Automatic Space Bound Analysis for Functional Programs with Garbage Collection

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Abstract
This article introduces a novel system for deriving upper bounds on the heap-space requirements of functional programs with garbage collection. The space cost model is based on a perfect garbage collector that immediately deallocates memory cells when they become unreachable. Heap-space bounds are derived using type-based automatic amortized resource analysis (AARA), a template-based technique that efficiently reduces bound inference to linear programming. The first technical contribution of the work is a new operational cost semantics that models a perfect garbage collector. The second technical contribution is an extension of AARA to take into account automatic deallocation. A key observation is that deallocation of a perfect collector can be modeled with destructive pattern matching if data structures are used in a linear way. However, the analysis uses destructive pattern matching to accurately model deallocation even if data is shared. The soundness of the extended AARA with respect to the new cost semantics is proven in two parts via an intermediate linear cost semantics. The analysis and the cost semantics have been implemented as an extension to Resource Aware ML (RaML). An experimental evaluation shows that the system is able to derive tight symbolic heap-space bounds for common algorithms. Often the bounds are asymptotic improvements over bounds that RaML derives without taking into account garbage collection.

1 Introduction
The memory footprint of a program is an important performance metric that determines if a program can be safely executed on a given system. Ideally, developers should describe or approximate the memory footprint of programs as functions of the inputs. However, such memory bounds are often difficult to derive and to prove sound. To assist programmers with deriving memory bounds, the programming language community has developed automatic and semi-automatic analysis techniques [24, 12, 2]. These systems are often special cases of more general resource bound analyses that are based on abstract interpretation [18, 7, 37], recurrence solving [16, 1, 14, 28], type systems [27, 22, 29, 43, 42, 15], program logics [5, 10, 9, 35], proof assistants [33, 11], and term rewriting [6, 34, 17].

This article introduces a novel type system for automatically deriving upper bounds on the heap-space requirements of functional programs with garbage collection (GC). Due to the challenges of modeling and predicting garbage collection, most existing techniques for automating and guiding the derivation of bounds on the heap memory requirements assume manual memory management or simply ignore deallocation in the analysis [24, 26, 36, 13, 12, 2]. As a result, the derived bounds are not accurate when the underlying system employs garbage collection. The only exceptions we are aware of are the works by Albert et al. [3, 4], Braberman et al. [8], and Umnikrishnan et al. [40, 39]. They analyze the heap-space usage of programs with GC in two steps. First, they make the deallocation of GC explicit; for example with a static analysis for estimating object lifetimes [4] or with a program translation [39]. Second, they extract and solve recurrence relations to derive a bound. The difference of our work is that our technique is based on a type system, which is proved sound with respect to a formal cost semantics. Advantages of a type-based approach include natural compositionality and the use of type derivations as certificates for resource bounds.

We model the (highwater mark) memory usage based on a perfect garbage collector that immediately deallocates memory cells when they become unreachable. The bounds that are derived with respect to this cost model are not only a good theoretical measure of the heap-space consumption of the program but also have practical relevance. Consider a function \( f : A \rightarrow B \) and assume we derived a bound \( b_f : \|A\| \rightarrow \mathbb{N} \). In an execution of \( f(a) \), we can then keep track of the memory usage and start the garbage collector whenever
the bound $b_f(a)$ is reached. It is then guaranteed that the evaluation will succeed using $b_f(a)$ heap-memory cells. To improve performance, we could trigger GC more often (to compactify the heap) or allow memory use of more than $b_f(a)$ cells (to amortize the cost of garbage collection).

The first technical contribution of the work is a new operational cost semantics that models a perfect garbage collector. The cost semantics is a big-step (or natural) semantics that keeps track of the reachable memory cells in the style of Spoonhower et al. [38] and Minamide [30]. Operationally, this cost is the high-water mark on the heap usage, or the maximum number of cells used in the mutable store during evaluation. If we traverse the evaluation tree in preorder and view each node as a “step” of the computation, then a cell is used in the current node if it is reachable from the reminder of the computation. Our formalization of reachability is identical to the concept that garbage collectors implement to decide if a cell can be freed during evaluation. For simplicity, we assume that evaluation of the cons node allocates one fresh heap cell and that all other operations do not allocate heap cells. However, the semantics can be instantiated with more realistic cost metrics. A difference to existing formulations of cost semantics with GC [31, 38, 30] is that we update the high-water mark when reachability changes at inner nodes of the derivation of the evaluation judgement instead of at leaves. Moreover, we use a freelist, which represents named cells available for evaluation. This alternative formulation is equivalent to the existing semantics and mainly motivated by the soundness proof of our type system for bound analysis. However, the cost semantics is a natural approach and different enough from its predecessors [38, 30] to be of interest in its own right.

Our second technical contribution is the type system for deriving bounds on the heap-space for programs with perfect GC. The type system is an extension of type-based automatic amortized resource analysis (AARA) [24, 27, 41, 21, 22, 32]. AARA is a template-based technique that introduces potential functions to efficiently and automatically reduce bound inference to linear programming. Existing type systems based on AARA can derive bounds on the highwater mark of the heap usage for programs with manual deallocation [27], but can only derive a bound on the number of total heap allocations for programs with GC [22]. This is usually a gross over-approximation of the actual memory requirement. Our extension is based on the observation that deallocation of a perfect collector can be modeled with destructive pattern matching (deallocating the matched cell) if data structures are used in a linear way. In the type system, we extend this observation to non-linear programs and use destructive pattern matching to accurately model deallocation even if data is shared.

The third technical contribution is to prove the soundness of the extended AARA with respect to the GC-based cost semantics. The proof is non-trivial and proceeds in two parts: First, we prove the soundness of the type system with respect to a semantics that copies data structures if they are shared. Second, we prove for all programs that our GC semantics uses less memory than this copying semantics. While the proofs are relatively standard, many details—like relating program states of the two semantics in the simulation proof—are quite involved. Briefly, we have to provide and maintain a mapping $\gamma$ from the heap used in the GC semantics $H_{gc}$ to subsets of the heap used in the copying semantics $H_{copy}$ such that the image of $H_{gc}$ under $\gamma$ forms a partition on the second heap. The intuition is that given a cell $l \in H_{gc}$, there must be multiple cells $\gamma(l) \in H_{copy}$ that were allocated during sharing, and thus “morally the same” as $l$.

The analysis and the cost semantics have been implemented as an extension to Resource Aware ML (RaML) [21, 22]. RaML is an implementation of AARA for a subset of OCaml that can derive multivariate polynomial bounds. However, we restrict the technical development in this paper to a simple first-order language with tuples and lists. The proofs and ideas carry over to the more complex case of RaML. An experimental evaluation shows that the system is able to derive tight symbolic heap-space bounds for common algorithms. Our results suggest that our new analysis provides asymptotic bound improvements to several classes of commonly used functions and programming patterns. We examine the reasons for these improvements and design decisions throughout the system.

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1. We are not considering memory fragmentation, which can be avoided using a copying collector.
2. An exception are function closures that we discuss in the Section 7.
BTypes $\tau ::= \\
| \text{nat} \quad \text{nat} \quad \text{var}(x) \quad x \\
| \text{unit} \quad \text{unit} \quad \text{nat}[n] \quad \pi \\
| \text{bool} \quad \text{bool} \quad \text{unit} \quad () \\
| \text{prod}(\tau_1; \tau_2) \quad \tau_1 \times \tau_2 \quad T \quad T \\
| \text{list}(\tau) \quad L(\tau) \quad F \quad F \\
| \text{if}(x; e_1; e_2) \quad \text{if } x \text{ then } e_1 \text{ else } e_2 \\
| \text{arr}(\tau_1; \tau_2) \quad \tau_1 \rightarrow \tau_2 \\
| \text{match}\rho(x_1, x_2, e_1) \quad \text{match } \rho \{ (x_1; x_2) \mapsto e_1 \} \\
| \text{let}(1; x; \tau, e_2) \quad \text{let } x = e_1 \text{ in } e_2 \\
| \text{share}(x; x_1, x_2, e) \quad \text{share } x \text{ as } x_1, x_2 \text{ in } e \\

FTypes $\rho ::= \\
| \text{ap}(f; x) \quad f(x) \\
| \text{tpl}(x_1; x_2) \quad (x_1, x_2) \\
| \text{match}_\rho(x_1, x_2, e_1) \quad \text{match } \rho \{ (x_1; x_2) \mapsto e_1 \} \\
| \text{cons}(x_1; x_2) \quad x_1 :: x_2 \\
| \text{match}(l) \{ (e_1; x, x_1, x_2, e_2) \} \quad \text{match } l \{ \text{nil } \mapsto e_1 | \text{cons}(x; x_1, x_2) \mapsto e_2 \} \\
| \text{let}(1; x; \tau, e_2) \quad \text{let } x = e_1 \text{ in } e_2 \\
| \text{share}(x; x_1, x_2, e) \quad \text{share } x \text{ as } x_1, x_2 \text{ in } e \\

Val v ::= \\
| \text{nil} \\
| \text{cons}(x_1; x_2) \quad x_1 :: x_2 \\
| \text{match}(l) \{ (e_1; x, x_1, x_2, e_2) \} \quad \text{match } l \{ \text{nil } \mapsto e_1 | \text{cons}(x; x_1, x_2) \mapsto e_2 \} \\
| \text{let}(1; x; \tau, e_2) \quad \text{let } x = e_1 \text{ in } e_2 \\
| \text{share}(x; x_1, x_2, e) \quad \text{share } x \text{ as } x_1, x_2 \text{ in } e \\

\begin{align*}
\text{Figure 1: Simple Types, Values, and Expressions}
\end{align*}

2 Setting the Stage

In the technical part of the paper, we focus our attention to a first-order, strictly evaluated functional language. One can think of this language as a simple subset of OCaml or SML. The only recursive data type in the language is the list type. However, our work extends to the expected algebraic data types definable in RaML. Being first order, the language does not allow arbitrary local functional definitions. Instead, all functions are defined at the top level and are mutually recursive by default. The types of these functions form a signature for the program, and the semantics and typing judgments will be indexed by this signature. Thus, the function types of the language can be expressed as arrows between zero-order (base) types. Types are formally defined in Figure 1. Like in all grammars, we provide the abstract (left) and concrete (right) syntax for every type former [19]. A signature $\Sigma : \text{Var} \rightarrow \text{FTypes}$ is a map from variables to first-order types. A program $P$ is a $\Sigma$ indexed map from $\text{Var}$ to pairs $(y_f, e_f)_{f \in \Sigma}$, where $\Sigma(y_f) = \tau \rightarrow \tau'$, and $\Sigma; y_f : \tau \vdash e_f : \tau'$ (the type system is discussed in Section 4). We write $P : \Sigma$ to mean $P$ is a program with signature $\Sigma$.

To simplify the presentation, the expressions of our language (see Figure 1) are in \textit{let normal form} (also \textit{A normal form}). The one nonstandard construct is \textit{share} $x$ as $x_1, x_2$ in $e$, which we explain in more detail in the following sections. We introduce two distinct notions of \textit{linearity}, one on the syntactic level, and one on the semantic level. Syntactic linearity is linearity in expression variables, while semantic linearity is linearity in locations (defined below). We say that a semantics is linear if it respects semantic linearity.

In line with previous works on space cost semantics [38, 30], we employ a heap, which persistently binds locations to values (normalized terms). As usual, we derive the cost of a (terminating) program from the number of heap locations used during execution, which in our case is the \textit{maximum difference} between the sizes of the initial and final freelist. We let $\text{Loc}$ be an infinite set of names for addressing the heap. For the rest of the paper, we use the following: $\text{Stack} \triangleq \{ V | V \rightarrow \text{Var} \}$ and $\text{Heap} \triangleq \{ H | H \rightarrow \text{Loc} \rightarrow \text{Val} \}$ for the set of stacks and heaps respectively.

Reachability Before we define the rules for the cost semantics, we relate the heap locations to values with the 3-place reachability relation $\text{reach}(H, v, L)$ on $\text{Heap} \times \text{Val} \times \varphi(\text{Loc})$, where $\varphi$ is the powermultiset. This is read as “under heap $H$, the value $v$ reaches the multiset of locations $L$”. Write $L = \text{reach}_H(v)$ to indicate this is a functional relation justified by the (valid) mode $(+, +, -)$. We say that the reachable set of $v$ is $L$.

\[
\begin{align*}
A &= \text{reach}_H(v_1) \\
B &= \text{reach}_H(v_2) \\
A \uplus B &= \text{reach}_H((v_1, v_2)) \\
A &= \text{reach}_H(l) \\
\{l\} \uplus A &= \text{reach}_H(l) \\
\emptyset &= \text{reach}_H(v) \\
\end{align*}
\]
In the rules, $\psi$ is multiset union. $L$ is a multiset because we need to keep track of the number of ways a location might be reached in order to prove soundness. However, the cost semantics can be read by truncating any multiset to a set. Furthermore, we will sometimes mix multiset and set operations as the situation calls for. For example, we will write $l \in S$ for a multiset $S$ if $S(l) \geq 1$. Complete definitions and notations can be found in the appendix.

The notion of reachability naturally lifts to expressions and contexts:

$$reach_H(V) = \biguplus_{x \in \text{dom}(V)} reach_H(V(x))$$

$$\text{locs}_{V,H}(e) = reach_H(V \upharpoonright_{\text{FV}(e)})$$

Where $FV : \text{Exp} \to \mathcal{P}(\text{Var})$ denotes the set of free-variables of expressions as usual.

Towards the Garbage Collection Cost Semantics  
Now we are ready to give a first attempt to modeling the cost semantics for a tracing garbage collector. Before we present our new semantics, we explain an existing cost semantics we experimented with [30]. Judgements have the form $V, H, R \vdash e \downarrow^s v, H'$, which can be read as follows. Under stack $V \in \text{Stack}$, heap $H \in \text{Heap}$, and continuation set $R \subseteq \text{Loc}$, $e$ evaluates to $v$ and $H'$ using $s$ heap locations. The idea is that $R$ keeps track of the set of locations necessary to complete the evaluation after $e$ is evaluated (hence the name continuation). For example, we have the let rule:

\[
V, H, R \uplus \text{locs}_{V,H}(x,e_2) \vdash e_1 \downarrow^{s_1} v_1, H_1 \quad V[x \mapsto v_1], H, R \vdash e_2 \downarrow^{s_2} v_2, H_2 \\
\Rightarrow \\
V, H, R \vdash \text{let}(e_1; x : \tau.e_2) \downarrow^{\max(s_1,s_2)} v_2, H_2
\]

Notice that to evaluate $e_1$, we have to extend the continuation $R$ with locations in $e_2$, which will be used after $e_1$ is evaluated. The total space used is the max of the component, indicating that locations used for $e_1$ can be reused for $e_2$. This is clear when we look at the variable rule:

\[
V(x) = v \\
\vdash V, H, R \vdash x \downarrow^{\|\text{dom}(R \uplus \text{reach}(H))\|} v, H
\]

It states that evaluating a variable $x$ requires the locations reachable from $x$ as well as the continuation set $R$. While this way of counting heap locations does model a tracing garbage collector, it is not compatible with the existing type systems for amortized analysis. In these systems, such as RaML, the type rules count the heap locations as data is created, i.e. at each data constructor. Thus looking up a variable incurs no cost, since it was accounted for during creation. On the other hand, the cost of indexing a variable in the semantics includes the cost of the entire continuation set, which is potentially unbounded. This mismatch between the dynamics and statics of language prevents us from proving the soundness of the analysis. We give a new cost semantics that is 1) compatible with the type system and 2) also a more concrete model of a garbage collector since costs are realized with explicit locations.

3 Garbage Collection Cost Semantics

In this section, we present our novel cost semantics by combining freelist semantics from [25] with the cost semantics for modeling perfect GC [30] that we discussed in the previous section. The resulting semantics, called $\mathcal{E}_g$, is well suited for proving the soundness of the novel type-based bound analysis.

The garbage collection cost semantics $\mathcal{E}_g$ is defined by a collection of judgement of the form

$$C \vdash_{P, \Sigma} e \downarrow v, H', F'$$

Where $C \in \text{Stack} \times \text{Heap} \times \varphi(\text{Loc}) \times \mathcal{P}(\text{Loc})$ is a configuration usually written with variables $V, H, R, F$. Because the signature $\Sigma$ for the mapping of function names to first-order functions does not change during evaluation, we drop the subscript $P : \Sigma$ from $\vdash_{P, \Sigma}$ when the context of evaluation is clear. Given a configuration $C = (V, H, R, F)$, the evaluation judgment states that under stack $V$, heap $H$, continuation (multi)set
$R$, freelist $F$, and program $P$ with signature $\Sigma$, the expression $e$ evaluates to value $v$, and engenders a new heap $H'$ and freelist $F'$. In comparison with the attempt from the previous section, the key ingredient we added is the freelist, which serves as the set of available locations. Similar to the predicate $\text{reach}$, we call $R$ a (multi)set since the fact that it’s a multiset is only useful during the soundness proof. For evaluation, it is convenient to just view $R$ as a set. Define a computation as a pair $(C,e)$ of a configuration $C$ and an expression $e$. Next, we give some coherence conditions to a configuration. For a configuration $(V,H,R,F)$, denote the garbage w.r.t. a set of locations $L$ as $\text{collect}(R,L,H,F) = \{ l \in H \mid l \notin F \cup R \cup L \}$.

**Definition 1.** A configuration $(V,H,R,F)$ is well-formed if

1. $\text{dom}(H) \subseteq \text{reach}_H(V) \cup R \cup F$
2. $\text{reach}_H(V) \cup R \subseteq \text{dom}(H) \setminus F$
3. $\text{collect}(R, \text{reach}_H(V), H, F) = \emptyset$

Furthermore, this condition is invariant under evaluation:

**Lemma 1.** Given a well-formed context $(V,H,R,F)$ and $V,H,R,F \vdash e \downarrow v$, $H',F'$, we have $\text{dom}(H') \subseteq \text{reach}_H(v) \cup R \cup F'$, $\text{reach}_H(v) \cup R \subseteq \text{dom}(H') \setminus F'$, and $\text{collect}(R, \text{reach}_H(v), H', F') = \emptyset$.

The well-formed conditions ensure the stack and continuation sets are within the active region of the heap $H \setminus F$, and that the active region of the heap does not contain garbage – all garbage locations are already in the freelist. From now on, all configurations are implicitly assumed to be coherent in the sense defined above.

The semantics $\mathcal{E}_{\text{gc}}$ is designed to model the heap usage of a program running with a tracing counting garbage collector: whenever a heap cell becomes unreachable from the root set, it becomes collected and added to the freelist as available for reallocation. As before, the continuation set $R$ represents the set of locations required to compute the continuation excluding the current expression. We define the root set as the union of the locations in the continuation set $R$ and the locations in the current expression $e$.

The inference rules for the semantics are given in Figure 2. For example, the rule $F:\text{CondT}$ states that, to evaluate a conditional, look in the stack for the value of the branching boolean. In the case it is true, we proceed to evaluate the first branch. Furthermore, we collect cells in the heap that are not reachable from the root set $(R \cup \text{locs}_{V',H(e_1)})$ or already in the current free-list $F$, and add them (g) to the available cells for evaluating $e_1$.

Another example is the rule $F:\text{Let}$ for let expressions: to evaluate the expressions $\text{let}(e_1; x: \tau.e_2)$, we evaluate the first expression with the corresponding restricted stack $V_1$ and an expanded continuation set $R'$. The extra locations come from the free variables of $e_2$ (not including the bound variable $x$), which we cannot collect during the evaluation of $e_1$. Next, we restrict the extended stack to only free variables of $e_2$, and evaluate $e_2$ with this stack and the original continuation set $R$. The other rules are similar.

Note that in contrast to the semantics in the previous section, evaluating a variable does not incur any cost. This ensures that we will be able prove the soundness of the type system. Also, since we don’t allow local function definitions, we do not create closures during evaluation. Also note that we restrict the domain of the stack to the appropriate variables during evaluation. This is only to facilitate the proof of the linearity of the copying semantics introduced later, and not necessary for the implementation.

For example, we can implement the $\text{append}$ and $\text{appTwice}$ function, which has variable sharing. First, we analyze the heap usage of $\text{append}$ under $\mathcal{E}_{\text{gc}}$. We case on the first component of the input. In case it’s nil, we just return 12, and there are no allocations or deallocations. In case it’s cons of $x$ and $xs$, we need to allocate one heap location for the cons cell binding $x$ and the recursive result, for which we can use the just matched-on cell. Again, the net overhead is zero. Thus, the total space overhead of $\text{append}$ is zero.

For $\text{appTwice}$, we first share the list 1 as 11 and 12. In the first $\text{let}$, the locations in 12 are added to the continuation set, which prevents the first call to $\text{append}$ from destructing 11. Thus size of 11 new locations are allocated from the freelist to construct 11'. The second call has no net increase in heap allocations since 12 can be destructed along the way. The return value is a pair which is stack-allocated and doesn’t require a heap allocation. Thus, the total space overhead for $\text{appTwice}$ is size of the input list 1.
We will use the allocation/heap metric which simply counts the number of cons constructor calls during the evaluation cost of the next step and the potential of the next program state. The idea of AARA \[24, 27, 21, 22\] is to automate Amortization with GC.

\[V_1 = V \mid_{FV(e_1)} \quad R' = R \cup \text{locsv}_{V,H}(\text{lam}(x : \tau,e_2)) \]
\[V_1, H, R', F \vdash e_1 \Downarrow v_1, H_1, F_1 \]
\[V_2 = (V[x \mapsto v_1]) \mid_{FV(e_2)} \quad g = \{ l \in H \mid l \notin F \cup R \cup \text{locsv}_{V,H}(e_2) \} \]
\[V_2, H_1, R, F_1 \cup g \vdash e_2 \Downarrow v_2, H_2, F_2 \]
\[V, H, R, F \vdash \text{let}(e_1; x : \tau,e_2) \Downarrow v_2, H_2, F_2 \]

(F:Let)

\[V(x) = \top \quad V' = V \mid_{FV(e_1)} \quad V', H, R, F \vdash e_1 \Downarrow v, H', F' \]
\[V, H, R, F \vdash \text{if}(x; e_1; e_2) \Downarrow v, H', F' \]

(F:CondT)

\[V(x) = F \quad V' = V \mid_{FV(e_2)} \quad V', H, R, F \vdash e_2 \Downarrow v, H', F' \]
\[V, H, R, F \vdash \text{if}(x; e_1; e_2) \Downarrow v, H', F' \]

(F:CondF)

\[V' = ([y \mapsto v']) \mid_{FV(e_f)} \quad g = \{ l \in H \mid l \notin F \cup R \cup \text{locsv}_{V,H}(e_f) \} \quad V', H, R, F \vdash e_f \Downarrow v, H', F' \]
\[V, H, R, F \vdash \text{app}(f; x) \Downarrow v, H', F' \]

(F:App)

\[V, H, R, F \vdash \text{null} \Downarrow (\text{val}(\text{Null}), H, F) \]
\[V(x) = \text{null} \quad g = \{ l \in H \mid l \notin F \cup R \cup \text{locsv}_{V,H}(e_1) \} \quad V', H, R, F \vdash e_1 \Downarrow v, H', F' \]
\[V, H, R, F \vdash \text{match}\{x\}(e_1; x_h, x_2, e_2) \Downarrow v, H', F' \]

(F:MatNil)

\[V(x) = v \quad V, H, R, F \vdash x \Downarrow v, H, F \]

(F:Var)

\[V(x) = l \quad H(l) = \langle v_h, v_i \rangle \quad V' = (V[x_h \mapsto v_h, x_i \mapsto v_i]) \mid_{FV(e_2)} \quad g = \{ l \in H \mid l \notin F \cup R \cup \text{locsv}_{V,H}(e_2) \} \quad V', H, R, F \vdash e_2 \Downarrow v, H', F' \]
\[V, H, R, F \vdash \text{match}\{x\}(e_1; x_h, x_2, e_2) \Downarrow v, H', F' \]

(F:MatCons)

\[V(x) = v' \quad V' = (V[x_1 \mapsto v', x_2 \mapsto v']) \mid_{FV(e)} \quad g = \{ l \in H \mid l \notin F \cup R \cup \text{locsv}_{V,H}(e) \} \quad V', H, R, F \vdash e \Downarrow v, H', F' \]
\[V, H, R, F \vdash \text{share}(x; x_1, x_2, e) \Downarrow v, H', F' \]

(F:Share)

Figure 2: Cost Semantics for Perfect Garbage Collection

From this, we see that the minimum size for the initial freelist to successively evaluate a call to \texttt{appTwice} is exactly the length of the input. In general, we define the cost of a closed program to be the minimum size of the initial freelist that guarantees successful evaluation, which is equivalent to the cost annotation in the previous cost semantics introduced in Section 2.

4 Automatic Amortized Heap-Space Analysis with GC

Automatic Amortized Resource Analysis (AARA) The idea of AARA \[24, 27, 21, 22\] is to automate the potential method of amortized analysis using a type system. Types introduce potential functions that map data structures of the given type to non-negative numbers. The type rules ensure that there is always sufficient potential to cover the evaluation cost of the next step and the potential of the next program state.

To illustrate the idea, we informally explain the linear potential method for the functions in Figure 3. We will use the allocation/heap metric which simply counts the number of cons constructor calls during the
evaluation. With this metric, the cost of evaluating \texttt{append(11,12)} is \(m\), where \(m\) is the number of cons constructors in 11, and the resource annotated type of \texttt{append} is \(L^1(\text{int}) \times L^0(\text{int}) \xrightarrow{0/0} L^0(\text{int})\). This type says that to type \texttt{append(11,12)}, we need 11 to have 1 potential per element, 12 to have 0 per element, and the result will be a list with 0 potential per element. Additionally, the function uses 0 constant potential, and leaves 0 constant potential after evaluating. This translates to a bound which states that the number of allocations \texttt{append} makes is bounded by 1 times size of the first list. For \texttt{appTwice(1)}, the cost under the heap metric is \(2m\), where \(m\) is the number of cons constructors in 1. This is because we have to share the input list across two calls of \texttt{append}, which each requires lists with unit potential per element. For example, if \(l : L^2(\text{int})\), then 11 and 12 both get 1 potential per element so that \(11 : L^1(\text{int}), 12 : L^1(\text{int})\), which covers the cost of the next 2 calls to \texttt{append}, and the resulting pair of lists both have 0 potential per element.

More generally, we can give the following types to \texttt{append} and \texttt{appTwice}:

\[
\text{append} : L^p(\text{int}) \times L^q(\text{int}) \xrightarrow{r/q'} L^s(\text{int}), \text{where } p \geq s + 1, q \geq s \text{ and } r \geq r'
\]

\[
\text{appTwice} : L^p(\text{int}) \xrightarrow{q'/r'} L'^s(\text{int}) \times L^q(\text{int}), \text{where } p \geq r + s + 2 \text{ and } q \geq q'
\]

Notice that the constant potentials \(r\) and \(q\) are unconstrained since the functions don’t use any potential in the base cases. With AARA, the type system keeps track of this collection of constraints on resource annotations and passes them to an off-the-shelf LP-solver which finds the minimum solution. This is then translated to concrete resource bounds like the ones we derived by hand. It has been shown that this technique can be extended to polynomial potential functions, user-defined data types, and higher-order functions while still relying on linear constraint solving [21, 22].

**Linear Potential Functions** Before giving the type rules, we need to formalize linear potential as explained above. Since potential is associated with the structure of a value and not the particular heap locations, it is helpful to introduce a mapping from heap values to semantic values of a type. First, we give a denotational semantics for (define the structures of) the first-order types:

\[
\begin{align*}
[\text{unit}] &= \{\text{val(Null)}\} \\
[\text{bool}] &= \{\text{val(T), val(F)}\} \\
[\text{nat}] &= \mathbb{N} \\
[\text{int}] &= \mathbb{Z} \\
[\text{arr}] &= \{\text{cons(a;l): } a \in [A] \text{ and } l \in [L(A)]\}
\end{align*}
\]

The meaning of each type is the least set such that the above holds. As usual, we write \([a_1,\ldots,a_n]_{\text{cons}}\) for \(\text{cons}(a_1;\ldots,\text{cons}(a_n;\text{nil}))\).

In Figure 4 we give the judgements relating heap values to semantic values, in the form \(H \models v \mapsto a : A\), which can be read as follows: Under heap \(H\), heap value \(v\) defines the semantic value \(a \in [A]\). Given a stack \(V\), we write \(H \models V : \Gamma\) if \(\text{dom}(V) \subseteq \text{dom}(\Gamma)\) and for every \(x \mapsto v \in V\), \(H \models V(x) \mapsto a : \Gamma(x)\) for some \(a \in [A]\).

We introduce linear potential for structures corresponding to the base types. The definition of linear potential is standard [20]. Below is the grammar for resource-annotated types:

\[
\begin{align*}
\text{BTypes } A &::= \\
\text{FTypes } &::= \rho \ ::= \\
& \dots \\
& \text{list}^p(A) \quad L^p(A)
\end{align*}
\]

\[
\begin{align*}
\text{arr}(A_1; A_2; p; q) & \quad A_1 \frac{p/q}{A_2}
\end{align*}
\]

\footnote{This is in contrast to the highwater mark for the GC semantics \(E_{\text{gc}}\) that is targeted by our new analysis.}
The intended meaning is that a list of $L^p(A)$ has $p$ units of potential per cons cell, and a function of type $A \rightarrow B$ takes constant potential $p$ to run and $q$ is the constant potential left afterwards.

With linear potential, each component of a structure is associated with a constant amount of potential. Given a structure $a$ in a heap $H$, where $H \models v \mapsto a : A$, we define its potential $\Phi_H(a : A)$ by recursion on $A$:

$$\Phi_H(v : A) = 0$$

if $A \in \{\text{unit, bool, nat}\}$

$$\Phi_H([v_1, v_2] : A_1 \times A_2) = \Phi_H(v_1 : A_1) + \Phi_H(v_2 : A_2)$$

$$\Phi_H(l : L^p(A)) = p + \Phi_H(v_h : A) + \Phi_H(v_T : L^p(A))$$

if $H(l) = [v_h, v_T]$.

Write $\Phi_{V,H}(\Gamma)$ for $\Sigma_{x \in \text{dom}(V)} \Phi_H(V(x) : \Gamma(x))$.

Now define $A \gamma A_1, A_2, n$ as the sharing relation for resource-annotated types:

$$L^p(A) \gamma L^q(A_1), L^r(A_2)$$

if $p = q + r + n$ and $A \gamma A_1, A_2$

$$A \times B \gamma A_1 \times B_1, A_2 \times B_2$$

if $A \gamma A_1, A_2$ and $B \gamma B_1, B_2$

$$A \gamma A, A$$

if $A \in \{\text{unit, bool, nat}\}$.

The sharing relation captures the amount of potential needed to copy a type $A$ where each cons node in any structure in $[[A]]$ has a copying overhead $n$.

**Type Rules** The type system $\text{FO}^{gc}$ consists of rules of the form

$$\Sigma; \Gamma \vdash \frac{q}{q'} e : A$$

read as under signature $\Sigma : \text{Var} \rightarrow \text{FTypes}$, typing environment $\Gamma : \text{Var} \rightarrow \text{BTypes}$, $e$ has type $A$ starting with $q$ units of constant potential and ending with $q'$ units.

Our type system is based on the one of classic linear AARA [24]. We give a review of the rules in Figure 5. Since we are interested in the number of heap locations, there is an implicit side condition in all rules which ensures all constants are assumed to be nonnegative.

For example, $L:\text{Cons}$ states that to add an element to a list with $p$ potential per element, we need $p + 1$ units of constant potential: $p$ to maintain the potential of the list, and 1 for allocating the cons cell. $L:\text{MatL}$ states that matching on a list with type $L^p(A)$, we need to type the nil case with the same constant potentials, and we need to type the cons case with an additional $p$ units of constant potential, since we get the spill of $p$ from the definition of linear potential. As the last example, we look at $L:\text{Share}$, which states that to share a variable $x$ of type $A$, we need to split the potential between $A_1$ and $A_2$, and type the rest of the expression with the two new variables $x_1 : A_1$, $x_2 : A_2$.

**New Rules** The new type system for programs with garbage collection replaces the rules $L:\text{MatL}$ and $L:\text{Share}$. The observation is that if we ensure that locations are used linearly, we can use destructive pattern
the copying semantics because the often contains an "overhead" of 1 for every cons node in any structure in J.

In previous versions of AARA [27, 21], the typing judgment and cost semantics are refinements to model local garbage collection by returning the potential associated with the constructor location (notice the extra +1 in the second premise):

\[
\Sigma; \Gamma; \tilde{\cdot} \vdash e_1 : B \\
\Sigma; \Gamma, x_h : A, x_t : L^p(A) \frac{\langle q+p+1 \rangle_1}{q} e_2 : B \\
\Sigma; \Gamma, x : L^p(A) \frac{\langle q \rangle_1}{q} \text{match } \{ \text{nil} \leftrightarrow e_1 | \text{cons}(x_h; x_t) \leftrightarrow e_2 \} : B 
\]

This is validated by the fact (Lemma 10) that in the auxiliary copying semantics (introduced in later), once a cons-cell is matched on, there can be no live references from the root set to it, and thus we are justified in restituting the potential to type the subexpression e2.

However, the rule L:MatLD is not sound for programs with aliasing of data. We address this issue by replacing the rule L:Share with the rule L:ShareCopy:

\[
A \forall A_1, A_2 \\
\Sigma; \Gamma, x_1 : A_1, x_2 : A_2 \frac{\langle q \rangle_1}{q} e : B \\
\Sigma; \Gamma, x : A \frac{\langle q \rangle_1}{q} \text{share } \text{as } x_1, x_2 \text{ in } e : B 
\]

To share a variable of type A, we need to split the potential between two new annotated types A1 and A2 as usual. In addition, we have to pay an “overhead” of 1 for every cons node in any structure in [A]. The idea is that we treat data as if it is actually copied. This is sound w.r.t. the copying semantics because the size of the domain of the reachable set of a value v is exactly the linear potential of v : A with all resource annotations set to 1.

For example, Figure 6 contains derivations for append and appTwice. Here, A is short for int and \( \Sigma = \text{[append \rightarrow L^p(A) \times L^q(A) \frac{\langle q \rangle_1}{q} L^p(A)]} \) is the program signature.

From these derivations, we get the improved space overhead bound to append and appTwice:

\[
\text{append : } L^p(\text{int}) \times L^q(\text{int}) \xrightarrow{r/r'} L^s(\text{int}), \text{ where } p \geq s, q \geq s, \text{ and } r \geq r' \\
\text{appTwice : } L^p(\text{int}) \xrightarrow{q/q'} L^r(\text{int}) \times L^s(\text{int}), \text{ where } p \geq r + s + 1 \text{ and } q \geq q' 
\]

Cost Metrics In previous versions of AARA [27, 21], the typing judgment and cost semantics are parametrized by a cost metric \( m : \text{res} \times \text{const} \rightarrow \mathbb{Q} \), which assigns a constant cost to each step in the semantics. Recall the heap metric introduced above; formally, this is the function \( k \mapsto 1_{k=k_{\text{const}}} \). We instantiate
Theorem 2 (Soundness). Let $H = V; \Gamma$, $\Sigma; \Gamma \vdash_{q/r} e : B$, and $V, H \vdash e \downarrow v, H'$. Then for all configurations $W, Y, F, R$, if $V, H \leadsto W, Y$ and $|F| \geq \Phi_{V,H}(\Gamma) + q$, there exists a value $w$, and a freelist $F'$ such that

$$W, Y, R, F \vdash e \downarrow w, Y', F' \quad \text{and} \quad v \leadsto_{F'} w.$$  

Here, $\mathcal{E}_{\text{oper}}$ is a standard big-step semantics, with judgments of the form $\overline{V, H \vdash e \downarrow v, H'}$ derived from $\mathcal{E}_{gc}$. $V, H \leadsto W, Y$ is context equivalence, and $v \leadsto_{F'} w$ is value equivalence (these are defined below). The
Theorem states that, given a terminating expression and a freelist that is sufficiently large (as predicated by the type derivation), a run with $E_{gc}$ will normalize to an equivalent value.

To facilitate the proof, we define an intermediate semantics $E_{copy}$ which is semantically linear. The proof has two stages: First, we show $E_{copy}$ over-approximates $E_{gc}$, meaning that any computation that succeeds with $E_{gc}$ will succeed with an equally-sized or smaller freelist with $E_{copy}$. Then we show $FO^{gc}$ is sound with respect to $E_{copy}$, and thus by the previous step sound with respect to $E_{gc}$.

As mentioned above, we introduce a big step semantics $E_{oper}$ that does not use freelists or account for garbage collection. We use it to characterize expressions that normalize to values when initialized with a sufficient freelist. This technique has also been employed in earlier work on AARA [25]. In the judgment $V, H \vdash e \downarrow v, H'$, the “freelist” is the whole ambient set of locations $Loc$, thus we never run out of locations during evaluation. This introduces a problem for value and context equivalence: when comparing evaluation results between a run with $E_{copy}$ and $E_{oper}$, the return values might not be syntactically equal. Consider the following expression $e = \text{let } _- = [4] \text{ in } [5]$. Let $Loc = \mathbb{N}$, the natural numbers. Consider the evaluation $\emptyset, \emptyset, \emptyset, \{1\} \vdash_{E_{copy}} e \downarrow v_1, H_1, F_1$. First, 1 is allocated and mapped to [4]. Then, since the first subexpression [4] is not used afterwards, we collect 1, and reuse it and map again to [5]. Thus $v_1 = 1$. In an evaluation $\emptyset, \emptyset, \emptyset, \{1\} \vdash_{E_{copy}} e \downarrow v_2, H_2, F_2$, we also first map 1 to [4], but then allocate a new location, say 2, and map it to [5], and $v_2 = 2$. Due to the difference in allocation strategies and the fact that both are nondeterministic, we need a more robust notion of equality for values. Luckily, the structures from the denotational semantics (defined in Section 4) does the job. In both runs, the return value maps to the semantic value [5]. Thus we use semantic equality as the basis for value and context equivalence:

**Definition 2 (Value Equivalence).** Two values $v_1, v_2$ are equivalent (with the presupposition that they are well-formed w.r.t. heaps $H_1, H_2$), iff $H_1 \models v_1 \mapsto a : A$ and $H_2 \models v_2 \mapsto a : A$. Write value equivalence as $v_1 \overset{H_1}{\sim} v_2$.

**Definition 3 (Context Equivalence).** Two contexts $(V_1, H_1), (V_2, H_2)$ are equivalent iff $dom(V_1) = dom(V_2)$ and for all $x \in dom(V_1)$, $V_1(x) \overset{H_1, H_2}{\sim} V_2(x)$. Write context equivalence as $(V_1, H_1) \overset{H_2}{\sim} (V_2, H_2)$.

Stated simply, two contexts are equivalent when they have the same domain and equal variables bind equal semantic values.

**Linear Garbage Collection Cost Semantics** To establish the soundness of the type system, we need an intermediary semantics $E_{copy}$, which is *semantically linear*. As mentioned in Section 2, this means that locations are treated linearly, that is, no location can be used twice in a program. Variable sharing is achieved via *copying*: the shared value is obtained by allocating a fresh set of locations from the freelist and copying the locations of the original value one by one. This is also sometimes referred to as *deep copying*. Let $copy(H, L, v, H', v')$ be a 5-place relation on $Heap \times P(\text{Loc}) \times Val \times Heap \times Val$. Similar to reachability, we write this as $H', v = copy(H, L, v)$ to signify the intended mode for this predicate: $(+, +, -, -, -)$.

\[
\begin{align*}
& v \in \{n, T, F, \text{Null}\} \\
& H, v = copy(H, L, v) \\
\end{align*}
\]

\[
\begin{align*}
& \quad l' \in L \\
& \quad H', v = copy(H, L \setminus \{l'\}, H(l)) \\
\end{align*}
\]

\[
\begin{align*}
& \quad |L_1| = |\text{dom}(\text{reach}_H(v_1))| \\
& \quad H_1, v_1 = copy(H, L_1, v_1) \\
\end{align*}
\]

\[
\begin{align*}
& \quad H_2, (v_1', v_2') = copy(H, L, (v_1, v_2)) \\
\end{align*}
\]

Primitives require no cells to copy; a location value is copied recursively; a pair of values is copied sequentially, and the total number of cells required is the size of the reachable set of the value. Now, consider $E_{gc}$ with the share rule $F:\text{Share}$ replaced with the following rule:

\[
\begin{align*}
& \quad L \subseteq F \\
& \quad |L| = |\text{dom}(\text{reach}_H(v'))| \\
& \quad H', v'' = copy(H, L, v') \\
& \quad V' = (V[x_1 \mapsto v', x_2 \mapsto v'']) |_{FV(e)} \\
& \quad \quad V', H', R, F' \vdash \text{share } x \text{ as } x_1, x_2 \text{ in } \emptyset \vdash v, H'', F'' \quad (E:\text{Share})
\end{align*}
\]
To share a variable, we first copy the shared value. The number of cells required is equal to the size of the reachable set from the value. This copying sharing semantics is what justifies the analysis to restitute the potential when matching on a cons node, since even if the node was shared, we had to pay for the cost by copying the node when sharing the original value. Next, we restrict the stack to the appropriate variables. Lastly, any locations not reachable from the current subexpression $e$ are collected. This is for the case when a variable is shared but not used later.

Recall that a computation is a pair $(C, e)$ consisting of a configuration $C = (V, H, R, F)$ and an expression $e$. Since the cost semantics can only preserve the linearity of a computation, we restrict our attention to computations that are linear initially, and show that $\mathcal{E}_{\text{copy}}$ respects the linearity of any initially linear computation. This motivates the following definitions:

**Definition 4.** (Linear context) Given a context $(V, H)$, let $x, y \in \text{dom}(V)$, $x \neq y$, and $r_x = \text{reach}_H(V(x))$, $r_y = \text{reach}_H(V(y))$. It is linear given that $\text{set}(r_x), \text{set}(r_y)$, and $r_x \cap r_y = \emptyset$.

Where $\text{set}(S)$ means $S$ a proper set $(\forall x, S(x) \leq 1)$. Denote this by $\text{linearCtxt}(V, H)$. Whenever $\text{linearCtxt}(V, H)$ holds, there is at most one path from a variable on the stack $V$ to any location in $H$. Now we can formalize our intuition for linear computations:

**Definition 5** (Linear computation). Given a configuration $C = (V, H, R, F)$ and an expression $e$, we say the 5-tuple $(C, e)$ is a computation; it is a linear computation given that $\text{dom}(V) = \text{FV}(e)$, $\text{linearCtxt}(V, H)$, and $\text{disjoint}(R, F, \text{locs}_V, H(e))$. And we write $\text{linearComp}(V, H, R, F, e)$ (equivalently $\text{linearComp}(C, e)$) to denote this fact.

Intuitively, we expect that any terminating computation with $\mathcal{E}_{\text{copy}}$ has a corresponding run with $\mathcal{E}_{\text{gc}}$ that can be instantiated with an equally-sized or smaller freelist. Although this seems quite straightforward to prove, a complete characterization of the relationship between the space allocations of two runs with each semantics is necessary. To demonstrate the difficulties involved, consider the following proof attempt:

**Attempt 1.** Let $C_2 = (V, H, R, F)$ be a configuration and $(C_2, e)$ be a linear computation. Given that $C_2 \vdash \mathcal{E}_{\text{copy}} e \downarrow v, H', F'$, for all configurations $C_1 = (W, Y, R, M)$ such that $W, Y \sim V, H$ and $|M| = |F|$, there exists a triple $(w, Y', M') \in \text{Val} \times \text{Heap} \times \text{Loc}$ such that

$$C_1 \vdash \mathcal{E}_{\text{copy}} e \downarrow w, Y', M' \quad \text{and} \quad v \sim_{V, H'} w \quad \text{and} \quad |M'| \geq |F'| .$$

We proceed with induction on the derivation of the judgment $\mathcal{E}_{\text{copy}}$. Almost every case goes through, save for $\text{E}$. Let. First, we get $W_1, Y \sim V_1, H$ and we have the following from induction on the first premise:

$$W_1, Y, R', M \vdash \mathcal{E}_{\text{copy}} e \downarrow w_1, Y_1, M_1 \quad \text{and} \quad v_1 \sim_{V_1} w_1 \quad \text{and} \quad |M_1| \geq |F_1|$$

To instantiate the induction hypothesis on the second premise, we need to show that, among other things, $|M_1 \cup j| \geq |F_1 \cup g|$, where $j$ is the set of collected locations in the $\mathcal{E}_{\text{gc}}$ judgment. We cannot show this precisely because $g$ might contain more cells then $j$ due to the linearity of $\mathcal{E}_{\text{copy}}$, thus preventing a piecewise comparison. But of course $|j|$ is always less than $|g|$, since $\mathcal{E}_{\text{gc}}$ doesn’t copy to share values! This shows that there is a mismatch between the induction hypothesis and the relationship between the sizes of the respective freelists and the garbage sets. Specifically, we need to know exactly how much larger $M_1$ is compared to $F_1$ at any given step.

Having a sense of what is missing, we formulate the criteria which characterize the required equivalence between two configurations, which we call copy extension.

**Definition 6.** A configuration $C_2 = (V_2, H_2, R_2, F_2)$ is a copy extension of another configuration $C_1 = (V_1, H_1, R_1, F_1)$ iff

1. $V_1, H_1 \sim V_2, H_2$
2. There is a proper partition $\gamma : \text{dom}(H_1) \setminus F_1 \to \mathcal{P}(\text{dom}(H_2) \setminus F_2)$ such that for all $l \in \text{dom}(\gamma)$, $|\gamma(l)| = \text{reach}_{H_1}(V_1)(l) + R_1(l)$
3. For all $l \in \text{dom}(\gamma)$, $x \in \text{dom}(V_1)$, sequence of directions $P$ which is valid w.r.t. $V_1(x)$, $\text{reach}_H(V_2(x; P)) \cap \gamma(l) = \text{reach}_H(V_1(x; P))(l)$. 

4. For all $l \in \text{dom}(\gamma)$, $|\gamma(l) \cap R_2| = R_1(l)$

5. $|F_1| = |F_2| + |\ominus(\gamma)|$, where $\ominus(\gamma) = \bigcup_{P \in ec(\gamma)} P \setminus \text{rep}(P)$

Write this as $C_1 \leq C_2$.

The intention is that $C_2$ is a configuration for an evaluation using $\mathcal{C}_{\text{copy}}$, and $C_1$ a configuration for $\mathcal{C}_{\text{gc}}$. The first condition is the straightforward context equivalence. The second condition requires the existence of a mapping $\gamma$ that tells us given a location in $H_1 \setminus F_1$, which locations in $H_2 \setminus F_2$ are shared instances.

For example, consider the expression $\text{share } x$ as $x_1, x_2$ in $e$ and assume the stack is $[x \mapsto 1]$, and the heap equals $[1 \mapsto \langle 0, \text{null} \rangle]$, i.e. $x$ is the list $[0]$. In an evaluation with $\mathcal{C}_{\text{gc}}$, the stack becomes $[x_1 \mapsto 1, x_2 \mapsto 1]$, and the heap does not change. With $\mathcal{C}_{\text{copy}}$, we allocate a new location in the heap: $[1 \mapsto \langle 0, \text{null} \rangle, 2 \mapsto \langle 0, \text{null} \rangle]$, and the stack changes accordingly: $[x_1 \mapsto 1, x_2 \mapsto 2]$. Now $\gamma$ would map 1 to $\{1, 2\}$, since both are shared instances of the former.

Thus, the image of $\gamma$ is a collection of disjoint subsets whose union is $\text{dom}(H_2) \setminus F_2$, and each location in $\text{dom}(H_2) \setminus F_2$ belongs to a unique class whose preimage is the unique representative in $\text{dom}(H_1) \setminus F_1$. Furthermore, we noticed it is crucial to include the fact that the size of $\gamma(l)$ must be the sum of the number of references from the stack and the continuation set. Furthermore, we also require each subset $\gamma(l)$ (also referred to as class) to be nonempty (this is the proper partition condition).

While $\gamma$ gives us a relation between the active regions of two respective heaps, we still need to know exactly how variables on the stack factor in this relationship. Let $l \in H_1$. Specifically, we need to know that the number of references to $l$ from every subvalue in $V_1$ is equal to the size of the corresponding part of the class $\gamma(l)$. First, we need to access subvalues of a value using directions:

Definition 7. Let $\text{Dir}$ be the set $\{L, R, N\}$, denoting left, right, and next respectively. We define the function $\text{get}_H : (1 \oplus \text{Val}) \times \text{Dir} \rightarrow 1 \oplus \text{Val}$ which indexes values via directions:

$$\text{get}_H(\text{Just}((v_1, v_2)), L) = \text{Just}(v_1) \quad \text{get}_H(\text{Just}(l), N) = \text{Just}(l)$$

$$\text{get}_H(\text{Just}((v_1, v_2)), R) = \text{Just}(v_2) \quad \text{get}_H(\ldots, \ldots) = \text{None}$$

Let $P \in S(\text{Dir})$, where $S(X)$ denotes the set of sequence with elements from $X$. We define $\text{find}_H : (1 \oplus \text{Val}) \times S(\text{Dir}) \rightarrow 1 \oplus \text{Val}$ extending $\text{get}_H$ to sequences of directions:

$$\text{find}_H(v, D :: P) = \text{find}_H(\text{get}_H(v, D), P)$$

$$\text{find}_H(v, []) = v$$

Call $P$ valid w.r.t. a value $v$ if $\text{find}_H(v, P) = \text{Just}(v')$ for some $v'$. Given a valid sequence $P$ w.r.t. $V(x)$, write $V_H(x; P)$ for $\text{fromJust}(\text{find}_H(V(x; P)))$ and $\text{reach}_H(V(x; P))$ for $\text{reach}_H(V_H(x; P))$. A map $m_V : X \rightarrow S(\text{dir})$ is a subvalue map given that $X \subseteq \text{dom}(V)$ and each $x \in X$ is mapped to a sequence $P$ which is valid w.r.t. $V(x)$. Given a subvalue map $m_V$, define its action as $\text{reachPath}_{V,H}(X, m) = \bigcup_{x \in X} \text{reach}_H(V(x; m(x)))$.

With this, the third condition gives us a more fine grained restriction: for any subvalue in $V_1$, the number of references from it to $l$ is equal to the size of the intersection of the reachable set of the corresponding subvalue in $V_2$ with the appropriate class $\gamma(l)$.

The next condition simply states that the continuation sets respect $\gamma$. Lastly, we have that $F_1$ is greater than $F_2$, with the overhead $\ominus(\gamma)$ being exactly the sum $\sum_{l \in \gamma} |\gamma(l)| - 1$. Here $ec(\gamma)$ is the image of $\gamma$: $\{\gamma(l) \mid l \in \text{dom}(\gamma)\}$. Since each class $\gamma(l)$ is non-empty, we use $\text{rep}(l)$ to choose an arbitrary element from the class.

Below are some expected properties of a copy extension:

Lemma 3. Let $V_1, H_1 \sim V_2, H_2$. Then for all $x \in \text{dom}(V_1)$ and sequence of directions $P$, Either $\text{find}_H(V_1(x), P) = \text{find}_H(V_2(x), P) = \text{None}$ or $\text{find}_H(V_1(x), P) = v_1$, $\text{find}_H(V_2(x), P) = v_2$ and $v_1 \sim_{H_1} v_2$.
**Lemma 4.** Let $V_2, H_2, R_2, F_2 \vdash \text{copy } e \downarrow v, H', F'$, and $V_1, H_1, R_1, F_1 \leq V_2, H_2, R_2, F_2$, where $\gamma$ is the partition satisfying the copy extension property. Then for all $l \in \text{dom}(\gamma)$ and subvalue map $m_{V_1} : X \rightarrow S(\text{dir})$, $\gamma(l) \subseteq \text{collect}(R_2, \text{reachPath}_{V_2, H_2}(X, m), H_2, F_2)$ if $l \in \text{collect}(R_1, \text{reachPath}_{V_1, H_1}(X, m), H_1, F_1)$.

Now we can state the key lemma:

**Lemma 5.** Let $(C_2, e) = (V, H, R, F, e)$ be a linear computation. Given that $C_2 \vdash \text{copy } e \downarrow v, H', F'$, for all configurations $C_1$ such that $C_1 \leq C_2$, there exists a triple $(w, Y', M') \in \text{Val } \times \text{Heap } \times \text{Loc}$ and $\gamma' : \text{dom}(Y') \setminus M' \rightarrow \mathcal{P}(\text{dom}(H') \setminus F')$ s.t.

1. $C_1 \vdash \text{copy } e \downarrow w, Y', M'$
2. $v \uparrow_{Y'} w$
3. $\gamma'$ is a proper partition, and for all $l \in \text{dom}(\gamma')$, $|\gamma'(l)| = |\text{reach}_{Y_1}(w_1)(l)| + S(l)$
4. For all $l \in \text{dom}(\gamma')$ and $P \in \mathcal{S}(\text{Dir})$ that is valid w.r.t. $v$, $|\text{reach}_{H'}(\text{find}_{H'}(v; P)) \cap \gamma'(l)| = \text{reach}_{Y_1}(\text{find}_{Y_1}(v; P))(l)$
5. For all $l \in \text{dom}(\gamma')$, $\gamma'(l) \cap R = \gamma(l) \cap R$
6. $|M'| = |F'| + |\odot(\gamma')|$

**Proof.** Induction on the evaluation judgment $\vdash \text{copy } e$. We illustrate the ideas with the case CondT:

Case: $V' = V \upharpoonright_{FV(e_1)}$ 
\[
\begin{array}{c}
V(x) = T \\
V, H, R, F \vdash \text{find}(x; e_1; e_2) \downarrow v, H', F'
\end{array}
\]

Let $W, Y, S, M$ be a configuration such that $W, Y, S, M \leq V, H, R, F$. Define $W' = W \upharpoonright_{\text{dom}(V')}$ and $j = \{l \in Y' \setminus M \cup \text{loc}_{V_1, H_1}(e_1)\}$. To instantiate the induction hypothesis, we need to show that $W', Y, S, M \cup j \leq V', H, R, F \cup g$, giving us 5 obligations:

1. $(\gamma'}(l) = \gamma \upharpoonright_{\text{dom}(Y)} \setminus (M \cup j)) \setminus g$. First, we show that $\gamma'$ is a partition. Let $l, l' \in \text{dom}(Y) \setminus (M \cup j)$ be two arbitrary locations. Then $\gamma'(l) \cap \gamma'(l') = \emptyset$ since $\gamma$ is a partition. Now consider the image of $\gamma'$:

\[
\gamma'(\text{dom}(Y) \setminus (M \cup j)) = \gamma(\text{dom}(Y) \setminus (M \cup j)) \setminus g = (\bigcup_{l \in \text{dom}(Y) \setminus M} \gamma(l)) \setminus g
\]

\[
= (\text{dom}(H) \setminus F) \setminus g
\]

\[
\text{dom}(H) \setminus (F \cup g)
\]

Hence $\gamma'$ is a partition. Next we need to show it is proper, or that every class is nonempty. Let $l \in \text{dom}(Y) \setminus (M \cup j)$ be any location. Since $\gamma'(l) = \gamma(l) \setminus g$, it suffices to show that the class $\gamma(l)$ is not all collected. For the sake of contradiction, assume $\gamma(l) \subseteq g$. But then $l \in j$ by Lemma 4, and we have a
contradiction since we assumed \( l \notin j \). Lastly, \(|\gamma'(l)| = \text{reach}_Y(W')(l) + S(l)\) follows from the definition of \( g \) and the second and third condition of copy extension. Conditions (3) and (4) follow similarly.

Lastly, we need to show that the overhead is preserved: \(|M| = |F| + |g| + |\cup \cup (\gamma')|\). By assumption, \(|M| = |F| + |g\cup| \cup (\gamma)|\), so it suffices to show \(|g| + |\cup \cup (\gamma')| = |g\cup| \cup (\gamma)|\). Since \( g \) and the image of \( \gamma' \) are disjoint, it suffices to define a bijection \( f : j \cup \cup (\gamma) \rightarrow g \cup \cup (\gamma') \). First, we separate the classes \( ec(\gamma) \) into those that are completely collected into \( g \) and those that are only partially collected: \( C_1 = \{ \gamma(l) | l \in j \} \) and \( C_2 = ec(\gamma) \setminus C_1 \). Further, let \( D_1 = \bigcup_{C \subseteq C_2} C \setminus \{ \text{rep}(C) \} \) and \( D_2 = \bigcup_{C \subseteq C_2} C \setminus \{ \text{rep}(C) \} \). Then we have \( \cup \cup (\gamma) = D_1 \cup D_2 \) and by Lemma 4, \( g = (\bigcup_{C \subseteq C_1} C) \sqcup L \) for some \( L \). Therefore, we need to find a bijection \( f : j \cup D_1 \cup D_2 \rightarrow (\bigcup_{C \subseteq C_2} C) \sqcup L \cup \cup (\gamma') \). It suffices to find the bijections \( f_1 : j \cup D_1 \rightarrow \bigcup_{C \subseteq C_1} C \) and \( f_2 : D_2 \rightarrow L \cup \cup (\gamma') \). First, we define \( f_1 \):

\[
f_1(x) = \begin{cases} 
\text{rep}(\gamma(l)) & x = (\text{inl}, l) \\
\text{inr} & x = (\text{inr}, l)
\end{cases}
\]

Note that \( f_1 \) simply maps the set of representatives into their respective classes and is the identity on the rest of the class. Thus \( f_1 \) is a bijection, and \(|j| + |D_1| = |\bigcup_{C \subseteq C_1} C|\). Next, note that

\[
|C_2| = |ec(\gamma) \setminus \{ \gamma(l) | l \in j \}| = |ec(\gamma \setminus \{ \text{dom}(Y) \setminus (M \cup j) \})| = |ec(\gamma')|
\]

Which means that \( C_2 \) has the same number of classes as \( \gamma' \) (even though the actual classes might be different). Since both \( \gamma \) and \( \gamma' \) are proper partitions, we can keep class representatives for each class when defining the bijection:

\[
|D_2| = |L \cup \cup (\gamma')| \\
\iff |\bigcup_{C \subseteq C_2} C \setminus \{ \text{rep}(C) \}| = |L \cup \bigcup_{C \subseteq ec(\gamma')} C \setminus \{ \text{rep}(C) \}| \\
\iff |\bigcup_{C \subseteq C_2} C| = |L \cup \bigcup_{C \subseteq ec(\gamma')} C|
\]

In fact, the latter two sets are equal. Let \( l \in \bigcup_{C \subseteq C_2} C \). If \( l \) is collected into \( g \), \( l \) must be in \( L \) since it is not from a class in \( C_1 \). If \( l \) is not collected, it remains in \( \bigcup_{C \subseteq ec(\gamma')} C \). For the other direction, let \( l \in L \cup \bigcup_{C \subseteq ec(\gamma')} C \). If \( l \in L \), then it is in a class that was not completely collected, which means \( l \in \bigcup_{C \subseteq C_2} C \). Otherwise, \( l \) is in a class that is disjoint from \( g \), which means again that \( l \in \bigcup_{C \subseteq C_2} C \). Hence \( f_2 \) is simply the identity.

Finally, we have the copy extension \( W', Y, S, F \cup j \equiv \gamma', H, \tilde{R}, F \cup g \). Now we instantiate the induction hypothesis to obtain the triple \( \langle w, Y', M' \rangle \) and new partition \( \gamma'' \) with the expected properties. Applying the rule F:CondT to the evaluation \( W', Y, S, F \cup j \equiv e_1 \iff w, Y', M' \) given by the first property from induction, we get \( W, Y, S, F \vdash \text{if } (x; e_1; e_2) \Downarrow w, Y', M' \), which inherits the required conditions (2) - (6) from induction. □

Thus, we have shown that we can execute a computation using \( E_{gc} \) given that the computation succeeded in a run with \( E_{copy} \), which means that \( E_{copy} \) is an over approximation of \( E_{gc} \).

**Soundness of FO\(^{gc}\)** For the second part of the proof, we show FO\(^{gc}\) is sound w.r.t. \( E_{copy} \). Below are selected lemmas used in the soundness theorem:

**Definition 8 (Stability).** Given heaps \( H, H' \), a set of locations is stable if \( \forall l \in R. H(l) = H'(l) \). Denote this by \( \text{stable}(R, H, H') \).

**Lemma 6.** Let \( H \models v \rightarrow a : A \). For all sets of locations \( R \), if \( \text{reach}_H(v) \subseteq R \) and \( \text{stable}(R, H, H') \), then \( H' \models v \rightarrow a : A \) and \( \text{reach}_H(v) = \text{reach}_{H'}(v) \).

**Lemma 7** (Stability of copying). Let \( H, v' = \text{copy}(H, L, v) \). For all \( l \in H \), if \( l \notin L \), then \( H(l) = H'(l) \). Further, \( \text{reach}_{H'}(v') \subseteq L \).

**Lemma 8** (Copy is copy). Let \( H, v' = \text{copy}(H, L, v) \). If \( H \models v \rightarrow a : A \), then \( H' \models v' \rightarrow a : A \).
Lemma 9. Let \( A \equiv A_1, A_2, H \vdash v : A, v \sim_H v_1, \) and \( v \sim_H v_2. \) Then \( \Phi_H(v : A) = \Phi_H(v_1 : A_1) + \Phi_H(v_2 : A_2) + n \cdot |\text{dom}(\text{reach}_H(v))| \)

Lemma 10 (Linearity of \( \mathcal{E}_{\text{copy}} \)). Let \( C \) be a configuration, \( C \vdash \mathcal{E}_{\text{copy}} e \Downarrow v, H', F', \) and \( \Sigma; \Gamma \vdash e : B. \) Given that \( \text{linearComp}(C, e), \) we have that \( \text{set}(\text{reach}_H(v)) \) and \( \text{disjoint}({R,F,\text{reach}_H(v)})\)

Theorem 11 (Soundness). Let \( H_o \vdash V_o : \Gamma, \Sigma; \Gamma \vdash \frac{q}{q'} e : B, V_o, H_o \vdash e \Downarrow v_o, H'_o. \) Then \( \forall C \in \mathbb{Q}^+ \) and configuration \((V, H, R, F)\) s.t.

1. \( V_o, H_o \sim V, H \)
2. \( \text{linearComp}(V, H, R, F, e) \)
3. \( |F| \geq \Phi_{V,H}(\Gamma) + q + C \)
then there exists a triple \((v, H', F')\), and a freelist \( F' \) s.t.

1. \( V, H, R, F \vdash \mathcal{E}_{\text{copy}} e \Downarrow v, H', F' \)
2. \( v_o \sim_{H'_o} v \)
3. \( |F'| \geq \Phi_{H'}(v : B) + q' + C \)

In other words, given a terminating expression (verified by succeeding with the run using \( \mathcal{E}_{\text{oper}} \)) and given a freelist that is sufficiently large (as predicated by the type derivation), a run with \( \mathcal{E}_{\text{copy}} \) will normalize to an equivalent value, and the resulting freelist will be sufficiently large (as predicated by the type derivation).

6 Implementation and Evaluation

Implementation We have implemented the novel cost semantics and the type system in Resource Aware ML (RaML). The implementation covers full RaML, including user-defined data types, higher-order functions, and polynomial potential functions. However, there is no destructive match for function closures and analyzing the heap-space usage of closures still amounts to counting allocations only. The main changes that where necessary have been in the rules for sharing and pattern matching as described earlier. We also needed to change some elaboration passes that were no longer cost preserving with the GC cost model.

The garbage collection cost semantics is implemented as an alternative evaluation module inside RaML. As mentioned before, RaML leverages the syntax of OCaml programs. First, we take the OCaml type checked abstract syntax tree and perform a series of transformations. The evaluation modules operate on the resulting RaML syntax tree. In the gc evaluation module, evaluate has the following signature:

\[
\text{evaluate} : (\langle a, \text{unit} \rangle \text{Expressions.expression} \rightarrow \text{int} \rightarrow ((\langle \text{a value} * \langle \text{a heap} * \text{Int.Set.t} \rangle \text{option})
\]

Here, the second argument int specifies the size of the initial freelist. The result is an option triple of the return value, heap, and freelist; None is returned in case the freelist was not sufficient for the evaluation. Whereas the normal evaluation boxes every value (everything evaluates to a location), the gc module follows the cost semantics and only boxes data constructors. The rationale is that the size for other values can be computed statically and thus stack allocated. One difference between the cost semantics and its implementation is that while in the language presented here list is the only data type, our implementation supports user defined data types. The extension is straightforward except the treatment of the nil constructor, or generally “empty” constructors that have arity zero. For simplicity of presentation, we evaluate all nil constructors to the same null value in the cost semantics. This is natural for lists because all nil constructors are the same, and every list has at most one nil node. However, for custom data types that have more than one kind of empty constructor, it is not possible to map every constructor to the same null value. Thus, the implementation treats all constructors uniformly, so each empty constructor also costs one heap location.

As mentioned before, all functions used in a program are declared in a global mutually recursive block, and we do not account for the constant space overhead for this block in the cost semantics. In order to implement
this global function block, we allow closure creation during program evaluation. However, we allocate all closures from a separate freelist into a separate heap. This ensures that data constructors are allocated from the correct freelist and no space overhead is created by allocating closures for function declarations.

**Evaluation** We evaluated our new analysis on a number of functions. Table 1 contains a representative compilation. It shows the type signature for each function. Table 2 presents the test data that showcase the difference between RaML\textsubscript{heap}, the previous RaML type system instantiated with the heap metric (the old analysis which only counts heap allocations), and FO\textsuperscript{gc}, which includes deallocations and copying cost for sharing. For each type system, we show the heap space bound computed by RaML, the number of constraints generated, and the time elapsed during analysis. The last column gives the expression for the exact heap high watermark derived by hand and verified by running the cost semantics.

<table>
<thead>
<tr>
<th>function</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>quicksort</td>
<td>['a -\rightarrow' a -\rightarrow bool; 'a list -\rightarrow 'a list</td>
</tr>
<tr>
<td>mergesort</td>
<td>['a; 'a] -\rightarrow bool; 'a list -\rightarrow 'a list</td>
</tr>
<tr>
<td>ocamlsort</td>
<td>['a; 'a] -\rightarrow bool; 'a list -\rightarrow 'a list</td>
</tr>
<tr>
<td>selection sort</td>
<td>int list -\rightarrow int list</td>
</tr>
<tr>
<td>eratosthenes</td>
<td>int list -\rightarrow int list</td>
</tr>
<tr>
<td>dfs</td>
<td>btree; int] -\rightarrow btree option</td>
</tr>
<tr>
<td>bfs</td>
<td>btree; int] -\rightarrow btree option</td>
</tr>
<tr>
<td>transpose</td>
<td>'a list list -\rightarrow 'a list list</td>
</tr>
<tr>
<td>map_it</td>
<td>['a -\rightarrow 'b; 'a list list] -\rightarrow 'b list list * 'b list list</td>
</tr>
<tr>
<td>pairs</td>
<td>'a list -\rightarrow ('a * 'a) list</td>
</tr>
</tbody>
</table>

Table 1: Signature of Test Functions

Except for bfs and dfs, all functions in the table take a *principal* argument of type list. The variables in the table refer to this argument (for example, the type of the principal argument of quicksort is 'a list). In general, M refers to the number of cons constructors of the principal argument (or the number of outer cons nodes in case of nested lists); L refers to the maximum number of cons nodes of the inner lists.

For the sorting functions, aside from mergesort, the new analysis using the gc metric derived asymptotically better bounds when compared to the heap metric. Furthermore, all bounds are *exact* with respect to the cost semantics. In regards to mergesort, the analysis was not able to derive a tight bound due to the limitations of AARA in deriving logarithmic bounds. A particularly nice result is that for quicksort, we derive that the space usage is exactly 0, which justifies its use as a zero space-overhead (or "in place") sorting algorithm.

Next, we have have the graph search algorithms operating on a binary tree. Again, the gc metric was able to derive exact space overheads, while the heap metric derived linear bounds for both. For transpose, the gc metric derived an asymptotically better bound, but was not able to derive the exact overhead. We implement matrices as lists-of-lists in row-major order. The transpose function is implemented tail-recursively, with the accumulator starting as the empty list. When “flipping” the first row \( r \) of the input and appending this to the accumulator, we need to create \(|r|\) many new nil and cons constructors to store the row as a column. While this overhead only occurs once, RaML is unable to infer this from the source code, and thus the cost is repeated over the entire input matrix, resulting in the linear bound (w.r.t. the size of the matrix). This artifact is unrelated to the new extension; it is a limitation due to the implementation of RaML.

The last two functions demonstrate how the gc metric performs when there is variable sharing. map_it maps the input function across each list in the principal argument twice, returning a tuple of nested lists. The gc metric dictates that every outer data constructor in the principal argument needs to be copied, and thus gives the linear bound \( M + 1 \). In this case, the bound is exact. The function “pairs” takes a list and outputs all pairs of the input list which are ordered ascending in input position. For example, pairs \([1;2;3;4] = [(1,2);(1,3);(1,4);(2,3);(2,4);(3,4)]\). For pairs, the gc metric derived a bound that is asymptotically the same as the heap metric, but with better constants. An exact bound could not be derived because the deallocation potential from the pattern match in the definition of pairs is wasted since the matched body could already be typed with zero cost. However, this deallocation is used as usual in the cost semantics. Thus the slack in the bound totals to the size of the input.
7 Conclusion and Future Work

In this article, we introduced a novel operational cost semantics that models a perfect tracing garbage collector and an extension to AARA that is sound with respect to the new semantics. We implemented the new semantics and analysis as modules in RaML and found through experimental testing that the extended AARA was able to derive asymptotically better bounds for several commonly used functions and programming patterns; often, the bounds are optimal with respect to the cost semantics.

One direction for future work is using the cost free metric $cf$ to model global garbage collection. In $cf$, all resource constants, including constructor nodes, are set to 0. A cost-free typing judgment then captures how an expression manipulates the structures in the context into the structure induced by its type. Using this fact, we could express the maximum space usage in the sequential composition $\text{let}(e_1; x : \tau.e_2)$ by analyzing $e_1$ twice—once with the cost-free metric and once with the regular metric—and assign potential to $x$ using the result type in the cost-free typing. In prior work [23], the authors have successfully employed this cost-free metric to analyze parallel programs. Here, the difficulty is showing the simultaneous soundness of both destructive pattern matching and the cost-free composition. Another complication is the choice between local variable sharing and global context sharing. We leave the exploration of this area to future work.

Another direction for future work are function closures. The current treatment in our implementation is unsatisfactory since there is no equivalent to the destructive pattern match for closures. As a result, the GC metric in RaML only accounts for allocation of closures, which is not an improvement over the existing implementation. Ideally, we would like to account for deallocation at function applications and treat closures similar to other data structures in sharing. However, the size of closures cannot be determined easily statically and closures can not capture potential and are currently shared freely in RaML. As a result, the techniques we developed here do not directly carry over to closures.

Finally, we are interested in exploring if our work can be used to improve the efficiency of garbage collection in languages like OCaml. A guaranteed upper bound on the heap space can be used in different ways to control the frequency of the collections and the total memory that is requested from the operating system.

Acknowledgments

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References


A Notation

For a finite mapping $f : A \rightarrow B$, we write $\text{dom}(f)$ for the defined values of $f$. Sometimes we shorten $x \in \text{dom}(f)$ to $x \in f$. We write $f[x \mapsto y]$ for the extension of $f$ where $x$ is mapped to $y$, with the constraint that $x \notin \text{dom}(f)$. 
Given possibly non-disjoint sets $A, B$, let the disjoint union be $A \oplus B$ defined by $\{(\mathtt{inl}, a) \mid a \in A\} \cup \{(\mathtt{inr}, b) \mid b \in B\}$.

Let a multiset be a function $S : A \to \mathbb{N}$, i.e., a map of the multiplicity of each element in the domain. Write $x \in S$ if $S(x) \geq 1$. If for all $s \in S$, $\mu(s) = 1$, then $S$ is a property set, and we denote this by $\text{set}(S)$. Additionally, $A \uplus B$ denotes counting union of sets where $(A \uplus B)(s) = A(s) + B(s)$, similarly, $(A \cap B)(s) = \min A(s), B(s)$. Furthermore, $A \cup B$ denotes the usual union where $(A \cup B)(s) = \max (A(s), B(s))$. For the union of disjoint multi- sets $A$ and $B$, we write $A \sqcup B$ to emphasize the disjointness. For a collection of pairwise disjoint multi- sets $\mathcal{C}$, i.e., $\forall X, Y \in \mathcal{C}$. $X \cap Y = \emptyset$, we write $\text{disjoint}(\mathcal{C})$.

In the rest of the paper, we sometimes treat a set $A$ sets as multiset $A : A \to \mathbb{N}$ via $x \mapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{o.w.} \end{cases}$ when convenient. For instance, if an operation defined on multisets is used on sets and multisets, the set is thus promoted.

Given a set $A$, let $\mathcal{P}(A)$ be the powerset of $A$. Given a multiset $A$, let $\varphi(A)$ be the power multiset of $A$, i.e. the set of all submultisets of $A$.

For a partition $f : A \to \mathcal{P}(B)$, we write the set of equivalence classes as $\text{ec}(f) = \{f(x) \mid x \in A\} = f(A)$, i.e. the image of $f$ on its domain $A$. Furthermore, a partition is proper if for any $x \in A, f(x) \neq \emptyset$.

Given a proper partition $f : A \to \mathcal{P}(B)$, for every $a \in A$, we can choose an arbitrary $b \in f(a)$ to be the representative for that part; call this $\text{rep}(a)$.

## B Linearity of Copy Semantics

In the soundness proof of $\text{FO}^{gc}$, we used an important lemma: that $\mathcal{E}_{\text{copy}}$ is semantically linear, i.e. locations are used linearly. To see why, consider the second premise in the rule $\text{L:MatLD}$. In addition to the $p$ units of potential justified by the definition of linear potential, we get 1 unit from deallocating the cons cell itself. This is only sound if in the corresponding rule in $\mathcal{E}_{\text{copy}}$, a location was actually collected. Consider the evaluation in question:

\[
\begin{align*}
V(x) &= l \\
H(l) &= \langle v_h, v_i \rangle \\
V'' &= V[x_h \mapsto v_h, x_i \mapsto v_i] \downarrow_{fV(e_2)} \\
g &= \{l \in H \mid l \notin F \cup R \cup \text{locs}_{V'', H}(e_2)\} \\
V'', H, R, F \uplus g \downarrow v, H', F'\quad (S_1)
\end{align*}
\]

If all the variables in $V$ was mapped to values with disjoint reachable sets, then we see that $l$ is only in the reachable set of $x$ (assuming that well-typed expressions don’t have duplicate occurrences of variables, i.e. $x \notin fV(e_1) \cup fV(e_2)$). Then it follows that $l \notin g$ given that locations in $V$, $R$, and $F$ are also all disjoint, and this is what we needed to justify the rule $\text{L:MatL}$. Thus we have to show that $\mathcal{E}_{\text{copy}}$ preserves the linearity invariant: given a linear computation, the evaluation result is also linear.

First, we characterize semantically linear contexts:

**Definition 9.** (Linear context) Given a context $(V, H)$, let $x, y \in \text{dom}(V)$, $x \neq y$, and $r_x = \text{reach}_H(V(x))$, $r_y = \text{reach}_H(V(y))$. It is linear given that:

1. $\text{set}(r_x), \text{set}(r_y)$
2. $r_x \cap r_y = \emptyset$

Denote this by $\text{linearCxt}(V, H)$.

Whenever $\text{linearCxt}(V, H)$ holds, visually, one can think of the stack as a collection of disjoint, directed trees with locations as nodes: consequently, there is at most one path from a variable on the stack $V$ to any location in $H$. Now we can formalize our intuition for linear computations:

**Definition 10** (Linear computation). Given a configuration $C = (V, H, R, F)$ and an expression $e$, we say the 5-tuple $(C, e)$ is a computation; it is a linear computation given the following:
1. \( \text{dom}(V) = FV(e) \)
2. \( \text{linearCtx}(V, H) \)
3. \( \text{disjoint}(\{R, F, \text{locs}_V, H(e)\}) \)

And we write \( \text{linearComp}(V, H, R, F, e) \) to denote this fact.

Given a semantically linear computation (one with no sharing between the underlying locations), the resulting value is linear (expressed by item 1. and 2. below):

**Lemma 12** (Linearity of \( E_{\text{copy}} \)). For all stacks \( V \) and heaps \( H \), let \( V, H, R, F \vdash e \Downarrow v, H', F' \) and \( \Sigma; \Gamma \vdash e : B \). Then given that \( \text{linearComp}(V, H, R, F, e) \), we have the following:

1. \( \text{set}(\text{reach}_{H'}(v)) \)
2. \( \text{disjoint}(\{R, F', \text{reach}_{H'}(v)\}) \), and
3. \( \text{stable}(R, H, H') \)

Where \( \text{stable} \) is a predicate on \( \mathcal{P}(\text{Loc}) \times \text{Heap} \times \text{Heap} \), defined below. The premises of this lemma is a subset of the premises of the soundness theorem. Thus, we could have merged the proof of this lemma directly into the soundness proof. However, this makes the presentation clearer; furthermore, the linearity of \( E_{\text{copy}} \) is an interesting in itself, regardless of the accompanying type system. Some auxiliary lemmas:

Define \( \hat{\jmath} : L^p(A) \to L(A) \) as the map that erases resource annotations. This gives a simplified judgment \( \Sigma^!; \Gamma^! \vdash e : B^! \) used in proofs where the resource annotations are not necessary.

**Lemma 13.** If \( \Sigma; \Gamma \vdash^q e : B \), then \( \Sigma^!; \Gamma^! \vdash e : B^! \).

*Proof.* Induction on the typing judgement. \( \square \)

Define \( \text{FV}^* : \text{Exp} \to \wp(\text{Var}) \), the multiset of free variables of expressions, as the usual \( \text{FV} \) inductively over the structure of \( e \). This version of \( \text{FV} \) reflects the multiplicity of variable occurrences.

**Lemma 14.** If \( \Sigma; \Gamma \vdash^q e : B \), then \( \text{set}(\text{FV}^*(e)) \) and \( \text{dom}(\Gamma) = \text{FV}(e) \).

*Proof.* Induction on the typing judgement. \( \square \)

**Definition 11** (Stability). Given heaps \( H, H' \), a set of locations is \( \text{stable} \) if \( \forall l \in R. H(l) = H'(l) \). Denote this by \( \text{stable}(R, H, H') \).

**Lemma 15.** Let \( H \vdash v \mapsto a : A \). For all sets of locations \( R \), if \( \text{reach}_H(v) \subseteq R \) and \( \text{stable}(R, H, H') \), then \( H' \vdash v \mapsto a : A \) and \( \text{reach}_{H'}(v) = \text{reach}_H(v) \).

*Proof.* Induction on the structure of \( H \vdash v \mapsto a : A \). \( \square \)

**Corollary 16.** Let \( H \vdash V : \Gamma \). For all sets of locations \( R \), if \( \bigcup_{x \in V} \text{reach}_H(V(x)) \subseteq R \) and \( \text{stable}(R, H, H') \), then \( H' \vdash V : \Gamma \).

*Proof.* Follows from Lemma 6. \( \square \)

**Lemma 17** (stability of copying). Let \( H', v' = \text{copy}(H, L, v) \). For all \( l \in H \), if \( l \notin L \), then \( H(l) = H'(l) \). Further, \( \text{reach}_{H'}(v') \subseteq L \).

**Lemma 18** (copy is copy). Let \( H', v' = \text{copy}(H, L, v) \). If \( H \vdash v \mapsto a : A \), then \( H' \vdash v' \mapsto a : A \).

**Lemma 19.** Let \( A \vdash^n A_1, A_2, H \vdash v_1 : A_1, H \vdash v_2 : A_2, \) and \( H \vdash v : A \). Then \( \Phi_H(v : A) = \Phi_H(v_1 : A_1) + \Phi_H(v_2 : A_2) + n \cdot |\text{dom}(\text{reach}_H(v))| \)
Lemma 20. Let $H \vdash V : \Gamma$, $\Sigma; \Gamma \vdash e : A$, and $V, H \vdash e \downarrow v, H'$. Then $H' \vdash v : A$.

Now the proof for linearity of $E_{\text{copy}}$:

Proof. Nested induction on the evaluation judgement and the typing judgement.

Case 1: $E_{\text{Var}}$

Suppose $H \vdash V : \Gamma$, $\text{dom}(V) = \text{FV}(e)$, $\text{linearCtx}(V, H)$, $\text{disjoint} \{ \{R, F, \text{reach}_H(v)\} \}$

set(reach$_H(v)$) \hspace{1cm} (linearCtx($V, H$))

disjoint($\{R, F, \text{reach}_H(v)\}$) \hspace{1cm} (disjoint($\{R, F, \text{loc}_{V, H}(e)\}$))

linearCtx($V, H$) \hspace{1cm} (Sp.)

stable($R, H, H'$) \hspace{1cm} ($H = H'$)

Case 2: $E_{\text{Const}}^*$ Due to similarity, we show only for $E_{\text{ConstI}}$

Suppose $H \vdash V : \Gamma$, $\text{dom}(V) = \text{FV}(e)$, $\text{linearCtx}(V, H)$, $\text{disjoint} \{ \{R, F, \text{loc}_{V, H}(e)\} \}$

set(\text{reach}_H(v)) \hspace{1cm} (\text{reach}_H(v) = \emptyset)

disjoint($\{R, F, \emptyset\}$) \hspace{1cm} (disjoint($R, F$))

linearCtx($V, H$) \hspace{1cm} (Sp.)

stable($R, H, H'$) \hspace{1cm} ($H = H'$)

Case 4: $E_{\text{App}}$

Case 5: $E_{\text{CondT}}$ Similar to $E_{\text{MatNil}}$

Case 6: $E_{\text{CondF}}$ Similar to $E_{\text{CondT}}$

Case 7: $E_{\text{Let}}$

$V, H, R, F \vdash \text{let}(e_1; x : \tau.e_2) \downarrow v_2, H_2, F_2$ \hspace{1cm} (case)

$V_1, H, R', F \vdash e_1 \downarrow v_1, H_1, F_1$ \hspace{1cm} (ad.)

$\Sigma; \Gamma_1, \Gamma_2 \vdash \text{let}(e_1; x : \tau.e_2) : B$ \hspace{1cm} (case)

$\Sigma; \Gamma_1 \vdash e_1 : A$ \hspace{1cm} (ad.)

Suppose $H \vdash V : \Gamma$, $\text{dom}(V) = \text{FV}(e)$, $\text{linearCtx}(V, H)$, $\text{disjoint} \{ \{R, F, \text{loc}_{V, H}(e)\} \}$

$H \vdash V : \Gamma_1$ \hspace{1cm} (def of W.D.E and Lemma 14)

By IH, we have invariant on the first premise

N TS (1) - (3) to instantiate invariant on the first premise

(1) $\text{dom}(V_1) = \text{FV}(e_1)$ \hspace{1cm} (def of $V_1$)

(2) $\text{linearCtx}(V_1, H)$ \hspace{1cm} (linearCtx($V, H$) and $V_1 \subseteq V$)

(3) $\text{disjoint} \{R', F, \text{loc}_{V, H}(e_1)\}$

Thus we have disjoint($R', F, \text{loc}_{V, H}(e_1)$)
By IH,
1.(1) \( \text{set}(\text{reach}_{H_1}(v_1)) \)
1.(2) \( \text{disjoint}\{(R,F_1,\text{reach}_{H_1}(v_1))\} \)
1.(3) \( \text{stable}(R',H,H_1) \)
\( V'_2,H_1,R,F_1 \uplus g \vdash e_2 \downarrow v_2,H_2,F_2 \) \quad \text{(ad.)}
\( \Sigma; \Gamma_2,x : A \vdash e_2 : B \) \quad \text{(ad.)}
\( H_1 \vdash V'_2 : (\Gamma_2,x : A) \) \quad \text{(Lemma ??)}
By IH, we have invariant on the second premise
NTS (1) - (3) to instantiate invariant on the second premise
(1) \( \text{dom}(V'_2) = FV(e_2) \) \quad \text{(def of} \ V'_2) 
(2) \( \text{linearCtxt}(V'_2,H_1) \)
Let \( x_1,x_2 \in V'_2', x_1 \neq x_2 \) be arb.
\textbf{case: } \( x_1 \neq x, x_2 \neq x \)
\( \text{reach}_{H_1}(V'_2(x_1)) \subseteq R' \)
\( \text{reach}_{H_1}(V'_2(x_2)) \subseteq R' \)
\( \text{reach}_{H_1}(V'_2(x_1)) = \text{reach}_{H_1}(V'_2(x_1)), \text{reach}_{H_1}(V'_2(x_2)) = \text{reach}_{H_1}(V'_2(x_2)) \)
(\( \text{stable}(R',H,H_1) \) and Lemma 6)
\( \text{reach}_{H_1}(V'_2(x_1)) = \text{reach}_{H_1}(V(x_1)), \text{reach}_{H_1}(V'_2(x_2)) = \text{reach}_{H_1}(V(x_2)) \)
(\( \text{stable}(R',H,H_1) \) and Lemma 6)
\( \text{linearCtxt}(V'_2,H_1) \)
(\( \text{linearCtxt}(V,H) \))
\textbf{case: } \( x_1 = x, x_2 \neq x \)
\( \text{reach}_{H_1}(V'_2(x_1)) = \text{reach}_{H_1}(v_1) \)
\( \text{reach}_{H_1}(V'_2(x_2)) \subseteq R' \)
\( \text{set}(\text{reach}_{H_1}(v_1)) \)
\( \text{reach}_{H_1}(V'_2(x_2)) = \text{reach}_{H}(V(x_2)) \)
(\( \text{same as above} \))
\( \text{set}(\text{reach}_{H_1}(V'_2(x_2))) \)
\( \text{reach}_{H_1}(V'_2(x_1)) \cap \text{reach}_{H_1}(V'_2(x_2)) = \emptyset \)
(\( \text{disjoint}(\{R',\text{reach}_{H_1}(v_1)\}) \))
Thus we have \( \text{linearCtxt}(V'_2,H_1) \)
(\( \text{disjoint}(\{R',F_1\}) \) from 1.2 and \( R \subseteq R' \))
\( R \cap F_1 = \emptyset \)
(\( \text{disjoint}(\{R',F_1\}) \) from 1.2 and \( R \subseteq R' \))
\( R \cap (F_1 \uplus g) = \emptyset \)
(\( \text{def of} \ g \))
NTS \( (F_1 \uplus g) \cap \text{locs}_{V'_2,H_1}(e_2) = \emptyset \)
Let \( l \in \text{locs}_{V'_2,H_1}(e_2) \) be arb.
l \( \in \text{reach}_{H_1}(V'_2(x')) \) for some \( x' \in V'_2 ' \)
\textbf{case: } \( x' \neq x \)
\( \text{reach}_{H_1}(V'_2(x')) = \text{reach}_{H_1}(V'_2(x')) \)
(\( \text{same as above} \))
\( \text{reach}_{H_1}(V'_2(x')) \subseteq R' \)
(\( \text{def of} \ R' \))
\( \text{reach}_{H_1}(V'_2(x')) \cap F_1 = \emptyset \)
(\( \text{disjoint}(\{R',F_1\}) \) from 1.2)
\textbf{case: } \( x' = x \)
\( \text{reach}_{H_1}(V'_2(x')) = \text{reach}_{H_1}(v_1) \)
(\( \text{def of} \ V'_2 \))
\( \text{reach}_{H_1}(V'_2(x')) \cap F_1 = \emptyset \)
(\( \text{disjoint}(\{F_1,\text{reach}_{H_1}(v_1)\}) \) from 1.2)
Case 11: E:Cons
Case 10: E:Nil
Similar to E:Const*
Case 9: E:MatP
Similar to E:MatCons
Similar to E:Var

H 
Suppose \( V,H,R,F \vdash \) reach \( \text{stable} \) set (1) - (3) holds after evaluation
Thus \( \text{reach}_H(V'(x')) \cap (F_1 \cup g) = \emptyset \)
\( \text{NTS } R \cap \text{locsv}_{V',H}(e_2) = \emptyset \)
Let \( l \in \text{locsv}_{V',H}(e_2) \) be arb.
l \in \text{reach}_H(V'(x')) for some \( x' \in V'_2 \)

\( \text{case: } x' \neq x \)
\( \text{reach}_H(V'_2(x')) = \text{reach}_H(v_1) \) (def of \( V'_2 \))
\( \text{reach}_H(V'_2(x')) \cap R = \emptyset \) (disjoint(\{R', \text{reach}_H(v_1)\}) from 1.2 and \( R \subseteq R' \))

Thus \( \text{reach}_H(V'_2(x')) \cap R = \emptyset \)
Hence we have (3) disjoint(\( R, F_1 \cup g, \text{locsv}_{V',H}(e_2) \))

By instantiating the invariant on the second premise, we have
\( 2.(1) \) set(\( \text{reach}_H(v_2) \))
\( 2.(2) \) disjoint(\{R, F_2, \text{reach}_H(v_2)\})
\( 2.(3) \) stable(R, H_1, H_2)

Lastly, showing (1) - (3) holds for the original case:
\( 1 \) set(\( \text{reach}_H(v_2) \)) (By 2.1)
\( 2 \) disjoint(\{R, F_2, \text{reach}_H(v_2)\}) (By 2.2)
\( 3 \) stable(R, H_1, H_2)

Let \( l \in R \) be arb.
\( H(l) = H_1(l) \) (stable(\( R', H, H_1 \)) from 1.3)
\( H_1(l) = H_2(l) \) (stable(\( R, H_1, H_2 \)) from 2.3)
\( H(l) = H_2(l) \)
Hence stable(R, H, H_2)

Case 8: E:Pair Similar to E:Var
Case 9: E:MatP Similar to E:MatCons
Case 10: E:Nil Similar to E:Const*
Case 11: E:Cons

\( V, H, R, F \vdash e \downarrow l, H'', F' \)
Suppose \( H \vdash V : \Gamma, \text{dom}(V) = FV(e), \text{linearCtx}(V, H), \text{disjoint}(\{R, F, \text{locsv}_{V,H}(e)\}) \)
\( \text{NTS } (1) - (3) \) holds after evaluation
\( 1 \) set(\( \text{reach}_{H''}(l) \))
\( \text{stable}(\{\text{locsv}_{V,H}(e)\}, H, H'') \) (disjoint(\{F, \text{locsv}_{V,H}(e)\}) and copy only updates \( l \in L \subseteq F \))
\( \text{reach}_H(V(x_i)) = \text{reach}_{H''}(V(x_i)) \) (\( \text{reach}_H(V(x_i)) \subseteq \text{locsv}_{V,H}(e) \) and \( 6 \) for \( i = 1, 2 \))
Case 12: E:MatNil

Suppose \( H \models V : \Gamma, \text{dom}(V) = F V(e), \text{linearCtx}(V, H), \text{disjoint}([R, F, \text{locs}_{V, H}(e)]) \)

\[ \Sigma; \Gamma' \vdash e_1 : B \] (ad.)

\[ V', H, R, F \cup g \vdash e_1 \downarrow v, H', F' \] (ad.)

\[ H \models V' : \Gamma' \] (def of W.D.E)

By IH, we have invariant on the premise

NTS (1) - (3) to instantiate invariant on the premise

(1) \( \text{dom}(V') = F V(e_1) \) (def of \( V' \))

(2) \( \text{linearCtx}(V', H) \) (linearCtx\( (V, H) \) and \( V' \subseteq V \))

(3) \( \text{disjoint}([R, F, \text{locs}_{V', H}(e_1)]) \) (disjoint\( ([R, F, \text{locs}_{V, H}(e)] \) and \( \text{locs}_{V', H}(e_1) \subseteq \text{locs}_{V, H}(e) \))

Instantiating invariant on premise,

(1) \( \text{set}(\text{reach}_H(v)) \)

(2) \( \text{disjoint}([R, F_1, \text{reach}_H(v)]) \)

(3) \( \text{stable}(R, H, H') \) (since copy only updates \( l \in L \subseteq F \) and \( F \cap R = \emptyset \))

Case 13: E:MatCons

\[ V(x) = l \] (ad.)

\[ H(l) = \langle v_h, v_t \rangle \] (ad.)

\[ \Gamma = \Gamma', x : L(A) \] (ad.)

\[ \Sigma; \Gamma', x_h : A, x_t : L(A) \vdash e_2 : B \] (ad.)

\[ V'', H, R, F \cup g \vdash e_2 \downarrow v_2, H_2, F' \] (ad.)

Suppose \( H \models V : \Gamma, \text{dom}(V) = F V(e), \text{linearCtx}(V, H), \text{disjoint}([F, R, \text{locs}_{V, H}(e)]) \)

\[ H \models V(x) : L(A) \] (def of W.D.E)

\[ H \models v_h : A, H \models v_t : L(A) \]

\[ H \models V'' : \Gamma', x_h : A, x_t : L(A) \] (def of W.D.E)

By IH, we have invariant on the premise

NTS (1) - (3) to instantiate invariant on the premise

(1) \( \text{dom}(V'') = F V(e_2) \) (def of \( V'' \))

(2) \( \text{linearCtx}(V'', H) \)

Let \( x_1, x_2 \in V'', x_1 \neq x_2, r_{x_1} = \text{reach}_H(V''(x_1)), r_{x_2} = \text{reach}_H(V''(x_2)) \)

\textbf{case: } \( x_1 \notin \{x_h, x_t\}, x_2 \notin \{x_h, x_t\} \)

(1), (2) from \( \text{linearCtx}(V, H) \)
\textbf{case:} \(x_1 = x_h, x_2 \notin \{x_h, x_t\}\)
\begin{align*}
& \text{set}(r_{x_1}) \quad (\text{since set}(reach_H(V(x))) \text{ from linearCtxt}(V,H)) \\text{set}(r_{x_2}) \quad (\text{since linearCtxt}(V,H)) \\
& x_2 \in FV(e) \quad \text{(def of FV)} \\
& reach_H(V(x)) \cap r_{x_2} = \emptyset \quad \text{(def of reach and linearCtxt)} \\
& \text{hence } r_{x_1} \cap r_{x_2} = \emptyset
\end{align*}

\textbf{case:} \(x_1 = x_h, x_2 = x_t\)
\begin{align*}
& \text{set}(r_{x_1}) \text{ since set}(reach_H(V(x))) \text{ from linearCtxt}(V,H) \\
& \text{set}(r_{x_2}) \text{ since set}(reach_H(V(x))) \text{ from linearCtxt}(V,H) \\
& r_{x_1} \cap r_{x_2} = \emptyset \quad \text{(set}(reach_H(V(x)))
\end{align*}

\textbf{case: otherwise}
\begin{itemize}
  \item similar to the above
\end{itemize}

Thus we have linearCtxt\(V''\), \(H\)
\begin{align*}
(3) & \text{ disjoint}\{R, F \cup g, locs_{V''}, H(e_2)\} \quad (\text{since } F \cap R = \emptyset \text{ and by def of } g) \\
\end{align*}

NTS \(R \cap locs_{V''}, H(e_2) = \emptyset\)

Let \(l' \in locs_{V''}, H(e_2)\) be arb.
\begin{itemize}
  \item case: \(l' \in reach_H(V''(x'))\) for some \(x' \in FV(e_2)\) where \(x' \notin \{x_h, x_t\}\)
    \begin{align*}
    x' & \in V \\
    l' & \in reach_H(V(x')) \\
    x' & \in FV(e) \\
    l' & \in locs_{V,H}(e) \\
    l' & \notin R \quad \text{(def of } V''\text{)} \\
    
    \end{align*}
    (def of FV)
  \item case: \(l' \in reach_H(V''(x_h))\)
    \begin{align*}
    l' & \in reach_H(v_h) \\
    l' & \in reach_H(V(x)) \\
    l' & \in locs_{V,H}(e) \\
    l' & \notin R \quad \text{(since disjoint}(\{F, R, locs_{V,H}(e)\})\text{)}
    \end{align*}
  \item case: \(l' \in reach_H(V''(x_t))\)
    \begin{align*}
    l' & \in reach_H(v_t) \\
    l' & \in reach_H(V(x)) \\
    l' & \in locs_{V,H}(e) \quad \text{(def of reach)} \\
    l' & \notin R \quad \text{(def of locs_{V,H})} \quad \text{(since disjoint}(\{F, R, locs_{V,H}(e)\})\text{)}
    \end{align*}
  \item similar to above
\end{itemize}

Hence \(R \cap locs_{V''}, H(e_2) = \emptyset\)
\begin{align*}
F \cap locs_{V''}, H(e_2) & = \emptyset \quad \text{(Similar to above)} \\
g \cap locs_{V''}, H(e_2) & = \emptyset \quad \text{(def. of } g) \\
(F \cup g) \cap locs_{V''}, H(e_2) & = \emptyset
\end{align*}

Thus disjoint\(\{R, F \cup g, locs_{V''}, H(e_2)\}\)

Instantiating invariant on the premise,
\begin{align*}
(1) & \text{ set}(reach_H(v)) \\
(2) & \text{ disjoint}\{\{R, F', reach_H(v)\}\} \\
(3) & \text{ stable}(R, H, H')
\end{align*}
Case 13: E:Share

\[ e = \text{share } x \text{ as } x_1, x_2 \text{ in } e' \]

Suppose \( H \vDash V : \Gamma, \text{dom}(V) = FV(e), \text{linearCtx}(V, H), \text{disjoint}({R, F, \text{locs}_{V,H}(e)}) \) (def. of wfc)

Let \( V_2 = (V[x_1 \mapsto v', x_2 \mapsto v'']) \upharpoonright FV(e') \)

We show the subsequent computation is also well-formed to invoke the IH:

1. \( \text{dom}(V_2) = FV(e') \) (\( \text{dom}(V) = FV(e) \) and def of \( FV \))
2. \( \text{linearCtx}((V[x_1 \mapsto v', x_2 \mapsto v'']) \upharpoonright FV(e'), H) \)
   
   Let \( x' \mapsto v''' \in V'[x_1 \mapsto v'], \text{reach}_{H'}(v''') \cap \text{reach}_{H'}(v'') = \emptyset \) (definition of copy)
   
   \( \text{reach}_{H'}(v'') \subseteq L \subseteq F \) (By Lemma 7)
   
   but since \( F \cap \text{locs}_{V,H}(e) = \emptyset \), we have the result. (linearComp(\( V, H, R, F, e \))
3. \( \text{disjoint}({R, F \setminus L, \text{locs}_{V_2,H'}(e')})) \)
   
   Disjointedness involving \( F \) follows from assumption. Last one follows since \( \text{locs}_{V_2,H'}(e') \subseteq \text{locs}_{V,H}(e) \cup L \)

By IH:

1. \( \text{set}(\text{reach}_{H'}(v)) \)
2. \( \text{disjoint}({R, F', \text{reach}_{H'}(v)}) \)
3. \( \text{stable}(R, H', H'') \)

STS stable(\( R, H, H' \)), which follows from \( L \cap R = \emptyset \) and Lemma 7

C Soundness

Theorem 21 (Soundness). let \( V_o, H_o \vdash V_o : \Gamma, \Sigma; e : B, V_o, H_o \vdash e \Downarrow v_o, H'_o \). Then \( \forall C \in \mathbb{Q}^+ \) and configuration \( V, H, R, F \) s.t.

1. \( V_o, H_o \sim V, H \)
2. \( \text{dom}(V) = FV(e) \)
3. \( \text{linearCtx}(V, H) \)
4. \( \text{disjoint}({R, F, \text{locs}_{V,H}(e)}) \)
5. \( |F| \geq \Phi_{V,H}(\Gamma) + q + C \)

then there exists a triple \( (v, H', F') \), and a freelist \( F' \) s.t.

1. \( V, H, R, F \vdash e \Downarrow v, H', F' \)
2. \( v_o \sim_{H'_o} v \)
3. \( |F'| \geq \Phi_{H'}(v : B) + q' + C \)

Call this the existence clause.

Proof. Nested induction on the evaluation judgement and the typing judgement.
Case 1: E:Var

\[ V_o, H_o \vdash x \downarrow V_o(x), H_o \quad \text{(case)} \]
\[ \Sigma; x : B \models x : B \quad \text{(case)} \]

Let \( C \in \mathbb{Q}^+ \), well-formed configuration \( V, H, R, F \) s.t. \( V_o, H_o \sim V, H \) and \( |F| \geq \Phi_{V,H}(x : B) + q + C \)

NTS the conclusions for the existence clause:

(1) \( V, H, R, F \vdash e \downarrow V(x), H, F \quad \text{(E:Var)} \)

(2) \( V_o(x) \sim_{H_o} V(x) \)

(3) And we have \( F \geq \Phi_{V,H}(x : B) + q + C \)

\[ = \Phi_H(V(x) : B) + q + C \quad \text{(definition of \( \Phi \))} \]

Case 2: E:Const* Similar to E:Var

Case 4: E:App

\[ V_o, H_o \vdash f(x) \downarrow v_o, H_o' \quad \text{(case)} \]
\[ V_o(x) = v_o' \quad \text{(admissibility)} \]

Let \( P(f) = (y_f, e_f) \)

\[ [y_f \mapsto v_o'], H_o \vdash e_f \downarrow v_o, H_o' \quad \text{(admissibility)} \]
\[ \Sigma; x : A \models^{q/q'} f(x) : B \quad \text{(case)} \]
\[ \Sigma(f) = A \models^{q/q'} B \quad \text{(admissibility)} \]

Let \( C \in \mathbb{Q}^+ \), well-formed configuration \( V, H, R, F \) s.t. \( V_o, H_o \sim V, H \) and \( |F| \geq \Phi_{V,H}(x : A) + q + C \)

\[ \Sigma; y_f : A \models^{q/q'} e_f : B \]

Let \( V(x) = v' \), \( g = \text{collect}(R, \text{locs}_{V,H}(e_f), H, F) \)

By IH on \( [y_f \mapsto v'], H, R, F \cup g \), have the existence clause. NTS the following conditions:

(1) \( [y_f \mapsto v'], H_o \sim [y_f \mapsto v'], H \)

(2) - (4) Have by assumption

(5) NTS \( |F \cup g| \geq \Phi_{[y_f \mapsto v'], H}(x : A) + q + C \)

\[ \text{STS} \ |F| \geq \Phi_{[y_f \mapsto v'], H}(y_f : A) + q + C \]

\[ |F| \geq \Phi_{V,H}(x : A) + q + C \]

\[ = \Phi_H(v' : A) + q + C \]

\[ = \Phi_{[y_f \mapsto v'], H}(y_f : A) + q + C \]

Applying the existence clause and E:App, we’re done.

Case 5: E:CondT

\[ \Gamma = \Gamma', x : \text{bool} \quad \text{(ad.)} \]
\[ H \vdash V : \Gamma' \quad \text{(def of W.F.E)} \]
\[ \Sigma; \Gamma' \models^{q/q} e_t : B \quad \text{(ad.)} \]
\[ V, H, R, F \cup g \vdash e_t \downarrow v, H', F' \quad \text{(ad.)} \]
\[ |F \cup g| - |F'| \leq \Phi_{V,H}(\Gamma) + q - (\Phi_{H'}(v : B) + q') \quad \text{(IH)} \]
\[ |F| - |F'| \leq \Phi_{V,H}(\Gamma) + q - (\Phi_{H'}(v : B) + q') \]
Case 6: E:CondF  Similar to E:CondT

Case 7: E:Let

\[ V_o, H_o \vdash e \downarrow v''_o, H''_o \]  \hspace{1cm} (case)

\[ V'_o = V_o \upharpoonright \text{dom}(FV(e_1)) \]  \hspace{1cm} (ad.)

\[ V'_o, H_o \vdash e_1 \downarrow v'_o, H'_o \]  \hspace{1cm} (ad.)

\[ \Sigma; \Gamma \models^{g}_q \text{let}(e_1; x : \tau.e_2) : B \]  \hspace{1cm} (case)

\[ \Gamma = \Gamma_1 , \Gamma_2 \]  \hspace{1cm} (ad.)

\[ \Sigma; \Gamma_1 \models^{P} e_1 : A \]  \hspace{1cm} (ad.)

\[ H \vdash V'_o : \Gamma_1 \]  \hspace{1cm} (def of \(\models\))

Let \( C \in \mathbb{Q}^+ \), well-formed configuration \( V, H, R, F \) s.t. \( V_o, H_o \sim V, H \) and \(|F| \geq \Phi_{V,H}(\Gamma) + q + C\)

\(\text{NTF } v_2, H_2, F_2 \text{ s.t.}\)

1. \( V, H, R, F \vdash e \downarrow v_2, H_2, F_2 \) and

2. \( v''_o \sim^{H''_o} H_2 v_2 \)

3. \(|F_2| \geq \Phi_{H_1}(v_2 : B) + q' + C\)

Let \( R' = R \cup \text{locs}_{V,H}(FV(e_2) \setminus \{x\}) \)

\( \text{disjoint}([R', F, \text{locs}_{V,H}(e_1)]) \)  \hspace{1cm} (Similar to case in Lemma 10)

Let \( V_1 = V \upharpoonright \text{dom}(FV(e_1)) \)

Instantiate IH with \( C = \mathcal{C} + \Phi_{V_2,H}(\Gamma_2), V = V_1, H = H, F = F, R = R'\)

we get existence clause on the first premise

\(\text{NTS (1) - (4) to instantiate existence clause on the first premise:}\)

(1) \( V'_o, H_o \sim V_1, H \)  \hspace{1cm} (assumption)

(2) - (4) Same as in 10

(4) \(|F| \geq \Phi_{V_1,H}(\Gamma_1) + q + C + \Phi_{V,H}(\Gamma_2)\)  \hspace{1cm} \(|F| \geq \Phi_{V,H}(\Gamma) + q + C \text{ and } \Phi_{V,H}(\Gamma) \geq \Phi_{V_1,H}(\Gamma_1) + \Phi_{V,H}(\Gamma_2)\)

Instantiating existence clause on the first premise, we get \( v_1, H_1, F_1 \text{ s.t.}\)

Fact 1. \( V_1, H, R', F \vdash e_1 \downarrow v_1, H_1, F_1 \)

Fact 2. \( v'_o \sim^{H'_o} v_1 \)

Fact 3. \(|F_1| \geq \Phi_{H_1}(v_1 : A) + p + C + \Phi_{V_2,H_1}(\Gamma_2)\)

For the second premise:

\[ V''_o = (V_o[x \mapsto v_o]) \upharpoonright FV(e_2) \]  \hspace{1cm} (ad.)

\[ V''_o, H''_o \vdash e_2 \downarrow v''_o, H''_o \]  \hspace{1cm} (ad.)

\[ \Sigma; \Gamma_2, x : A \models^{P} e_2 : B \]  \hspace{1cm} (ad.)

\[ H_1 \vdash v'_o : A \]  \hspace{1cm} (By Lemma 6 and ??)

\[ H_1 \vdash V_2 : \Gamma_2, x : A \]  \hspace{1cm} (def of \(\models\))

Let \( V_2 = (V[x \mapsto v_1]) \upharpoonright FV(e_2) \)

Let \( g = \{ l \in H_1 \mid l \notin F_1 \cup R \cup \text{locs}_{V_2,H_1}(e_2) \} \)

Instantiate IH with \( C = C, V = V_2, H = H_1, F = F_1 \cup g, R = R, \) we get existence clause on the second premise

\(\text{NTS (1) - (4) to instantiate existence clause on the second premise:}\)

(1) \( V''_o, H''_o \sim V_2, H_1 \)  \hspace{1cm} (assumption and Fact 2.)

(2) - (4) Same as Lemma 10
Case 8: E:Pair Similar to E:Const*

Case 9: E:MatP Similar to E:MatCons

Case 10: E:Nil Similar to E:Const*

Case 11: E:Cons

\[ V_o, H_o \vdash \text{cons}(x_1; x_2) \downarrow l_o, H_o' \]  
\[ H_o'(l_o) = \langle V_o(x_1), V_o(x_2) \rangle \]  
\[ \Sigma; x_1 : A, x_2 : L^p(A) \frac{q+p+1}{q} \text{cons}(x_1; x_2) : L^p(A) \]  
Let \( C \in \mathbb{Q}^+ \), well-formed configuration \( V, H, R, F \) s.t.
\[ V_o, H_o \sim V, H \text{ and } |F| \geq \Phi_{V,H}(x_1 : A, x_2 : L^p(A)) + (q + p + 1) + C \]
Let \( v = \langle V(x_1), V(x_2) \rangle, l \in F, H' = H \{ l \mapsto v \}, F' = F \setminus \{ l \} \)
\[ (1) V, H, R, F \vdash \text{cons}(x_1; x_2) \downarrow l, H', F' \]  
\[ (2) l_o \sim_{H'} l \]  
\[ (3) |F'| \geq \Phi_{H'}(l : L^p(A)) + q + C \]  
\[ |F| \geq \Phi_{V,H}(x_1 : A, x_2 : L^p(A)) + (p + q + 1) + C \]
\[ = \Phi_{H'}(l : L^p(A)) + q + 1 + C \]
\[ |F'| = |F| - 1 \geq \Phi_{H'}(l : L^p(A)) + q + C \]

Case 12: E:MatNil Similar to E:Cond*

Case 13: E:MatCons

\[ V_o(x) = l_o \]  
\[ H_o(l_o) = \langle v'_o, v''_o \rangle \]  
\[ \Gamma = \Gamma', x : L^p(A) \]  
\[ \Sigma; \Gamma', x_h : A, x_t : L^p(A) \frac{q+p+1}{q} e_2 : B \]  
\[ V'_o = \langle V_o[x_h \mapsto v'_o, x_t \mapsto v''_o] \rangle_{e_2} \]  
\[ V'_o, H_o \vdash e_2 \downarrow v_o, H' \]  
\[ H \vdash V(x) : L^p(A) \]  
\[ H \vdash v_h : A, H \equiv v_1 : L^p(A) \]  
\[ H \vdash V'_o; \Gamma', x_h : A, x_t : L^p(A) \]  
\[ \text{(Inversion on} \equiv) \]
\[ \text{(def of} \equiv) \]
\[ \text{NTF} v, H', F' \text{ s.t.} \]
\[ 1. V, H, R, F \vdash e \downarrow v, H', F' \text{ and} \]
\[ 2. v_o \sim_{H'} v \]
2. $|F'| \geq \Phi_{H'}(v : B) + q' + C$

Let $V(x) = l$

$H(l) = (v_h, v_l)$

Let $V' = (V[x_h \mapsto v_h, x_t \mapsto v_l]) |_{FV(e)}$

Let $g = \{l \in H \mid l \notin F \cup R \cup \text{locs}_{V', H}(e_2)\}$

NTS $g$ nonempty, in particular, that $l \in g$

$l \notin F \cup R$ (by assumption)

AFSOC $l \in \text{locs}_{V', H}(e_2)$

Then $l \in \text{reach}_H(V'(x'))$ for some $x' \in \text{dom}(V')$

**Case** $x' \in \{x_h, x_t\}$:

WLOG let $x' = x_h$

But then $\text{reach}_H(V(x))(l) \geq 2$ and $\text{set}(\text{reach}_H(V(x)))$ doesn’t hold

**Case** $x' \notin \{x_h, x_t\}$:

$x' \in \text{dom}(V)$

$x = x'$

Contradiction, $x \notin \text{dom}(V')$

$l \notin \text{locs}_{V', H}(e_2)$

Hence $l \in g$

By IH with $C' = C, V = V', H = H, R = R, F_1 = F \cup g$ we have the existence clause. NTS the following:

(1) $V'_o, H_o \sim V', H$

(2) - (4) Same as Lemma 10

(5) $|F'_1| \geq \Phi_{V', H}(V', x_h : A, x_t : L^p(A)) + q + p + 1 + C$:

$|F'_1| = |F \cup g|$  
\[ \geq |F| + |g| \]
\[ \geq \Phi_{V, H}(\Gamma) + q + C + |g| \]
\[ = \Phi_{V, H}(\Gamma, x_h : A, x_t : L^p(A)) + q + p + C + |g| \]
\[ = \Phi_{V, H}(V', x_h : A, x_t : L^p(A)) + q + p + 1 + C \]

By existence clause have $v, H', F'$ s.t.

Fact 1. $V', H, R, F \cup g \vdash e_2 \Downarrow v, H', F'$

Fact 2. $|F'| \geq \Phi_{H'}(v : B) + q' + C$

Apply E:MatCons and we’re done

**Case 14: E:Share**

$V_o, H_o \vdash \text{share } x \text{ as } x_1, x_2 \text{ in } e' \Downarrow v_o, H'_o$  
(case)

$V_o(x) = v'_o$  
(ad.)

$V'_o = (V_o[x_1 \mapsto v'_o, x_2 \mapsto v'_o]) |_{FV(e')}$  
(ad.)

$V'_o, H_o \vdash e' \Downarrow v_o, H'_o$  
(ad.)

$\Sigma; \Gamma, x : A \downarrow_{q'} \text{share } x \text{ as } x_1, x_2 \text{ in } e' : B$  
(case)

$A \uparrow A_1, A_2, 1$  
(ad.)

$\Sigma; \Gamma, x_1 : A_1, x_2 : A_2 \downarrow_{q'} e' : B$  
(ad.)

Let $C \in \mathbb{Q}^+$, well-formed configuration $V, H, R, F$ s.t. $V_o, H_o \sim V, H$ and $|F| \geq \Phi_{V, H}(\Gamma, x : A) + q + C$
Given a well-formed context $H$, $reach_E$, $H''$, $F''$ s.t.
1. $V, H, R, F \vdash e \downarrow v, H'', F''$ and
2. $v_o \sim_{H''} v$
3. $|F''| \geq \Phi_{H''}(v : B) + q' + C$

Let $V(x) = v', L \subseteq F, |L| = |dom(reach_H(v'))|, H', v'' = copy(H, L, v'), V' = (V[x_1 \mapsto v', x_2 \mapsto v'']) |_{FV(e)}$

$F' = F \setminus L, g = \{ l \in H \mid l \notin F' \cup R \cup locs_{V', H'}(e) \}$

By IH with $C, V', H', R, F' \cup g$, have the existence clause. NTS the following:

1. $V''_o, H_o \sim V', H'$

(1) $V''_o, H_o \sim V', H'$

(By Lemma 7 and assumption)

(2) - (4) same as lemma 10

(5) $|F' \cup g| \geq \Phi_{V', H'}(\Gamma, x_1 : A_1, x_2 : A_2) + q' + C$

ST $|(F \setminus L) \cup g| \geq \Phi_{V', H'}(\Gamma, x_1 : A_1, x_2 : A_2) + q + C$

$\iff (|F| - |L|) + |g| \geq \Phi_{V', H'}(\Gamma) + \Phi_{V', H'}(x_1 : A_1) + \Phi_{V', H'}(x_2 : A_2) + q + C$

ST $|F| \geq \Phi_{V', H'}(\Gamma) + \Phi_{V', H'}(x_1 : A_1) + \Phi_{V', H'}(x_2 : A_2) + |L| + q + C$

$\iff |F| \geq \Phi_{V', H'}(\Gamma) + \Phi_{V', H'}(x : A) + q + C$

$\iff |F| \geq \Phi_{V', H'}(\Gamma, x : A) + q + C$

(By Lemma 9)

Instantiate the existence clause and apply E:Share, and we're done

\[ \square \]

### D \ \mathcal{E}_{\text{copy}} \ \text{over-approximates} \ \mathcal{E}_{\text{gc}}

**Definition 12.** A configuration $(V, H, R, F)$ is well-formed if

1. $dom(H) \subseteq reach_H(V) \cup R \cup F$
2. $reach_H(V) \cup R \subseteq dom(H) \setminus F$
3. $collect(R, reach_H(V), H, F) = \emptyset$

For a context $(V, H, R, F)$, denote the garbage w.r.t. a set of locations $L$ as $collect(R, L, H', F') = \{ l \in H' \mid l \notin F' \cup R \cup L \}$.

**Lemma 22.** Given a well-formed context $(V, H, R, F)$ and $V, H, R, F \vdash e \downarrow v, H', F'$, $dom(H') \subseteq reach_H(v) \cup R \cup F'$, $reach_H(v) \cup R \subseteq dom(H') \setminus F'$ and $collect(R, reach_H'(v), H', F') = \emptyset$.

Now consider two well-formed configurations $C_1 = (V_1, H_1, R_1, F_1), C_2 = (V_2, H_2, R_2, F_2)$.

A mapping $f : A \rightarrow \mathcal{P}(B)$ is a partition on $B$ the image of $A$ forms a disjoint union of $B$ (e.g. $\forall x, y \in A, f(x) \cap f(y) = \emptyset \land \bigcup f(A) = B$). Furthermore, a partition is proper if for any $x \in A$, $f(x) \neq \emptyset$.

Given a proper partition $f$, we can choose an arbitrary $b \in f(a)$ to be the representation for that part; call this singlet set $\{ b \}$ $rep(a)$.

A simple corollary is the fact that if $V_2, H_2$ is a linear context (e.g. $linearCtx(V_2, H_2)$ holds), then $|\gamma(l)| = |\{ reach_{H_1}(V_1) \}(l) |$, where $reach_{H_1}(V_1) = \bigcup_{x \in dom(V)} reach_{H_1}(x)$. In general for a multiset $S$, when we hold, say that this $\gamma$ is a counting partition for $S$.

For a partition $f : A \rightarrow \mathcal{P}(B)$, we write the set of equivalence classes as $\mathcal{E}(f) = \{ f(x) \mid x \in A \} = f(A)$, i.e. the image of $f$ on its domain $A$. 


**Lemma 24.** Let \( \text{dir} \) be the set \( \{\text{L}, \text{R}, \text{N}\} \), denoting left, right, and next respectively. We can index values via directions:

\[
\begin{align*}
g_{H}(\text{Just}(v_1, v_2), \text{L}) & = \text{Just}(v_1) \\
g_{H}(\text{Just}(v_1, v_2), \text{R}) & = \text{Just}(v_2) \\
g_{H}(\text{Just}(v_1, v_2), \_ ) & = \text{None} \\
g_{H}(\text{Just}(l), \text{N}) & = \text{Just}(H(l)) \\
g_{H}(\text{Just}(l), \_ ) & = \text{None} \\
g_{H}(r, \_ ) & = r
\end{align*}
\]

Let \( P \) be a sequence of directions. Extend \( \text{get} \) to sequence of directions:

\[
\begin{align*}
\text{find}_{H}(v, D :: P) & = \text{find}_{H}(\text{get}_{H}(v, D), P) \\
\text{find}_{H}(v, []) & = v
\end{align*}
\]

Call \( P \) valid w.r.t. a value \( \gamma \) if \( \text{find}_{H}(v, P) = \text{Just}(v') \) for some \( v' \). Write \( V_{H}(x; P) \) for from\( \text{Just}(\text{find}_{H}(V(x), P)) \) given a valid sequence \( P \) w.r.t. \( V(x) \), and \( \text{reach}_{H}(V(x), P) \) for \( \text{reach}_{H}(V(x); P) \). Given a map \( m : X \rightarrow S(\text{dir}) \) from variables to valid sequences of directions, Define \( \text{reachPath}_{V,H}(X, m) = \bigcup_{x \in X} \text{reach}_{H}(V(x; m(x))) \).

**Lemma 23.** Let \( V_1, H_1 \sim V_2, H_2 \). Then for all \( x \in \text{dom}(V_1) \) and sequence of directions \( P \), Either \( \text{find}_{H_1}(V_1(x), P) = \text{find}_{H_2}(V_2(x), P) = \text{None} \) or \( \text{find}_{H_1}(V_1(x), P) = v_1 \), \( \text{find}_{H_2}(V_2(x), P) = v_2 \) and \( v_1 \sim_{H_2} v_2 \)

**Proof.** Induction on length of \( P \) and then \( H \vdash v \rightarrow a : A \). \( \square \)

**Definition 14.** A configuration \( C_2 = (V_2, H_2, R_2, F_2) \) is a copy extension of another configuration \( C_1 = (V_1, H_1, R_1, F_1) \) iff

1. \( V_1, H_1 \sim V_2, H_2 \)
2. There is a proper partition \( \gamma : \text{dom}(H_1) \setminus F_1 \rightarrow \mathcal{P}(\text{dom}(H_2) \setminus F_2) \) such that for all \( l \in \text{dom}(\gamma) \), \( |\gamma(l)| = \text{reach}_{H_2}(V_1(l)) + R_1(l) \)
3. for all \( l \in \text{dom}(\gamma) \), \( x \in \text{dom}(V_1) \), valid sequence of directions \( P \) w.r.t. \( V_1(x) \), \( |\text{reach}_{H_2}(V_2(x; P)) \cap \gamma(l)| = \text{reach}_{H_1}(V_1(x; P)) \cap \gamma(l) \).
4. for all \( l \in \text{dom}(\gamma) \), \( |\gamma(l) \cap R_2| = R_1(l) \)
5. \( |F_1| = |F_2| + |\ominus(\gamma)| \), where \( \ominus(\gamma) = \bigcup_{P \in \text{ext}(\gamma)} C \setminus \text{rep}(C) \)

Write this as \( C_1 \preceq C_2 \).

Note that \( \preceq \) is reflexive. Now the key lemma:

**Lemma 24.** Let \( (C_2, e) \) be a linear computation. Given that \( C_2 \vdash_{\text{copy}} e \Downarrow v, H', F' \), for all well-formed configurations \( C_1 \) such that \( C_1 \preceq C_2 \), there is exists a triple \( (w, Y', M') \in \text{Val} \times \text{Heap} \times \text{Loc} \) and \( \gamma' : \text{dom}(Y') \setminus M' \rightarrow \mathcal{P}(\text{dom}(H') \setminus F') \) s.t.

1. \( C_1 \vdash_{\text{free}} e \Downarrow w, Y', M' \)
2. \( v \sim_{Y', H'} w \)
3. \( \gamma' \) is a proper partition, such that for all \( l \in \text{dom}(\gamma') \), \( |\gamma'(l)| = \text{reach}_{Y_1}(w_1)(l) + S(l) \)
4. For all \( P \) valid w.r.t. \( v \), \( |\text{reach}_{H'}(\text{find}_{H}(v; P)) \cap \gamma'(l)| = \text{reach}_{Y'}(\text{find}_{Y'}(w; P))(l) \)
5. \( l \in \text{dom}(\gamma') \), \( \gamma'(l) \cap R = \gamma(l) \cap R \)
Lemma 25. Let $V_2, H_2, R_2, F_2 \vdash_{\text{copy}} e \downarrow v, H', F'$, and $V_1, H_1, R_1, F_1 \leq V_2, H_2, R_2, F_2$ because $(-, \gamma, \eta, -)$. Then the following hold: for all $l \in \text{dom}(H_1) \setminus F_1$, $X \subseteq \text{dom}(V)$, $m : X \rightarrow S(\text{dir})$, $l \in \text{dom}(\gamma)$, $\gamma(l) \subseteq \text{collect}(R_2, C_2, \text{reachPath}_{V_2, H_2}(X, m), H_2, F_2)$ if $l \in \text{collect}(R_1, \text{reachPath}_{V_1, H_1}(X, m), H_1, F_1)$.

Proof. $\implies$

\[
\gamma(l) \cap (F_2 \cup R_2 \cup \text{reachPath}_{V_2, H_2}(X, m)) = \emptyset \quad \text{(definition of clean)}
\]

NTS $l \notin F_1 \cup R_1 \cup \text{reach}_1(X)$

(1) $l \notin F_1$

(2) $l \notin R_1$

\[
\text{Know } \gamma(l) \cap R_2 = \emptyset
\]

|$\gamma(l) \cap R_2| = R_1(l) = 0

$\gamma(l) \cap R_2 = \emptyset$

(3) $l \notin \text{reachPath}_{V_1, H_1}(X, m)$

\[
\text{Know } \gamma(l) \cap \text{reachPath}_{V_2, H_2}(X, m) = \emptyset
\]

Let $x \in X$

\[
\gamma(l) \cap \text{reach}_{H_2}(V_2(x; m(x))) = \emptyset
\]

|$\gamma(l) \cap \text{reach}_{H_2}(V_2(x; m(x)))| = \text{reach}_{H_2}(V_1(x; m(x)))(l)

|reach_{H_2}(V_1(x; m(x)))(l)| = 0

reach_{Path_{V_1, H_1}}(X, m)(l) = 0

$l \notin \text{reachPath}_{V_1, H_1}(X, m)$

Thus, $l \in \text{collect}(R_1, \text{reachPath}_{V_1, H_1}(X, m), H_1, F_1)$

$\Leftarrow$

\[
l \notin (F_1 \cup R_1 \cup \text{reachPath}_{V_1, H_1}(X, m))
\]

NTS $\gamma(l) \cap (F_2 \cup R_2 \cup \text{reachPath}_{V_2, H_2}(X, m)) = \emptyset$

(1) $\gamma(l) \cap F_2 = \emptyset$

(2) $\gamma(l) \cap R_2 = \emptyset$

|$\gamma(l) \cap R_2| = R_1(l) = 0

$\gamma(l) \cap R_2 = \emptyset$

(3) $\gamma(l) \cap \text{reachPath}_{V_2, H_2}(X, m) = \emptyset$

Let $x \in X$

|$\gamma(l) \cap \text{reach}_{H_2}(V_2(x; m(x)))| = \text{reach}_{H_2}(V_1(x; m(x)))(l)

$\gamma(l) \cap \text{reach}_{H_2}(V_2(x; m(x))) = \emptyset$

$\gamma(l) \cap \text{reachPath}_{V_2, H_2}(X, m) = \emptyset$

$\gamma(l) \cap (F_2 \cup R_2 \cup \text{reachPath}_{V_2, H_2}(X, m)) = \emptyset$

$\gamma(l) \in \text{collect}(R_2, \text{reachPath}_{V_2, H_2}(X, m), H_2, F_2)$

Lemma 26. Given a proper partition $\gamma : \text{dom}(Y) \setminus M \rightarrow \mathcal{P}(\text{dom}(H) \setminus F)$, equivalent values $v \sim_{\mathcal{H}} w$, and the following:
1. \( L \subseteq F \)

2. \(|L| = |\text{dom}(\text{reach}_H(v))|\)

3. \( H', v' = \text{copy}(H, L, v) \)

4. \( F' = F \setminus L \)

there is a proper partition \( \gamma' : \text{dom}(Y) \setminus M \rightarrow \mathcal{P}(\text{dom}(H) \setminus F') \) s.t. \( |\gamma'(l)| = |\gamma(l)| + \text{reach}_Y(w)(l) \), \( |\text{reach}_H(v) \cap \gamma'(l)| = |\text{reach}_H(v') \cap \gamma'(l)| = |\text{reach}_H(v) \cap \gamma(l)| \).

**Proof.** Induction on \( \text{copy} \).

Now the proving \( \mathcal{E}_{\text{copy}} \) over approximates \( \mathcal{E}_{\text{gc}} \):

**Proof.** Induction on the evaluation judgement.

**Case 1: E:Var** Trivial

**Case 2: E:Const** Trivival

**Case 4: E:App** Similar to E:CondT

**Case 5: E:CondT**

\[
V, H, R, F \vdash^{	ext{copy}} \text{if}(x; e_1; e_2) \Downarrow v, H', F' \quad \text{(case)}
\]

Let \( W, Y, S, M \preceq V, H, R, F \)

Let \( W' = W \mid_{\text{dom}(V')} \)

Let \( j = \{ l \in Y | l \notin M \cup S \cup \text{locs}_{W,Y}(e_1) \} \)

**NTS** \( W', Y, S, M \cup j \preceq V', H, R, F \cup g \)

(1) \( W', Y \sim V', H \)

(2) \( \text{NTF} \) a proper partition \( \gamma' : \text{dom}(Y) \setminus (M \cup j) \rightarrow \mathcal{P}(\text{dom}(H) \setminus (F \cup g)) \)

Let \( \gamma'(l) = \gamma \mid_{\text{dom}(Y) \setminus (M \cup j)}(l) \setminus g \)

First, show \( \gamma' \) is a partition

Let \( l, l' \in \text{dom}(Y) \setminus (M \cup j) \)

\( \gamma'(l) \cap \gamma'(l') = \emptyset \) \( \quad \text{(\( \gamma \) is partition)} \)

\[ \gamma'(\text{dom}(Y) \setminus (M \cup j)) = \gamma(\text{dom}(Y) \setminus (M \cup j)) \setminus g \]

\[ = \left( \bigsqcup_{l \in \text{dom}(Y) \setminus M} \gamma(l) \setminus g \right) \]

\[ = \left( \bigsqcup_{l \in j} \gamma(l) \right) \setminus g \]

\[ = \left( \text{dom}(H) \setminus F \right) \setminus g \]

\[ = \text{dom}(H) \setminus (F \cup g) \]

Hence \( \gamma' \) is a partition

\( \gamma' \) is also proper:

Let \( l \in \text{dom}(Y) \setminus (M \cup j) \)

\( \gamma'(l) = \gamma(l) \setminus g \)

AFSOC \( \gamma(l) \subseteq g \)

\( l \in j \) \( \quad \text{(By Lemma 4)} \)
Let \( \gamma' \) be a bijection such that \( \gamma' \) is the \( \gamma \) \( \in \mathcal{C} \) \( \cap \mathcal{C} \)

First, we know that \( \gamma' \) is a bijection since assumed \( l \notin j \)

Now, NTS \( |\gamma'(l)| = \text{reach}_{\gamma'}(W')(l) + S(l) \)

Let \( l \in \text{dom}(\gamma') \)

\[ \gamma'(l) = \gamma(l) \setminus g \]

(definition)

Thus, \( \gamma'(l) = \text{reach}_{\gamma'}(W')(l) + S(l) - \gamma(l) \cap g \)

\[ g = \text{reach}_{\gamma'}(W \mid_{\text{dom}(V) \setminus \text{FV}(e_1)}) \]

(definition of \( g \))

\[ \gamma(l) \cap g = \text{reach}_{\gamma'}(W \mid_{\text{dom}(W) \setminus \text{FV}(e_1)})(l) \]

(definition 3. of \( \leq \))

\[ |\gamma'(l)| = \text{reach}_{\gamma'}(W')(l) + S(l) - \text{reach}_{\gamma'}(W \mid_{\text{dom}(W) \setminus \text{FV}(e_1)})(l) \]

\[ = \text{reach}_{\gamma'}(W \mid_{\text{FV}(e_1)})(l) + S(l) \]

\( \therefore l \in \text{dom}(\gamma') \), \( x \in \text{dom}(W') \), P valid sequence w.r.t. \( W'(x) \), NTS

\[ |\text{reach}_{\gamma'}(V'(x; P)) \cap \gamma'(l)| = \text{reach}_{\gamma'}(W'(x; P))(l) \]

\[ \text{STS} \mid \text{reach}_{\gamma'}(V'(x; P)) \cap (\gamma(l) \setminus g) = \text{reach}_{\gamma'}(W'(x; P))(l) \]

\[ g \cap \text{reach}_{\gamma'}(V'(x; P)) = \emptyset \]

(definition of \( g \))

\[ |\text{reach}_{\gamma'}(V'(x; P)) \cap (\gamma(l) \setminus g)| = |(\text{reach}_{\gamma'}(V'(x; P)) \setminus g) \cap \gamma(l)| \]

\[ = |\text{reach}_{\gamma'}(V'(x; P)) \setminus \gamma(l)| \]

\[ = |\text{reach}_{\gamma'}(V(x; P)) \setminus \gamma(l)| \]

\[ = \text{reach}_{\gamma'}(W(x; P))(l) \]

(definition 3. of \( \leq \))

\( \therefore \gamma(l) \cap R = S(l) \)

\( \gamma(l) \cap R = S(l) \)

\( g \cap R = \emptyset \)

(definition 4. of \( \leq \))

\( \therefore \text{STS} |M \cup j| = |\text{F} \setminus g| + |\ominus \gamma'| | \)

\[ \text{STS} |M| + |j| = |\text{F}| + |g| + |\ominus \gamma'| | \)

By assumption \( |M| = |\text{F}| + |\ominus \gamma'| | \)

\[ \text{STS} |j| + |\ominus \gamma'| = |g| + |\ominus \gamma'| | \)

NTF a bijection \( f : j \ominus \gamma \rightarrow g \ominus \gamma \)

First, we know that \( g = \bigcup_{l \in j} (\gamma(l)) \cup L \) for some \( L \)

\( \therefore \text{By 4} \)

Let \( C_1 = \{ \gamma(l) \mid l \in j \} \), \( C_2 = \text{ec}(\gamma) \setminus C_1 \)

Clearly, \( \ominus \gamma = \bigsqcup_{C \in C_1} C \setminus \text{rep}(C) \cup \bigsqcup_{C \in C_2} C \setminus \text{rep}(C) \)

Let \( D_1 = \bigsqcup_{C \in C_1} C \setminus \text{rep}(C), D_2 = \bigsqcup_{C \in C_2} C \setminus \text{rep}(C) \)

We define the bijection \( f \) by parts: \( f_1 : j \ominus D_1 \rightarrow \bigsqcup_{C \in C_2} C, f_2 : D_2 \rightarrow L \cup \ominus \gamma \)

\[ f_1(x) = \begin{cases} \text{rep}(\gamma(l)) & x = (\text{inl}, l) \\ l & x = (\text{inr}, l) \end{cases} \]
Clearly, $f_1$ is a bijection, and $|j| + |D_1| = \bigcup_{C \in \mathcal{C}_1} |C|$

To avoid the problem of maintaining a single representative for a class (which might be collected), note the following:

$$|\mathcal{C}_2| = |\text{ec}(\gamma) \setminus \{\gamma(l) \mid l \in j\}|$$

$$= |\text{ec}(\gamma \upharpoonright \text{dom}(\gamma) \setminus (M \cup j))|$$

$$= |\text{ec}(\gamma')|$$

Meaning that $\mathcal{C}_2$ has the same number of classes as $\gamma'$ (note these classes might be different)

Since both $\gamma, \gamma'$ are proper partitions we have the following:

$$|D_2| = |L \cup \odot(\gamma')| \iff \bigcup_{C \in \mathcal{C}_2} C \setminus \text{rep}(C) = |L \cup \bigcup_{C \in \text{ec}(\gamma')} C \setminus \text{rep}(C)| \iff \bigcup_{C \in \mathcal{C}_2} C = |L \cup \bigcup_{C \in \text{ec}(\gamma')} C|$$

In fact, the latter two sets are equal:

let $l' \in \bigcup_{C \in \mathcal{C}_2} C$

$l' \in H \setminus F$  \hspace{1cm} (Def. of partition)

**case** $l \in g$

$l' \notin \bigcup_{l \in j} \gamma(l)$ \hspace{1cm} (Def. of $\mathcal{C}_2$)

$l' \in L$ \hspace{1cm} (Def. of $g$)

$l' \in L \cup \bigcup_{C \in \text{ec}(\gamma')} C$

**case** $l' \notin g$

$l' \in H \setminus (F \sqcup g)$

Exists $C \in \text{ec}(\gamma')$ s.t. $l' \in C$ \hspace{1cm} (Def. of partition)

$l' \in \bigcup_{C \in \text{ec}(\gamma')} C$

$l' \in L \cup \bigcup_{C \in \text{ec}(\gamma')} C$

For the other direction, let $l' \in L \cup \bigcup_{C \in \text{ec}(\gamma')} C$

**case** $l' \in L$

$l' \in H \setminus F$ \hspace{1cm} (Def. of $L$)

Exists $C \in \text{ec}(\gamma)$ s.t. $l' \in C$ \hspace{1cm} (Def. of partition)

$l' \in \bigcup_{C \in \text{ec}(\gamma)} C$

**case** $l' \in \bigcup_{C \in \text{ec}(\gamma)} C$

$l' \in H \setminus (F \sqcup g)$ \hspace{1cm} (Def. of partition)

$l' \in H \setminus F$

Exists $C \in \text{ec}(\gamma)$ s.t. $l' \in C$ \hspace{1cm} (Def. of partition)

$l' \in \bigcup_{C \in \text{ec}(\gamma)} C$
Hence we show that \(|D_2| = |L \cup \odot(\gamma')|\), and together with the previous equality, \(|j| + |\odot(\gamma)| = |g| + |\odot(\gamma')|\)

Thus we have \(W', Y, S, F \cup j \not\subseteq V', H, R, F \cup g\)

\(V', H, R, F \cup g \vdash e_1 \Downarrow v, H', F'\) (case)

By IH on \((V', H, R, F \cup g)\), we have \((w, Y', M', \gamma'')\) such that

1. \(W', Y, S, M \cup j \vdash e_1 \Downarrow w, Y', M'\)
2. \(v \sim_{V'} w\)
3. \(\gamma'\) is a proper partition, and \(|\gamma''(l)| = \text{reach}_{Y'}(w)(l) + R(l)\)
4. \(\gamma''(l) \cap R = \gamma'(l) \cap R\)
5. \(\gamma'(l) \cap R = (\gamma(l) \setminus g) \cap R = \gamma(l) \cap R\)

Apply F:CondT to (1), we are done.

**Case 6: E:CondF** Similar to E:CondT

**Case 7: E:Let**

\(V, H, R, F \vdash e_{\text{copy}} \, \text{let}(e_1; x: \tau.e_2) \Downarrow v_2, H_2, F_2\) (case)

\(W, Y, S, M \preceq V, H, R, F\) (assumption)

\(V_1, H, R', F \vdash e_1 \Downarrow v_1, H_1, F_1\) (admissibility)

Let \(W_1 = W \upharpoonright_{FV(e_1)}, S' = S \uplus \text{locs}_{W,Y}(\text{lam}(x: \tau.e_2))\)

\(\text{NTS } W_1, Y, S', M \preceq V_1, H, R', F\)

1. \(W_1, Y \sim V_1, H\) (condition 1. of \(\preceq\))
2. \(b\) NTS \(|\gamma(l)| = \text{reach}_Y(W')(l) + S'(l)\)
3. \(|\gamma(l)| = \text{reach}_Y(W)(l) + S(l)\)
4. \(|\gamma(l)| = \text{reach}_Y(W')(l) + \text{reach}_Y(W \upharpoonright_{FV(e_2)}(l)) + S(l)\)
5. \(|\gamma(l)| = \text{reach}_Y(W')(l) + S'(l)\)

Let \(l \in \text{dom}(\gamma), x \in \text{dom}(W'), P\) valid sequence of directions. Have

\(|\text{reach}_H(W'(x; P)) \cap \gamma(l)| = \text{reach}_Y(W'(x; P))(l)\) (condition 3. of \(\preceq\))

4. \(a\) NTS \(S' \subseteq \text{dom}(Y) \setminus M\)

\(S \subseteq \text{dom}(Y) \setminus M\) (condition 4. of \(\preceq\))

\(\text{locs}_{W,Y}(\text{lam}(x: \tau.e_2)) \subseteq \text{reach}_Y(W) \subseteq \text{dom}(Y) \setminus M\) (well-formed configuration)

\(S' \subseteq \text{dom}(Y) \setminus M\)

4. \(b\) Let \(l \in S'\). NTS \(|\gamma(l) \cap R'(l)| = S'(l)\)

\(\text{STS } |\gamma(l) \cap (R \cup \text{reach}_H(FV(e_2) \setminus \{x\}))| = S(l) + \text{reach}_Y(FV(e_2) \setminus \{x\})(l)\)

\(\text{STS } |(\gamma(l) \cap R) \uplus (\gamma(l) \cap \text{reach}_H(FV(e_2) \setminus \{x\}))| = S(l) + \text{reach}_Y(FV(e_2) \setminus \{x\})(l)\)

\(\text{STS } |(\gamma(l) \cap R) | \cup (\gamma(l) \cap \text{reach}_H(FV(e_2) \setminus \{x\}))| = S(l) + \text{reach}_Y(FV(e_2) \setminus \{x\})(l)\)

\(\text{STS } |\gamma(l) \cap \text{reach}_H(V(FV(e_2) \setminus \{x\}))| = \text{reach}_Y(W(FV(e_2) \setminus \{x\}))(l)\)

\(\text{STS } |\gamma(l) \cap \bigcup_{x' \in FV(e_2) \setminus \{x\}} \text{reach}_H(V(x'))| = \bigcup_{x' \in FV(e_2) \setminus \{x\}} \text{reach}_Y(W(x'))(l)\)

\(\text{Let } x' \in FV(e_2) \setminus \{x\}\)

\(|\gamma(l) \cap \text{reach}_H(V(x'))| = \text{reach}_Y(W(x'))(l)|\) (condition 3. of \(\preceq\))
(5) Have $|M| = |F| + |\otimes (\gamma)|$  

By IH on first premise, there is $(w_1, Y_1, M_1)$ and $\gamma_1$ s.t.

Fact 1. $W', Y, S', M \vdash_{\text{rw}} e \Downarrow \Gamma, Y_1, M_1$

Fact 2. $v \sim_{H_1}^e w$

Fact 3. $\gamma_1$ is a proper partition, such that for all $l \in \text{dom}(\gamma_1)$,

$$|\gamma_1(l)| = |\text{reach}_{Y_1}(w_1)(l)| + S(l)$$

Fact 4. For all $P, |\text{reach}_{H_1}(\text{find}_{H_1}(w_1; P)) \cap \gamma_1(l)| = |\text{reach}_{Y_1}(\text{find}_{Y_1}(w_1; P))(l)|$

Fact 5. $|\gamma_1(l) \cap R' = |\gamma(l) \cap R' \land S' \subseteq \text{dom}(Y_1) \setminus M_1$

Fact 6. $|M_1| = |F_1| + |\otimes (\gamma_1)|$

$V_2, H_1, R, F_1 \cup g \vdash e_2 \Downarrow v_2, H_2, F_2$  
(admissibility)

Let $W_2 = (W[x \mapsto w_1]) |_{FV(e_2)}, j = \{ l \in H_1 \mid l \notin M_1 \cup S \cup \text{locs}_{W_2, Y_1(e_2)} \}$

NTS $W_2, Y_1, S, M_1 \cup j \preceq V_2, H_1, R, F_1 \cup g$

(1) $W_2, Y_1 \sim V_2, H_1$  
(condition 1. of $\preceq$ and Fact 1)

(2) Let $\gamma_2 : \text{dom}(Y_1) \setminus (M_1 \cup j) \rightarrow \mathcal{P}((\text{dom}(H_1) \setminus (F_1 \cup g))$ be defined by $\gamma_2(l) = \gamma_1(l) \cap \gamma$

NTS $|\gamma_2(l)| = |\text{reach}_{Y_1}(W_2)(l) + S(l)$

STS $|\gamma_1(l)| = |\gamma_1(l) \cap \gamma + \text{reach}_{Y_1}(W_2)(l) + S(l)$

\textbf{case} $x \in FV(e_2)$:

$$g = \text{collect}(R', v_1, H_1, F_1)$$
$$= \emptyset$$  
(By 22)

$$|\gamma_1(l)| = |\text{reach}_{Y_1}(w_1)(l)| + S'(l)$$
$$= |\text{reach}_{Y_1}(w_1)(l)| + \text{reach}_{Y_1}(W_2)(l) + S(l)$$
$$= |\text{reach}_{Y_1}(W_2)(l) + S(l)|$$

\textbf{case} $x \notin FV(e_2)$:

$$g = \text{collect}(R', v_1, H_1, F_1) \cup \text{reach}_{H_1}(v_1)$$
$$= \text{reach}_{H_1}(v_1)$$  
(By 22)

$$|\gamma_1(l) \cap \gamma| = |\text{reach}_{Y_1}(w_1)|$$

STS $|\gamma_1(l)| = |\text{reach}_{Y_1}(w_1)(l)| + |\text{reach}_{Y_1}(W_2)(l) + S(l)|$
$$= |\text{reach}_{Y_1}(w_1)(l) + S'(l)|$$
$$= |\text{reach}_{Y_1}(w_1)(l) + \text{reach}_{Y_1}(W_2)(l) + S(l)|$$
$$= |\text{reach}_{Y_1}(W_2)(l) + S(l)|$$

(9) Let $l \in \text{dom}(\gamma_2), x' \in \text{dom}(W_2), P$ valid sequence w.r.t. $W_2(x')$. NTS

$|\text{reach}_{H_1}(V_2(x'; P)) \cap \gamma_2(l)| = |\text{reach}_{Y_1}(W_2(x'; P))(l)|$

\textbf{case} $x' = x$:

$$|\text{reach}_{H_1}(V_2(x'; P)) \cap \gamma_2(l)| = |\text{reach}_{Y_1}(W_2(x'; P))(l)|$$

(Fact 4)

\textbf{case} $x' \neq x$:

$$|\text{reach}_{H_1}(V_2(x'; P)) \cap \gamma_2(l)| = |\text{reach}_{Y_1}(W_2(x'; P))(l)|$$  
(stability and condition 3. of $\preceq$)

$$|\text{reach}_{H_1}(V_2(x'; P)) \cap \gamma_1(l)| = |\text{reach}_{Y_1}(W_2(x'; P))(l)|$$

(reach$_{H_1}(V_2(x'; P)) \subseteq R'$ and Fact 5.)

$$|\text{reach}_{H_1}(V_2(x'; P)) \cap \gamma_2(l)| = |\text{reach}_{Y_1}(W_2(x'; P))(l)|$$

(g $\cap R' = \emptyset$)

(4.a) NTS $S \subseteq \text{dom}(Y_1) \setminus (M_1 \cup j)$

$S' \subseteq \text{dom}(Y_1) \setminus M_1$  
(Fact 5.)
Case 8: E:Pair Similar to E:CondT*

Case 9: E:MatP Similar to E:CondT

Case 10: E:Nil Similar to E:CondT*

Case 11: E:Cons

\[ V, H, R, F \vdash \text{cons}(x_1; x_2) \downarrow l, H', F \setminus \{l\} \quad \text{(case)} \]

\[ W, Y, S, M \preceq V, H, R, F \quad \text{(assumption)} \]

Let \( w = \langle W(x_1), W(x_2) \rangle \)

Let \( m \in M, Y' = Y\{m \mapsto w\} \)

(1) \( V, H, R, F \vdash \text{cons}(x_1; x_2) \downarrow m, Y', M \setminus \{m\} \) \quad \text{(F:Cons)}

(2) \( v \sim^l_w \text{w} \quad \text{(assumption)} \)

(3) Let \( \gamma' : \text{dom}(Y') \setminus M' \rightarrow P(\text{dom}(H') \setminus F') \) be defined by \( \gamma'(l') = \gamma[m \mapsto \{l\}]_l(l') \)

\( \gamma' \) is a proper partition \( \quad \text{(\( \gamma \) proper and \( l \notin \text{dom}(H) \setminus F \))} \)

NTS \( |\gamma'(l')| = \text{reach}_{Y'}(m)(l') + S(l') \)

STS \( |\gamma[m \mapsto \{l\}]_l(l')| = \text{reach}_{Y'}(m)(l') + S(l') \)

\textbf{case} \( l' = m \):

\[ |\gamma[m \mapsto \{l\}]_l(l')| = |\{l\}| = 1 \]

\[ \text{reach}_{Y'}(m)(l') + S(l') = 1 + \text{reach}_{Y'}(w_1)(m) + \text{reach}_{Y'}(w_2)(m) + S(l') \]
$$= 1 + S(m)$$

$$= 1$$

**(case)** \(l' \neq m:\)

$$|\gamma[m \mapsto \{l\}](l')| = |\gamma(l')| = reach_H(W)(l') + S(l')$$

= \(reach_H'(w_1)(l') + reach_H'(w_2)(l') + S(l')\)

= \{m \mapsto 1\} \cap reach_H'(w_1)(l') + reach_H'(w_2)(l') + S(l')

= reach_H'(m)(l') + S(l')

(4) Let \(l \in dom(\gamma')\), \(P\) be a valid path w.r.t. \(l\). NTS \(|reach_H'(find_H'(l; P)) \cap \gamma'(l')| = reach_Y'(find_Y'(m; P))(l')\)

**(case)** \(P = \emptyset:\)

$$|reach_H'(find_H'(l; P)) \cap \gamma'(l')| = |reach_H'(l) \cap \gamma'(l')|$$

= \(|\{l\} \cup reach_H'(v_1) \cup reach_H'(v_2) \cap \gamma'(l')|\)

= \(|\{l\} \cap \gamma(l')| + |reach_H'(v_1) \cap \gamma'(l')| + |reach_H'(v_2) \cap \gamma'(l')|\)

= \(1_{v = m} + \gamma'(l') = reach_Y'(w_1)(l') + reach_Y'(w_2)(l')\)

(5) NTS \(S \subseteq \gamma(\gamma')\)

Let \(l' \in dom(\gamma')\). NTS \(\gamma(l') \cap R = \gamma(l) \cap R\)

STS \(\gamma(m \mapsto \{l\})(l') \cap R = \gamma(l') \cap R\)

**(case)** \(l' = m:\)

$$\gamma(m \mapsto \{l\})(m) \cap R = \{l\} \cap R$$

= \(\emptyset\)

= \(\gamma(l') \cap R\)

(6) NTS \(|M'| = |F'| \cap \gamma(\gamma')\)

$$|M'| = |M| - 1$$

$$|F'| = |F| - 1$$

$$\gamma(\gamma') = \gamma(m \mapsto \{l\})$$

$$= \bigcup C \in ec(\gamma') \setminus (rep(C))$$

= \((\{l\} \setminus \{l\}) \cup \bigcup C \in ec(\gamma') \setminus (rep(C))\)

$$= \bigcup C \in ec(\gamma') \setminus (rep(C))$$

$$= \bigcup C \in ec(\gamma') \setminus (rep(C))$$

= \(\gamma(\gamma')\)

STS \(|M| - 1 = |F| - 1 + \gamma(\gamma')\)

Have \(|M| = |F| + \gamma(\gamma')\)

**Case 12: E:MatNil** Similar to E:Cond*
Case 13: E:MatCons

\[ V, H, R, F \vdash_{\text{Eow}} \text{match } x \{ \text{nil} \mapsto e_1 | \text{cons}(x_h; x_t) \mapsto e_2 \} \downarrow v, H', F' \]  
(case)

\[ W, Y, S, M \preceq V, H, R, F \]  
(assumption)

\[ V', H, R, F \cup g \vdash e_2 \downarrow v, H', F' \]  
(admissibility)

Let \( W' = W \mid_{\text{dom}(V')} \)

Let \( j = \{ l \in Y \mid l \notin M \cup S \cup \text{locs}_{W', Y}(e_2) \} \)

NTS \( W', Y, S, M \cup j \preceq V', H, R, F \cup g \)

Let \( \gamma' : \text{dom}(Y) \setminus (M \cup j) \rightarrow \mathcal{P}(\text{dom}(H) \setminus (F \cup g)) \) be defined by \( \gamma'(l) = \gamma(l) \setminus g \)

(1) \( W', Y \sim V', H \)  
(similar to E:CondT)

Let \( W(x) = m \)

\[ H(m) = \langle w_h, w_t \rangle, v_h \sim_{Y} w_h, v_l \sim_{Y} w_l \]

(1) \( \gamma'(l) \) is a proper partition .  
(similar to E:CondT)

Let \( l' \in \text{dom}(\gamma') \)

Now, NTS \( |\gamma'(l')| = \text{reach}_Y(W')(|l|) + S(l') \)

Consider \( |\gamma(l') \cap \text{reach}_H(V(x))| \)

\[ = |\gamma(l') \cap \{ l \}| + |\gamma(l') \cap \text{reach}_H(w_h)| + |\gamma(l') \cap \text{reach}_H(v_l)| \]

\[ = \text{reach}_Y(W(x))(l') \]

\[ = \{ m \mapsto 1 \} \cup \text{reach}_Y(w_h) \cup \text{reach}_Y(w_t)(l') \]

\[ = \{ m \mapsto 1 \}(l') + \text{reach}_Y(w_h)(l') + \text{reach}_Y(w_t)(l') \]

Note \( |\gamma(l') \cap \text{reach}_H(v_h)| = |\gamma(l') \cap \text{reach}_H(V(x; [N, L]))| \)

\[ = \text{reach}_Y(W(x; [N, L]))(l') \]

\[ = \text{reach}_Y(w_h)(l') \]

Similarly, \( |\gamma(l') \cap \text{reach}_H(v_t)| = \text{reach}_Y(w_t)(l') \)

Thus \( |\gamma(l') \cap \{ l \}| = \{ m \mapsto 1 \}(l') = 1_{V = m} \)

Back to the NTS:

\[ \gamma'(l') = \gamma(l') \setminus g \]  
(definition)

\[ |\gamma'(l')| = |\gamma(l')| - |\gamma(l') \cap g| \]

\[ = \text{reach}_Y(W)(l') + S(l') - |\gamma(l') \cap g| \]

case \( \{ x_h, x_t \} \subseteq \text{FV}(e_2) : \)

\[ g = \{ l \} \cup \text{reach}_H(V \mid_{\text{dom}(V) \setminus \text{FV}(e_2) \cup \{ x \}}) \]  
(definition of \( g \) and 10)

\[ |\gamma(l') \cap g| = |\gamma(l') \cap \{ l \}| + |\gamma(l') \cap \text{reach}_H(V \mid_{\text{dom}(V) \setminus \text{FV}(e_2) \cup \{ x \}})| \]  
(condition 3. of \( \preceq \))

Thus, \( |\gamma'(l')| = \text{reach}_Y(W)(l') + S(l') - 1_{V = m} + \text{reach}_Y(W \mid_{\text{dom}(W) \setminus \text{FV}(e_2) \cup \{ x \}})(l') \)

\[ = \text{reach}_Y(W \mid_{\text{FV}(e_2) \cup \{ x \}})(l') + S(l') - 1_{V = m} \]

\[ = \text{reach}_Y(W \mid_{\text{FV}(e_2)})(l') + \text{reach}_Y(m)(l') + \text{reach}_Y(w_h)(l') + \text{reach}_Y(w_t)(l') + S(l') - 1_{V = m} \]

\[ = \text{reach}_Y(W \mid_{\text{FV}(e_2)})(l') + \text{reach}_Y(w_h)(l') + \text{reach}_Y(w_t)(l') + S(l') \]

\[ = \text{reach}_Y(W)(l') + S(l') \]

case \( x_t \in \text{FV}(e_2), x_h \notin \text{FV}(e_2) : \)

\[ g = \{ l \} \cup \text{reach}_H(V \mid_{\text{dom}(V) \setminus \text{FV}(e_2) \cup \{ x \}}) \cup \text{reach}_H(v_t) \]  
(definition of \( g \) and 10)

\[ |\gamma(l') \cap g| = |\gamma(l') \cap \{ l \}| + |\gamma(l') \cap \text{reach}_H(V \mid_{\text{dom}(V) \setminus \text{FV}(e_2) \cup \{ x \}})| + |\gamma(l') \cap \text{reach}_H(v_t)| \]  
(condition 3. of \( \preceq \))
Let \( x \) case apply IH and F:MatCons and we're done.

Similar to E:CondT (4)

\[ |\gamma'|(l') = reach_Y(W(l')) + S(l') - (1_{l'=m} + reach_Y(W \mid_{dom(W) \cup \{x_1\}})(l') + reach_Y(w_i)(l)) \]

Thus, \( |\gamma'(l')| = reach_Y(W(l')) + reach_Y(m)(l') + reach_Y(w_h)(l') + reach_Y(w_i)(l') + S(l') - 1_{l'=m} - reach_Y(w_i)(l) \)

\[ = reach_Y(W \mid_{FV(e_2)}(l') + reach_Y(w_h)(l') + reach_Y(w_i)(l') + S(l') \]

\[ = reach_Y(W'(l') + S(l') \]

\textbf{case} \( x_i \in FV(e_2), x_h \notin FV(e_2) \) and \( \{x_h, x_i\} \cap FV(e_2) = \emptyset \) : symmetric to above

(3) Let \( l' \in dom(\gamma'), x' \in dom(W'), P \) valid sequence w.r.t. \( W'(x') \). NTS

\[ |reach_H(V(x'; P)) \cap \gamma'(l')| = reach_Y(W'(x'; P))(l') \]

\textbf{case} \( x' \notin \{x_h, x_i\} \) :

\[ |reach_H(V'(x'; P)) \cap \gamma'(l')| = |reach_H(V'(x'; P)) \cap (\gamma(l') \setminus g)| \]

\[ = |reach_H(V'(x'; P)) \cap \gamma(l')| \]

\[ = reach_Y(W'(x; P))(l') \]

\textbf{case} \( x' = x_h \) :

\[ |reach_H(V'(x'; P)) \cap \gamma'(l')| = |reach_H(find_H(v_h, P)) \cap \gamma'(l')| \]

\[ = |reach_H(find_H(v_h, P)) \cap ((\gamma'(l') \setminus g))| \]

\[ = |reach_H(find_H(v_h, P)) \cap \gamma'(l')| \]

\[ = reach_Y(W(x; P))(l') \]

\[ = reach_Y(find_H(m, N :: L :: P))(l') \]

\[ = reach_Y(find_H(v_h, P))(l') \]

\[ = reach_Y(W'(x_h; P))(l') \]

(4) Similar to E:CondT

(5) Similar to E:CondT

Apply IH and F:MatCons and we're done.

\textbf{Case 14: E:Share}

\[ V, H, R, F \vdash_{funs} \text{share } x \text{ as } x_1, x_2 \text{ in } e \Downarrow v, H'', F' \]

(case)

\[ W, Y, S, M \leq V, H, R, F \]

(assumption)

\[ V', H', R, F' \sqcup g \vdash e \Downarrow v, H'', F' \]

(admissibility)

Let \( W(x) = w' \)

\[ v' \sim_H w' \]

(condition 1 of. \( \preceq \))

Let \( W' = (W[x_1 \mapsto w', x_2 \mapsto w']) \mid_{FV(e)} \)

Let \( j \in Y \mid l \notin M \cup S \cup \text{locs}_{W', Y}(e) \}

NTS \( W', Y, S, M \cup j \preceq V', H, R, F \cup g \)

Let \( \gamma : \text{dom}(Y) \setminus (M \cup j) \to \mathcal{P}(\text{dom}(H) \setminus (F \sqcup g)) \) be defined by \( \gamma'(l) = \gamma(l) \setminus g \)

(1) \( W', Y \sim V', H \)

(Similar to E:CondT)

(2) Have proper partition \( \gamma : \text{dom}(Y) \setminus (M \cup j) \to \text{dom}(H') \setminus (F' \cup g) \) by defined by \( \gamma''(l) = \gamma'(l) \setminus g \)

\( \gamma'' \) is a proper partition

(Similar to E:CondT)
Let $l \in \text{dom}(\gamma'')$

Now, NTS $|\gamma''(l)| = \text{reach}_Y(W')(l) + S(l')$

$\gamma''(l) = \gamma'(l) \setminus g$  

$|\gamma''(l)| = |\gamma'(l)| - |\gamma'(l) \cap g|$

$= |\gamma(l)| + \mathbb{1}_{l \in \text{reach}_Y(w')} - |\gamma'(l) \cap g|$

$= \text{reach}_Y(W)(l) + S(l) + \text{reach}_Y(w')(l) - |\gamma'(l) \cap g|$

\textbf{case} $\{x_1, x_2\} \subseteq \text{FV}(e)$:

$g = \text{collect}(R, \text{reach}_H(V), H, F) = \emptyset$  

(\text{well-formed configuration})

$|\gamma''(l)| = \text{reach}_Y(W)(l) + S(l) + \text{reach}_Y(w')(l)$  

(By 26)

$= \text{reach}_Y(W')(l) + S(l)$

\textbf{case} $x_1 \in \text{FV}(e), x_2 \not\in \text{FV}(e)$:

$g = \text{collect}(R, \text{reach}_H(V), H, F) \cup \text{reach}'_H(v') = \text{reach}_H(v')$  

(\text{well-formed configuration})

$|\gamma''(l)| = \text{reach}_Y(W)(l) + S(l) + \text{reach}_Y(w')(l) - |\gamma'(l) \cap \text{reach}_H(v')|$

$= \text{reach}_Y(W)(l) + S(l) + \text{reach}_Y(w')(l) - |\gamma(l) \cap \text{reach}_H(v')|$

(By 26)

$= \text{reach}_Y(W)(l) + S(l) + \text{reach}_Y(w')(l) - \text{reach}_Y(w')(l)$  

(condition 3. of $\preceq$)

$= \text{reach}_Y(W)(l) + S(l)$

\textbf{case} $x_2 \in \text{FV}(e), x_1 \not\in \text{FV}(e)$ and $\{x_1, x_2\} \cap \text{FV}(e) = \emptyset$: symmetric to above

(3) - (5) Similar to E:MatCons

Applying the IH then F:Share, and we’re done.

\qed