Resource Analysis: Problem Set 1

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Remark: The point of this problem set is to show you that asymptotic analysis and recurrence relations are not so much fun. You will learn much better alternatives in the remainder of the term. Stay tuned.

1.1 (12 Points) Asymptotic Behavior of Polynomials

Recall the following definitions.

Definition 1. *Let $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be two functions from natural number to non-negative reals. We write*

- $f(x) \in O(g(x))$ *if there exists* $C > 0$ *and* $N \in \mathbb{N}$ *such that* $f(x) \leq C \cdot g(x)$ *for all* $x \geq N$
- $f(x) \in \Omega(g(x))$ *if there exists* $C > 0$ *and* $N \in \mathbb{N}$ *such that* $f(x) \geq C \cdot g(x)$ *for all* $x \geq N$
- $f(x) \in \Theta(g(x))$ *if* $f(x) \in \Omega(g(x))$ *and* $f(x) \in O(g(x))$

Let now $a_d > 0$ and let

$$p(n) = \sum_{i=0}^{d} a_i n^i$$

be a polynomial of degree d. Prove the following statements.

a) If $k \geq d$ then $p(n) = O(n^k)$

b) If $k \leq d$ then $p(n) = \Omega(n^k)$

c) If $k = d$ then $p(n) = \Theta(n^k)$

1.2 (16 Points) Asymptotic Behavior and Multiple Variables

Asymptotic notations (such as Big-O) are abused in many technical and non-technical ways. One such abuse is that textbooks and courses on analysis of algorithms often define asymptotic notations only for univariate functions (e.g., functions with one argument) but use it to analyze functions with a multivariate resource behavior.

To start with, it is undefined what an expression like $O(nm)$ means. But more importantly, it is not easy to come up with a sensible definition. The following two definitions often appear in the literature. Let $f, g : \mathbb{N}^k \to \mathbb{R}^{\geq 0}$ be two non-negative real-valued functions.

Definition 2. *We write* $g(\vec{n}) \in O_\forall(f(\vec{n}))$ *if there exists* $c \in \mathbb{R}_{\geq 0}$ *and* $N \in \mathbb{N}$ *such that* $g(n_1, \ldots, n_k) \leq c \cdot f(n_1, \ldots, n_k)$ *for every* $\vec{n}$ *with* $n_1 \geq N$, …, *and* $n_k \geq N$.

Definition 3. *We write $g(\vec{n}) \in O_\exists(f(\vec{n}))$ if there exists $c \in \mathbb{R}_{\geq 0}$ and $N \in \mathbb{N}$ such that $g(n_1, \ldots, n_k) \leq c \cdot f(n_1, \ldots, n_k)$ for every $\vec{n}$ with $n_j \geq N$ for some positive integer $j \leq k$.*

However, it is unclear if these definitions have similar properties as the univariate version. Consider for example the following OCaml program and assume that $g(m,n)$ runs in time $O_\exists(mn)$ (and thus $O_\forall(nm)$).

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let f(m,n) = for i = 1 to m-1 do
               g(i,n)
             done
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Let $F(m,n)$ be the running time of $f(m,n)$. **Prove or disprove the following statements.**

a) $mn + 1 \in O_\forall(mn)$

b) $mn + 1 \in O_\exists(mn)$

c) $F(m,n) \in O_\forall(m^2 n)$

d) $F(m,n) \in O_\exists(m^2 n)$

e) Let $f_1, f_2 : \mathbb{N}^2 \to \mathbb{R}_{\geq 0}$. If there exists $N \in \mathbb{N}$ such that $f_1(n_1, n_2) \leq f_2(n_1, n_2)$ for all $n_1, n_2 \geq N$ then $f_1(\vec{n}) \in O_\forall(f_2(\vec{n}))$.

f) Let $f_1, f_2 : \mathbb{N}^2 \to \mathbb{R}_{\geq 0}$. If there exists $N \in \mathbb{N}$ such that $f_1(n_1, n_2) \leq f_2(n_1, n_2)$ for all $n_1, n_2 \geq N$ then $f_1(\vec{n}) \in O_\exists(f_2(\vec{n}))$.

Bonus questions: Which of the two definitions is more useful for analyzing programs? Can you define a better generalization of the univariate Big-O notation?

1.3 (12 Points) Exact Solutions of Recurrence Relations

Find exact (non-asymptotic) solutions $T(n)$ for the following recurrence relations and prove that your solutions satisfy the (in-)equalities.

a) $T(0) = 10$ and $T(n) = T(n-1) + 12n + 3$ for $n > 0$

b) $T(m,0) = 0$ and $T(m,n) = T(m, n-4) + 12(m + \binom{m}{2})$ for $n > 0$

c) $T_1(0) = T_2(0) = 0$, $T_1(n) \geq T_2(n-1) + 20(n-1)$, and $T_2(n) \geq T_1(n-1) + 8(n-1)$ for $n > 0$

d) $T(0) = T(1) = 0$, $T(2) = 1$, and $T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n - 1$ for $n > 0$