \vdash a} \qquad \dfrac{?}{(b \twoheadrightarrow d)c \vdash d}}{(a \twoheadrightarrow b \twoheadrightarrow d)(c \twoheadrightarrow a \bullet b)c \vdash d} \qquad \dfrac{\dfrac{?}{(c \twoheadrightarrow a \bullet b)c \vdash a} \qquad \dfrac{?}{(b \twoheadrightarrow d) \vdash d}}{(a \twoheadrightarrow b \twoheadrightarrow d)(c \twoheadrightarrow a \bullet b)c \vdash d}$$

and all three proof attemps quickly are seen to fail.

4 Why not Stack Logic?

One evident question to ask is why we have a right-biased ordered arrow and left-biased decomposition of propositions in the context. A left arrow and right decomposition also works, but just produces a syntactically symmetric queue. We have encountered difficulties in producing a 'native' logic of a stack-shaped context (as opposed to focusing calculi that may happen to treat the ordered asynchronous context as a stack) and in this section discuss why this might be.

Ideas implicit in some of our previous work [RP09] connect the existence of well-behaved polarized logical connectives directly with the existence of corresponding context-algebra operations, which in turn determine the displayability of hypotheses in the informal sense alluded to above. In particular, the polarized connective $P \twoheadrightarrow N$ is internalized as a context operation $f \odot p$ that takes a *frame* f (corresponding to the negative proposition N, and representing the shape of a context-with-hole) and a world p (corresponding to the positive proposition P, and representing the shape of a context) and yields a new frame, consisting of f with an additional piece of context data, p, attached to the right of its hole.

This operation is precisely what is invoked when the $\twoheadrightarrow$ right rule is executed — it is what adds a fresh hypothesis to the right of the current context — but it is also fundamentally what allows left inference rules to freely ignore a context occurring to the *right* of the proposition being decomposed. Or instead of 'ignore', we might well say: include into the frame that is merely along for the ride during an inference step.

In short, to define a right (resp. left) arrow is intimately connected to the permission for a left rule to allow an uninvolved context to the right (resp. left) of the principal formula. There is only really a *requirement* running in one direction, however. If we want to have a right rule, we must allow display of propositions with junk to the right, but we could also allow that display without actually getting around to defining right-arrow. Indeed that is just what happened in the fragment of ordered logic above. In any event it is certain that full ordered logic, with left and right arrow, must allow left sequent rules take place arbitrarily in the *middle* of the context.

For a somewhat more elementary analysis of why right arrow requires the displayability properties it does, simply consider that arrows are supposed to be invertible as conclusions. If we could not skip over hypotheses attached to the right of the context, decomposing a conclusion of $A \twoheadrightarrow B$ might inhibit some decomposition of a hypotheses blocked in by the addition of A, contradicting its invertibility.

References

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