The Complexity of Computing Minimal Unidirectional Covering Sets☆

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Abstract

A common thread in the social sciences is to identify sets of alternatives that satisfy certain notions of stability according to some binary dominance relation. Examples can be found in areas as diverse as voting theory, game theory, and argumentation theory. Brandt and Fischer [1] proved that it is NP-hard to decide whether an alternative is contained in some inclusion-minimal unidirectional (i.e., either upward or downward) covering set. For both problems, we raise this lower bound to the Θ_2^p level of the polynomial hierarchy and provide a Σ_2^p upper bound. Relatedly, we show that a variety of other natural problems regarding minimal or minimum-size unidirectional covering sets are hard or complete for either of NP, coNP, and Θ_2^p. An important consequence of our results is that neither minimal upward nor minimal downward covering sets (even when guaranteed to exist) can be computed in polynomial time unless P = NP. This sharply contrasts with Brandt and Fischer's result that minimal bidirectional covering sets are polynomial-time computable.

1. Introduction

A common thread in the social sciences is to identify sets of alternatives that satisfy certain notions of stability according to some binary dominance relation. Applications range from cooperative to non-cooperative game theory, from social choice theory to argumentation theory, and from multi-criteria decision analysis to sports tournaments (see, e.g., [3, 1] and the references therein).

In settings of social choice, the most common dominance relation is the pairwise majority relation, where an alternative x is said to dominate another alternative y if the number of individuals preferring x to y exceeds the number of individuals preferring y to x. McGarvey [4] proved that *every* asymmetric dominance relation can be realized via a particular preference

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profile, even if the individual preferences are linear. The dominance graph shown in Figure 1 may for example result from the individual preferences of six voters given in the following table where each column represents a number of voters with preferences given in decreasing order. For example, the first column represents two voters who rank the alternatives in alphabetical order.

2	1	1	1	1
a	d	c	b	d
b	a	d	c	a
c	b	b	d	c
d	c	a	a	b

A well-known paradox due to the Marquis de Condorcet [5] says that the majority relation may contain cycles and thus does not always admit maximal elements, even if all of the underlying individual preferences do. This means that the concept of maximality is rendered useless in most cases. For this reason, various alternative *solution concepts* that can be used in place of maximality for nontransitive relations (see, e.g., [3]) have been proposed. In particular, concepts based on *covering relations*—transitive subrelations of the dominance relation at hand—have turned out to be very attractive [6, 7, 8].

In this paper, we study the computational complexity of problems related to the notions of upward and downward covering sets in dominance graphs. An alternative x is said to *upward cover* another alternative y if x dominates y and every alternative dominating x also dominates y. The intuition is that x "strongly" dominates y in the sense that there is no alternative that dominates x but not y. Similarly, an alternative x is said to *downward cover* another alternative y if x dominates y and every alternative dominated by y is also dominated by x. The intuition here is that x "strongly" dominates y in the sense that there is no alternative dominated by y but not by x. A *minimal upward* or *minimal downward covering set* is defined as an inclusion-minimal set of alternatives that satisfies certain notions of internal and external stability with respect to the upward or downward covering relation [8, 1].

Recent work in computational social choice has addressed the computational complexity of most solution concepts proposed in the context of binary dominance (see, e.g., [9, 10, 11, 12, 1, 13]). In particular, Brandt and Fischer [1] have shown NP-hardness of both the problem of deciding whether an alternative is contained in some minimal upward covering set and the problem of deciding whether an alternative is contained in some minimal downward covering set. For both problems, we improve on these results by raising their NP-hardness lower bounds to the Θ_2^p level of the polynomial hierarchy, and we provide an upper bound of Σ_2^p. Moreover, we will analyze the complexity of a variety of other problems related to minimal and minimum-size upward and downward covering sets that have not been studied before. In particular, we provide hardness and completeness results for the complexity classes NP, coNP, and Θ_2^p. Remarkably, these new results imply that neither minimal upward covering sets nor minimal downward covering sets (even when guaranteed to exist) can be found in polynomial time unless P = NP. This sharply contrasts with Brandt and Fischer's result that minimal *bidirectional* covering sets are polynomial-time computable [1]. Note that, notwithstanding the hardness of computing minimal upward covering sets, the decision version of this search problem is trivially in P: Every dominance graph always contains a minimal upward covering set.

Our Θ_2^p-hardness results apply Wagner's method [14] that was useful also in other contexts (see, e.g., [14, 15, 16, 17, 18]). To the best of our knowledge, our constructions for the first time apply his method to problems defined in terms of minimality rather than minimum size of a solution.

2. Definitions and Notation

In this section, we define the necessary concepts from social choice theory and complexity theory.

Definition 2.1 (Covering Relations). *Let A be a finite set of alternatives, let $B \subseteq A$, and let $\succ \subseteq A \times A$ be a dominance relation on A, i.e., $\succ$ is asymmetric and irreflexive.*[1] *A dominance relation $\succ$ on a set A of alternatives can be conveniently represented as a* dominance graph*, denoted by $(A, \succ)$, whose vertices are the alternatives from A, and for each $x, y \in A$ there is a directed edge from x to y if and only if $x \succ y$.*

For any two alternatives x and y in B, define the following covering relations (see, e.g., [6, 7, 19]):

- x upward covers y in B*, denoted by $x\, C_u^B\, y$, if $x \succ y$ and for all $z \in B$, $z \succ x$ implies $z \succ y$, and*
- x downward covers y in B*, denoted by $x\, C_d^B\, y$, if $x \succ y$ and for all $z \in B$, $y \succ z$ implies $x \succ z$.*

When clear from the context, we omit mentioning "in B" explicitly and simply write $x\, C_u\, y$ rather than $x\, C_u^B\, y$, and $x\, C_d\, y$ rather than $x\, C_d^B\, y$.

Definition 2.2 (Uncovered Set). *Let A be a set of alternatives, let $B \subseteq A$ be any subset, let $\succ$ be a dominance relation on A, and let C be a covering relation on A based on $\succ$. The* uncovered set of B with respect to C *is defined as*

$$\mathrm{UC}_C(B) = \{x \in B \mid y\, C\, x \text{ for no } y \in B\}.$$

For notational convenience, let $\mathrm{UC}_x(B) = \mathrm{UC}_{C_x}(B)$ for $x \in \{u, d\}$, and we call $\mathrm{UC}_u(B)$ the upward uncovered set of B *and* $\mathrm{UC}_d(B)$ *the* downward uncovered set of B.

In the dominance graph $(A, \succ)$ in Figure 1, b upward covers c in A, and a downward covers b in A (i.e., $b\, C_u^A\, c$ and $a\, C_d^A\, b$), so $\mathrm{UC}_u(A) = \{a, b, d\}$ is the upward uncovered set and $\mathrm{UC}_d(A) = \{a, c, d\}$ is the downward uncovered set of A. For both the upward and the downward covering relation (henceforth both will be called *unidirectional covering relations*),

[1] In general, $\succ$ need not be transitive or complete. For alternatives x and y, $x \succ y$ (equivalently, $(x, y) \in \succ$) is interpreted as x being strictly preferred to y (and we say "x dominates y"), e.g., due to a strict majority of voters preferring x to y.

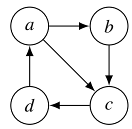


Figure 1: Dominance graph $(A, >)$.

transitivity of the relation implies nonemptiness of the corresponding uncovered set for each nonempty set of alternatives. The intuition underlying covering sets is that there should be no reason to restrict the selection by excluding some alternative from it (internal stability) and there should be an argument against each proposal to include an outside alternative into the selection (external stability).

Definition 2.3 (Minimal Covering Set). *Let A be a set of alternatives, let $>$ be a dominance relation on A, and let C be a covering relation based on $>$. A subset $B \subseteq A$ is a* covering set for A under C *if the following two properties hold:*

- Internal stability: $\mathrm{UC}_C(B) = B$.
- External stability: *For all $x \in A - B$, $x \notin \mathrm{UC}_C(B \cup \{x\})$.*

A covering set M for A under C is said to be (inclusion-)minimal *if no $M' \subset M$ is a covering set for A under C.*

Every upward uncovered set contains one or more minimal upward covering sets, whereas minimal downward covering sets may not always exist [1]. Dutta [8] proposed minimal covering sets in the context of tournaments, i.e., complete dominance relations. In tournaments, both notions of covering coincide because the set of alternatives dominating a given alternative x consists precisely of those alternatives not dominated by x. Minimal unidirectional covering sets are one of several possible generalizations to incomplete dominance relations (for more details, see [1]). Occasionally, it might be helpful to specify the dominance relation explicitly to avoid ambiguity. In such cases we refer to the dominance graph used and write, e.g., "M is an upward covering set for $(A, >)$." The unique minimal upward covering set for the dominance graph shown in Figure 1 is $\{b, d\}$, and the unique minimal downward covering set is $\{a, c, d\}$.

In addition to the (inclusion-)minimal unidirectional covering sets considered by Brandt and Fischer [1], we also consider *minimum-size* covering sets, i.e., unidirectional covering sets of smallest cardinality. For some of the computational problems we study, different complexities can be shown for the minimal and minimum-size versions of the problem (see Theorem 3.1 and Table 1). Specifically, we consider six types of computational problems, for both upward and downward covering sets, and for each both their "minimal" (prefixed by $\mathrm{MC_u}$ or $\mathrm{MC_d}$) and "minimum-size" (prefixed by $\mathrm{MSC_u}$ or $\mathrm{MSC_d}$) versions. We first define the six problem types for the case of minimal upward covering sets:

1. $\mathrm{MC_u}$-Size: Given a set A of alternatives, a dominance relation $>$ on A, and a positive integer k, does there exist some minimal upward covering set for A containing at most k alternatives?

2. $\mathrm{MC_u}$-Member: Given a set A of alternatives, a dominance relation $\succ$ on A, and a distinguished element $d \in A$, is d contained in some minimal upward covering set for A?

3. $\mathrm{MC_u}$-Member-All: Given a set A of alternatives, a dominance relation $\succ$ on A, and a distinguished element $d \in A$, is d contained in all minimal upward covering sets for A?

4. $\mathrm{MC_u}$-Unique: Given a set A of alternatives and a dominance relation $\succ$ on A, does there exist a unique minimal upward covering set for A?

5. $\mathrm{MC_u}$-Test: Given a set A of alternatives, a dominance relation $\succ$ on A, and a subset $M \subseteq A$, is M a minimal upward covering set for A?

6. $\mathrm{MC_u}$-Find: Given a set A of alternatives and a dominance relation $\succ$ on A, find a minimal upward covering set for A.

If we replace "upward" by "downward" above, we obtain the six corresponding "downward covering" versions, denoted by $\mathrm{MC_d}$-Size, $\mathrm{MC_d}$-Member, $\mathrm{MC_d}$-Member-All, $\mathrm{MC_d}$-Unique, $\mathrm{MC_d}$-Test, and $\mathrm{MC_d}$-Find. And if we replace "minimal" by "minimum-size" in the twelve problems just defined, we obtain the corresponding "minimum-size" versions: $\mathrm{MSC_u}$-Size, $\mathrm{MSC_u}$-Member, $\mathrm{MSC_u}$-Member-All, $\mathrm{MSC_u}$-Unique, $\mathrm{MSC_u}$-Test, $\mathrm{MSC_u}$-Find, $\mathrm{MSC_d}$-Size, $\mathrm{MSC_d}$-Member, $\mathrm{MSC_d}$-Member-All, $\mathrm{MSC_d}$-Unique, $\mathrm{MSC_d}$-Test, and $\mathrm{MSC_d}$-Find.

Note that the four problems $\mathrm{MC_u}$-Find, $\mathrm{MC_d}$-Find, $\mathrm{MSC_u}$-Find, and $\mathrm{MSC_d}$-Find are search problems, whereas the other twenty problems are decision problems.

We assume that the reader is familiar with the basic notions of complexity theory, such as polynomial-time many-one reducibility and the related notions of hardness and completeness, and also with standard complexity classes such as P, NP, coNP, and the polynomial hierarchy [20] (see also, e.g., the textbooks [21, 22]). In particular, coNP is the class of sets whose complements are in NP. $\Sigma_2^p = \mathrm{NP}^{\mathrm{NP}}$, the second level of the polynomial hierarchy, consists of all sets that can be solved by an NP oracle machine that has access (in the sense of a Turing reduction) to an NP oracle set such as SAT. SAT denotes the satisfiability problem of propositional logic, which is one of the standard NP-complete problems (see, e.g., Garey and Johnson [23]) and is defined as follows: Given a boolean formula in conjunctive normal form, does there exist a truth assignment to its variables that satisfies the formula?

Papadimitriou and Zachos [24] introduced the class of problems solvable in polynomial time via asking $O(\log n)$ sequential Turing queries to NP. This class is also known as the Θ_2^p level of the polynomial hierarchy (see Wagner [25]), and has been shown to coincide with the class of problems that can be decided by a P machine that accesses its NP oracle in a parallel manner (see [26, 27]). Equivalently, Θ_2^p is the closure of NP under polynomial-time truth-table reductions. It follows immediately from the definitions that $\mathrm{P} \subseteq \mathrm{NP} \cap \mathrm{coNP} \subseteq \mathrm{NP} \cup \mathrm{coNP} \subseteq \Theta_2^p \subseteq \Sigma_2^p$.

Θ_2^p captures the complexity of various optimization problems. For example, the problem of testing whether the size of a maximum clique in a given graph is an odd number, the problem of deciding whether two given graphs have minimum vertex covers of the same size, and the problem of recognizing those graphs for which certain heuristics yield good approximations for the size of a maximum independent set or for the size of a minimum vertex cover each are

Problem Type	MC_u	MSC_u	MC_d	MSC_d
SIZE	NP-complete	NP-complete	NP-complete	NP-complete
MEMBER	Θ_2^p-hard and in Σ_2^p	Θ_2^p-complete	Θ_2^p-hard and in Σ_2^p	coNP-hard and in Θ_2^p
MEMBER-ALL	coNP-complete [1]	Θ_2^p-complete	coNP-complete [1]	coNP-hard and in Θ_2^p
UNIQUE	coNP-hard and in Σ_2^p	coNP-hard and in Θ_2^p	coNP-hard and in Σ_2^p	coNP-hard and in Θ_2^p
TEST	coNP-complete	coNP-complete	coNP-complete	coNP-complete
FIND	not in polynomial time unless P = NP	not in polynomial time unless P = NP	not in polynomial time unless P = NP (follows from [1])	not in polynomial time unless P = NP

Table 1: Overview of complexity results for the various types of covering set problems. As indicated, previously known results are due to Brandt and Fischer [1]; all other results are new to this paper.

known to be complete for Θ_2^p (see [14, 16, 18]). Hemaspaandra and Wechsung [17] proved that the minimization problem for boolean formulas is Θ_2^p-hard. In the field of computational social choice, the winner problems for Dodgson [28], Young [29], and Kemeny [30] elections have been shown to be Θ_2^p-complete in the nonunique-winner model [15, 31, 32], and also in the unique-winner model [33].

3. Results and Discussion

Results. Brandt and Fischer [1] proved that it is NP-hard to decide whether a given alternative is contained in some minimal unidirectional covering set. Using the notation of this paper, their results state that the problems MC_u-MEMBER and MC_d-MEMBER are NP-hard. The question of whether these two problems are NP-complete or of higher complexity was left open in [1]. Our contribution is

1. to raise Brandt and Fischer's NP-hardness lower bounds for MC_u-MEMBER and MC_d-MEMBER to Θ_2^p-hardness and to provide (simple) Σ_2^p upper bounds for these problems, and
2. to extend the techniques we developed to apply also to the 22 other covering set problems defined in Section 2, in particular to the search problems.

Our results are stated in the following theorem.

Theorem 3.1. *The complexity of the covering set problems defined in Section 2 is as shown in Table 1.*

The detailed proofs of the single results collected in Theorem 3.1 will be presented in Section 5, and the technical constructions establishing the properties that are needed for these proofs are given in Section 4.

Discussion. We consider the problems of *finding* minimal and minimum-size upward and downward covering sets (MC_u-Find, MC_d-Find, MSC_u-Find, and MSC_d-Find) to be particularly important and natural.

Regarding upward covering sets, we stress that our result (see Theorem 5.7) that, assuming $P \neq NP$, MC_u-Find and MSC_u-Find are hard to compute does not seem to follow directly from the NP-hardness of MC_u-Member in any obvious way. The decision version of MC_u-Find is: Given a dominance graph, does it contain a minimal upward covering set? However, this question has always an affirmative answer, so the decision version of MC_u-Find is trivially in P. Note also that MC_u-Find can be reduced in a "disjunctive truth-table" fashion to the search version of MC_u-Member ("Given a dominance graph $(A, \succ)$ and an alternative $d \in A$, find some minimal upward covering set for A that contains d") by asking this oracle set about all alternatives in parallel. So MC_u-Find is no harder (with respect to disjunctive truth-table reductions) than that problem. The converse, however, is not at all obvious. Brandt and Fischer's results only imply the hardness of finding an alternative that is contained in *all* minimal upward covering sets [1]. Our reduction that raises the lower bound of MC_u-Member from NP-hardness to Θ_2^p-hardness, however, also allows us to prove that MC_u-Find and MSC_u-Find cannot be solved in polynomial time unless $P = NP$.

Regarding downward covering sets, the result that MC_d-Find cannot be computed in polynomial time unless $P = NP$ is an immediate consequence of Brandt and Fischer's result that it is NP-complete to decide whether there exists a minimal downward covering set [1, Thm. 9]. We provide an alternative proof based on our reduction showing that MC_d-Member is Θ_2^p-hard (see the proof of Theorem 5.13). In contrast to Brandt and Fischer's proof, our proof shows that MC_d-Find is hard to compute even when the existence of a (minimal) downward covering set is guaranteed. As indicated in Table 1, coNP-completeness of MC_u-Member-All and MC_d-Member-All was also shown previously by Brandt and Fischer [1].

As mentioned above, the two problems MC_u-Member and MC_d-Member were already known to be NP-hard [1] and are here shown to be even Θ_2^p-hard. One may naturally wonder whether raising their (or any problem's) lower bound from NP-hardness to Θ_2^p-hardness gives us any more insight into the problem's inherent computational complexity. After all, $P = NP$ if and only if $P = \Theta_2^p$. However, this question is a bit more subtle than that and has been discussed carefully by Hemaspaandra et al. [34]. They make the case that the answer to this question crucially depends on what one considers to be the most natural computational model. In particular, they argue that raising NP-hardness to Θ_2^p-hardness potentially (i.e., unless longstanding open problems regarding the separation of the corresponding complexity classes could be solved) is an improvement in terms of randomized polynomial time and in terms of unambiguous polynomial time [34].

4. Constructions

In this section, we provide the constructions that will be used in Section 5 to obtain the new complexity results for the problems defined in Section 2.

4.1. Minimal and Minimum-Size Upward Covering Sets

We start by giving the constructions that will be used for establishing results on the minimal and minimum-size upward covering set problems. Brandt and Fischer [1] proved the

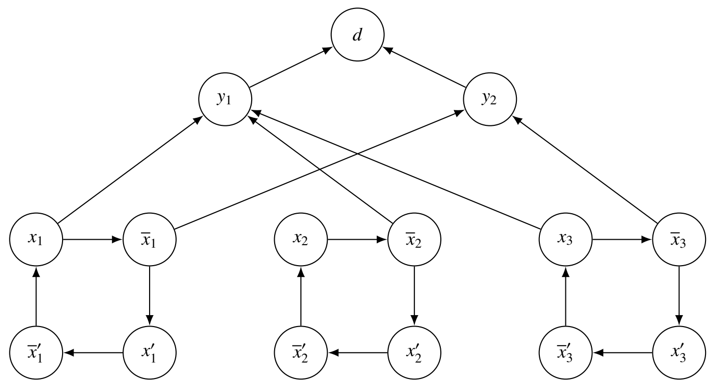


Figure 2: Dominance graph for Theorem 4.1, example for the formula $(v_1 \vee \neg v_2 \vee v_3) \wedge (\neg v_1 \vee \neg v_3)$.

following result. Since we need their reduction in Construction 4.7 and Section 5, we give a proof sketch for Theorem 4.1.

Theorem 4.1 (Brandt and Fischer [1]). *Deciding whether a designated alternative is contained in some minimal upward covering set for a given dominance graph is* NP-*hard. That is,* MC$_u$-Member *is* NP-*hard.*

Proof Sketch. NP-hardness is shown by a reduction from SAT. Given a boolean formula in conjunctive normal form, $\varphi(v_1, v_2, \ldots, v_n) = c_1 \wedge c_2 \wedge \cdots \wedge c_r$, over the set $V = \{v_1, v_2, \ldots, v_n\}$ of variables, construct an instance $(A, \succ, d)$ of MC$_u$-Member as follows. The set of alternatives is

$$A = \{x_i, \overline{x}_i, x_i', \overline{x}_i' \mid v_i \in V\} \cup \{y_j \mid c_j \text{ is a clause in } \varphi\} \cup \{d\},$$

where d is the distinguished alternative whose membership in some minimal upward covering set for A is to be decided, and the dominance relation $\succ$ is defined by:

- For each i, $1 \leq i \leq n$, there is a cycle $x_i \succ \overline{x}_i \succ x_i' \succ \overline{x}_i' \succ x_i$;
- if variable v_i occurs in clause c_j as a positive literal, then $x_i \succ y_j$;
- if variable v_i occurs in clause c_j as a negative literal, then $\overline{x}_i \succ y_j$; and
- for each j, $1 \leq j \leq r$, we have $y_j \succ d$.

As an example of this reduction, Figure 2 shows the dominance graph resulting from the formula

$$(v_1 \vee \neg v_2 \vee v_3) \wedge (\neg v_1 \vee \neg v_3),$$

which is satisfiable, for example via the truth assignment that sets each of v_1, v_2, and v_3 to false. Note that in this case the set $\{\overline{x}_1, \overline{x}_1', \overline{x}_2, \overline{x}_2', \overline{x}_3, \overline{x}_3'\} \cup \{d\}$ is a minimal upward covering set for A corresponding to the satisfying assignment, so there indeed exists a minimal upward covering set for A that contains the designated alternative d. In general, Brandt and Fischer [1] proved that there exists a satisfying assignment for φ if and only if d is contained in some minimal upward covering set for A. ❑

As we will use this reduction to prove results for both $\mathrm{MC_u}$-Member and some of the other problems stated in Section 2, we now analyze the minimal and minimum-size upward covering sets of the dominance graph constructed in the proof sketch of Theorem 4.1. Brandt and Fischer [1] showed that each minimal upward covering set for A contains exactly two of the four alternatives corresponding to any of the variables, i.e., either x_i and x_i', or $\overline{x_i}$ and $\overline{x_i}'$, $1 \leq i \leq n$. We now assume that if φ is not satisfiable then for each truth assignment to the variables of φ, at least two clauses are unsatisfied (which can be ensured, if needed, by adding two dummy variables). It is easy to see that every minimal upward covering set for A not containing alternative d must consist of at least $2n+2$ alternatives where $2n$ alternatives are from the variables and at least two, from the unsatisfied clauses. And every minimal upward covering set for A containing d consists of exactly $2n+1$ alternatives, where again $2n$ alternatives are from the variables, none from the clauses and alternative d. Thus, φ is satisfiable if and only if every minimum-size upward covering set consists of $2n+1$ alternatives. These minimum-size upward covering sets always include alternative d.

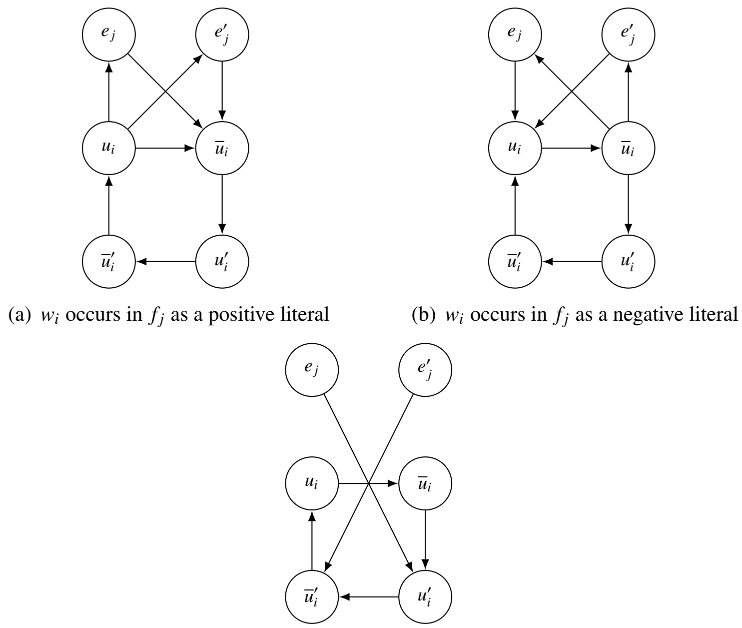
We now provide another construction that transforms a given boolean formula into a dominance graph with quite different properties.

Construction 4.2 (for coNP-hardness of upward covering set problems). *Given a boolean formula in conjunctive normal form, $\varphi(w_1, w_2, \ldots, w_k) = f_1 \wedge f_2 \wedge \cdots \wedge f_\ell$, over the set $W = \{w_1, w_2, \ldots, w_k\}$ of variables, we construct a set of alternatives A and a dominance relation $\succ$ on A. Without loss of generality, we may assume that if φ is satisfiable then it has at least two satisfying assignments. This can be ensured, if neeeded, by adding dummy variables.*

The set of alternatives is $A = \{u_i, \overline{u}_i, u_i', \overline{u}_i' \mid w_i \in W\} \cup \{e_j, e_j' \mid f_j \text{ is a clause in } \varphi\} \cup \{a_1, a_2, a_3\}$, and the dominance relation $\succ$ is defined by:

- *For each i, $1 \leq i \leq k$, there is a cycle $u_i \succ \overline{u}_i \succ u_i' \succ \overline{u}_i' \succ u_i$;*
- *if variable w_i occurs in clause f_j as a positive literal, then $u_i \succ e_j$, $u_i \succ e_j'$, $e_j \succ \overline{u}_i$, and $e_j' \succ \overline{u}_i$;*
- *if variable w_i occurs in clause f_j as a negative literal, then $\overline{u}_i \succ e_j$, $\overline{u}_i \succ e_j'$, $e_j \succ u_i$, and $e_j' \succ u_i$;*
- *if variable w_i does not occur in clause f_j, then $e_j \succ u_i'$ and $e_j' \succ \overline{u}_i'$;*
- *for each j, $1 \leq j \leq \ell$, we have $a_1 \succ e_j$ and $a_1 \succ e_j'$; and*
- *there is a cycle $a_1 \succ a_2 \succ a_3 \succ a_1$.*

Figure 3 shows some parts of the dominance graph that results from the given boolean formula φ. In particular, Figure 3(a) shows that part of this graph that corresponds to some variable w_i occurring in clause f_j as a positive literal; Figure 3(b) shows that part of this graph that corresponds to some variable w_i occurring in clause f_j as a negative literal; and Figure 3(c) shows that part of this graph that corresponds to some variable w_i not occurring in clause f_j.



(a) w_i occurs in f_j as a positive literal

(b) w_i occurs in f_j as a negative literal

(c) w_i does not occur in f_j

Figure 3: Parts of the dominance graph defined in Construction 4.2.

As a more complete example, Figure 4 shows the entire dominance graph that corresponds to the concrete formula $(\neg w_1 \vee w_2) \wedge (w_1 \vee \neg w_3)$, which can be satisfied by setting, for example, each of w_1, w_2, and w_3 to true. A minimal upward covering set for A corresponding to this assignment is $M = \{u_1, u'_1, u_2, u'_2, u_3, u'_3, a_1, a_2, a_3\}$. Note that neither e_1 nor e_2 occurs in M, and none of them occurs in any other minimal upward covering set for A either. For alternative e_1 in the example shown in Figure 4, this can be seen as follows. If there were a minimal upward covering set M' for A containing e_1 (and thus also e'_1, since they both are dominated by the same alternatives) then neither $\overline{u}_1$ nor u_2 (which dominate e_1) must upward cover e_1 in M', so all alternatives corresponding to the variables w_1 and w_2 (i.e., $\{u_i, \overline{u}_i, u'_i, \overline{u}'_i \mid i \in \{1, 2\}\}$) would also have to be contained in M'. Due to $e_1 > u'_3$ and $e'_1 > \overline{u}'_3$, all alternatives corresponding to w_3 (i.e., $\{u_3, \overline{u}_3, u'_3, \overline{u}'_3\}$) are in M' as well. Note that, e_2 and e'_2 are no longer upward covered and must also be in M'. The alternatives a_1, a_2, and a_3 are contained in every minimal upward covering set for A. But then M' is not minimal because the upward covering set M, which corresponds to the satisfying assignment stated above, is a strict subset of M'. Hence, e_1 cannot be contained in any minimal upward covering set for A.

We now show some properties of the dominance graph created by Construction 4.2 in general. We will need these properties for the proofs in Section 5. The first property, stated in Claim 4.3, has already been seen in the example above.

Claim 4.3. *Consider the dominance graph $(A, >)$ created by Construction 4.2, and fix any j, $1 \leq j \leq \ell$. For each minimal upward covering set M for A, if M contains the alternative e_j*

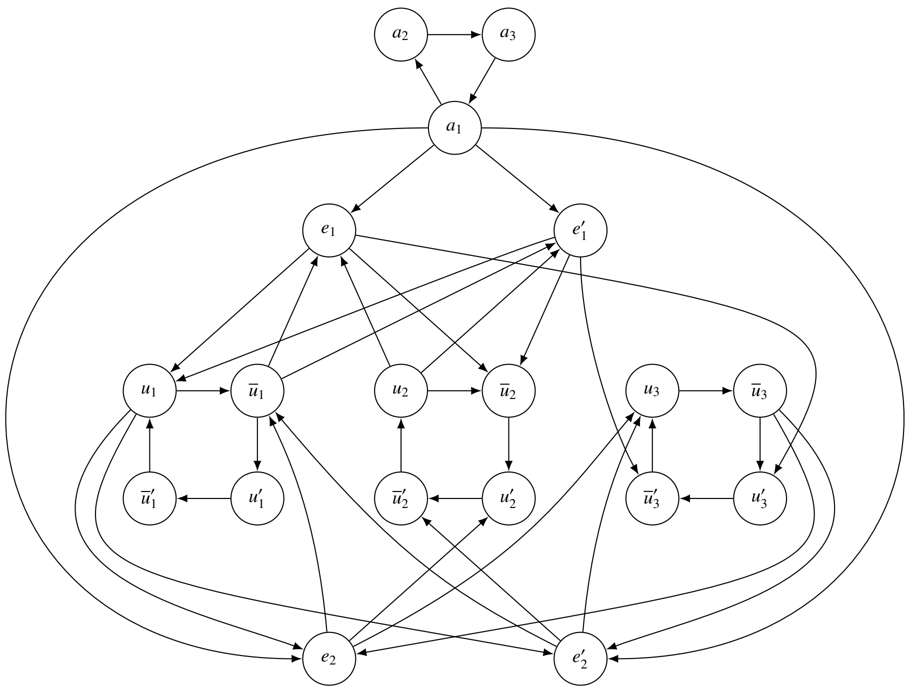


Figure 4: Dominance graph from Construction 4.2, example for the formula $(\neg w_1 \vee w_2) \wedge (w_1 \vee \neg w_3)$.

then all other alternatives are contained in M as well (i.e., $A = M$).

Proof. To simplify notation, we prove the claim only for the case of $j = 1$. However, since there is nothing special about e_1 in our argument, the same property can be shown by an analogous argument for each j, $1 \leq j \leq \ell$.

Let M be any minimal upward covering set for A, and suppose that $e_1 \in M$. First note that the dominators of e_1 and e'_1 are always the same (albeit e_1 and e'_1 may dominate different alternatives). Thus, for each minimal upward covering set, either both e_1 and e'_1 are contained in it, or they both are not. Thus, since $e_1 \in M$, we have $e'_1 \in M$ as well.

Since the alternatives a_1, a_2, and a_3 form an undominated three-cycle, they each are contained in every minimal upward covering set for A. In particular, $\{a_1, a_2, a_3\} \subseteq M$. Furthermore, no alternative e_j or e'_j, $1 \leq j \leq \ell$, can upward cover any other alternative in M, because $a_1 \in M$ and a_1 dominates e_j and e'_j but none of the alternatives that are dominated by either e_j or e'_j. In particular, no alternative in any of the k four-cycles $u_i \succ \overline{u}_i \succ u'_i \succ \overline{u}'_i \succ u_i$ can be upward covered by any alternative e_j or e'_j, and so they each must be upward covered within their cycle. For each of these cycles, every minimal upward covering set for A must contain at least one of the sets $\{u_i, u'_i\}$ and $\{\overline{u}_i, \overline{u}'_i\}$, since at least one is needed to upward cover the other one. [2]

Since $e_1 \in M$ and by internal stability, we have that no alternative from M upward covers e_1. In addition to a_1, the alternatives dominating e_1 are u_i (for each i such that w_i occurs as a positive literal in f_1) and $\overline{u}_i$ (for each i such that w_i occurs as a negative literal in f_1).

First assume that, for some i, w_i occurs as a positive literal in f_1. Suppose that $\{u_i, u_i'\} \subseteq M$. If $\overline{u}_i' \notin M$ then e_1 would be upward covered by u_i, which is impossible. Thus $\overline{u}_i' \in M$. But then $\overline{u}_i \in M$ as well, since u_i, the only alternative that could upward cover $\overline{u}_i$, is itself dominated by $\overline{u}_i'$. For the latter argument, recall that $\overline{u}_i$ cannot be upward covered by any e_j or e_j'. Thus, we have shown that $\{u_i, u_i'\} \subseteq M$ implies $\{\overline{u}_i, \overline{u}_i'\} \subseteq M$. Conversely, suppose that $\{\overline{u}_i, \overline{u}_i'\} \subseteq M$. Then u_i' is no longer upward covered by $\overline{u}_i$ and hence must be in M as well. The same holds for the alternative u_i, so $\{\overline{u}_i, \overline{u}_i'\} \subseteq M$ implies $\{u_i, u_i'\} \subseteq M$. Summing up, if $e_1 \in M$ then $\{u_i, u_i', \overline{u}_i, \overline{u}_i'\} \subseteq M$ for each i such that w_i occurs as a positive literal in f_1.

By symmetry of the construction, an analogous argument shows that if $e_1 \in M$ then $\{u_i, u_i', \overline{u}_i, \overline{u}_i'\} \subseteq M$ for each i such that w_i occurs as a negative literal in f_1.

Now, consider any i such that w_i does not occur in f_1. We have $e_1 > u_i'$ and $e_1' > \overline{u}_i'$. Again, none of the sets $\{u_i, u_i'\}$ and $\{\overline{u}_i, \overline{u}_i'\}$ alone can be contained in M, since otherwise either u_i or $\overline{u}_i'$ would remain upward uncovered. Thus, $e_1 \in M$ again implies that $\{u_i, u_i', \overline{u}_i, \overline{u}_i'\} \subseteq M$.

Now it is easy to see that, since $\bigcup_{1 \leq i \leq k} \{u_i, u_i', \overline{u}_i, \overline{u}_i'\} \subseteq M$ and since a_1 cannot upward cover any of the e_j and e_j', $1 \leq j \leq \ell$, external stability of M enforces that $\bigcup_{1 < j \leq \ell} \{e_j, e_j'\} \subseteq M$. Summing up, we have shown that if e_1 is contained in any minimal upward covering set M for A, then $M = A$. ❑

Claim 4.4. *Consider Construction 4.2. The boolean formula φ is satisfiable if and only if there is no minimal upward covering set for A that contains any of the e_j, $1 \leq j \leq \ell$.*

Proof. It is enough to prove the claim for the case $j = 1$, since the other cases can be proven analogously.

From left to right, suppose there is a satisfying assignment $\alpha : W \to \{0, 1\}$ for φ. Define the set

$$B_\alpha = \{a_1, a_2, a_3\} \cup \{u_i, u_i' \mid \alpha(w_i) = 1\} \cup \{\overline{u}_i, \overline{u}_i' \mid \alpha(w_i) = 0\}.$$

Since every upward covering set for A must contain $\{a_1, a_2, a_3\}$ and at least one of the sets $\{u_i, u_i'\}$ and $\{\overline{u}_i, \overline{u}_i'\}$ for each i, $1 \leq i \leq k$, B_α is a (minimal) upward covering set for A. Let M be an arbitrary minimal upward covering set for A. By Claim 4.3, if e_1 were contained in M, we would have $M = A$. But since $B_\alpha \subset A = M$, this contradicts the minimality of M. Thus $e_1 \notin M$.

From right to left, let M be an arbitrary minimal upward covering set for A and suppose $e_1 \notin M$. By Claim 4.3, if any of the e_j, $1 < j \leq \ell$, were contained in M, it would follow that $e_1 \in M$, a contradiction. Thus, $\{e_j \mid 1 \leq j \leq \ell\} \cap M = \emptyset$. It follows that each e_j must be

[2] The argument is analogous to that used in the construction of Brandt and Fischer [1] in their proof of Theorem 4.1. However, in contrast with their construction, which implies that *either* $\{x_i, x_i'\}$ *or* $\{\overline{x}_i, \overline{x}_i'\}$, $1 \leq i \leq n$, *but not both*, must be contained in any minimal upward covering set for A (see Figure 2), our construction also allows for both $\{u_i, u_i'\}$ and $\{\overline{u}_i, \overline{u}_i'\}$ being contained in some minimal upward covering set for A. Informally stated, the reason is that, unlike the four-cycles in Figure 2, our four-cycles $u_i > \overline{u}_i > u_i' > \overline{u}_i' > u_i$ also have incoming edges.

upward covered by some alternative in M. It is easy to see that for each j, $1 \leq j \leq \ell$, and for each i, $1 \leq i \leq k$, e_j is upward covered in $M \cup \{e_j\} \supseteq \{u_i, u'_i\}$ if w_i occurs in f_j as a positive literal, and e_j is upward covered in $M \cup \{e_j\} \supseteq \{\overline{u}_i, \overline{u}'_i\}$ if w_i occurs in e_j as a negative literal. It can never be the case that all four alternatives, $\{u_i, u'_i, \overline{u}_i, \overline{u}'_i\}$, are contained in M, because then either e_j would no longer be upward covered in M or the resulting set M was not minimal. Now, M induces a satisfying assignment for φ by setting, for each i, $1 \leq i \leq k$, $\alpha(w_i) = 1$ if $u_i \in M$, and $\alpha(w_i) = 0$ if $\overline{u}_i \in M$. ❑

Note that in Construction 4.2 every minimal upward covering set for A obtained from any satisfying assignment for φ contains exactly $2k+3$ alternatives, and there is no minimal upward covering set of smaller size for A when φ is unsatisfiable.

Claim 4.5. *Consider Construction 4.2. The boolean formula φ is not satisfiable if and only if there is a unique minimal upward covering set for A.*

Proof. Recall that we assumed in Construction 4.2 that if φ is satisfiable then it has at least two satisfying assignments.

From left to right, suppose there is no satisfying assignment for φ. By Claim 4.4, there must be a minimal upward covering set for A containing one of the e_j, $1 \leq j \leq \ell$, and by Claim 4.3 this minimal upward covering set for A must contain all alternatives. By reason of minimality, there cannot be another minimal upward covering set for A.

From right to left, suppose there is a unique minimal upward covering set for A. Due to our assumption that if φ is satisfiable then there are at least two satisfying assignments, φ cannot be satisfiable, since if it were, there would be two distinct minimal upward covering sets corresponding to these assignments (as argued in the proof of Claim 4.4). ❑

Wagner provided a sufficient condition for proving Θ_2^p-hardness that was useful in various other contexts (see, e.g., [14, 15, 16, 17, 18]) and is stated here as Lemma 4.6.

Lemma 4.6 (Wagner [14]). *Let S be some* NP*-complete problem and let T be any set. If there exists a polynomial-time computable function f such that, for all $m \geq 1$ and all strings $x_1, x_2, \ldots, x_{2m}$ satisfying that if $x_j \in S$ then $x_{j-1} \in S$, $1 < j \leq 2m$, we have*

$$\|\{i \mid x_i \in S\}\| \text{ is odd } \iff f(x_1, x_2, \ldots, x_{2m}) \in T, \tag{4.1}$$

then T is Θ_2^p-hard.

We will apply Lemma 4.6 as well. In contrast with those previous results, however, one subtlety in our construction is due to the fact that we consider not only minimum-size but also (inclusion-)minimal covering sets. To the best of our knowledge, our Construction 4.7 and Construction 4.17, which will be presented later, for the first time apply Wagner's technique [14] to problems defined in terms of minimality/maximality rather than minimum/maximum size of a solution:[3] In Construction 4.7 below, we define a dominance graph

[3]For example, recall Wagner's Θ_2^p-completeness result for testing whether the size of a maximum clique in a given graph is an odd number [14]. One key ingredient in his proof is to define an associative operation on graphs, $\bowtie$, such

based on Construction 4.2 and the construction presented in the proof sketch of Theorem 4.1 such that Lemma 4.6 can be applied to prove MC$_u$-MEMBER Θ_2^p-hard (see Theorem 5.2), making use of the properties established in Claims 4.3, 4.4, and 4.5.

Construction 4.7 (for applying Lemma 4.6 to upward covering set problems). *We apply Wagner's Lemma with the* NP*-complete problem* $S = \mathrm{SAT}$ *and construct a dominance graph. Fix an arbitrary* $m \geq 1$ *and let* $\varphi_1, \varphi_2, \ldots, \varphi_{2m}$ *be* $2m$ *boolean formulas in conjunctive normal form such that if* φ_j *is satisfiable then so is* φ_{j-1}*, for each* j*,* $1 < j \leq 2m$*. Without loss of generality, we assume that for each* j*,* $1 \leq j \leq 2m$*, the first variable of* φ_j *does not occur in all clauses of* φ_j*. Furthermore, we require* φ_j *to have at least two unsatisfied clauses if* φ_j *is not satisfiable, and to have at least two satisfying assignments if* φ_j *is satisfiable. It is easy to see that if* φ_j *does not have this property, it can be transformed into a formula that does have it, without affecting the satisfiability of the formula.*

We now define a polynomial-time computable function f*, which maps the given* $2m$ *boolean formulas to a dominance graph* $(A, \succ)$ *with useful properties for upward covering sets. Define* $A = \bigcup_{j=1}^{2m} A_j$ *and the dominance relation* $\succ$ *on* A *by*

$$\left(\bigcup_{j=1}^{2m} \succ_j\right) \cup \left(\bigcup_{i=1}^{m} \left\{(u'_{1,2i}, d_{2i-1}), (\overline{u}'_{1,2i}, d_{2i-1})\right\}\right) \cup \left(\bigcup_{i=2}^{m} \{(d_{2i-1}, z) \mid z \in A_{2i-2}\}\right),$$

where we use the following notation:

1. *For each* i*,* $1 \leq i \leq m$*, let* $(A_{2i-1}, \succ_{2i-1})$ *be the dominance graph that results from the formula* φ_{2i-1} *according to Brandt and Fischer's construction [1] given in the proof sketch of Theorem 4.1. We use the same names for the alternatives in* A_{2i-1} *as in that proof sketch, except that we attach the subscript* $2i-1$*. For example, alternative* d *from the proof sketch of Theorem 4.1 now becomes* d_{2i-1}*,* x_1 *becomes* $x_{1,2i-1}$*,* y_1 *becomes* $y_{1,2i-1}$*, and so on.*

2. *For each* i*,* $1 \leq i \leq m$*, let* $(A_{2i}, \succ_{2i})$ *be the dominance graph that results from the formula* φ_{2i} *according to Construction 4.2. We use the same names for the alternatives in* A_{2i} *as in that construction, except that we attach the subscript* $2i$*. For example, alternative* a_1 *from Construction 4.2 now becomes* $a_{1,2i}$*,* e_1 *becomes* $e_{1,2i}$*,* u_1 *becomes* $u_{1,2i}$*, and so on.*

3. *For each* i*,* $1 \leq i \leq m$*, connect the dominance graphs* $(A_{2i-1}, \succ_{2i-1})$ *and* $(A_{2i}, \succ_{2i})$ *as follows. Let* $u_{1,2i}, \overline{u}_{1,2i}, u'_{1,2i}, \overline{u}'_{1,2i} \in A_{2i}$ *be the four alternatives in the cycle corresponding to the first variable of* φ_{2i}*. Then both* $u'_{1,2i}$ *and* $\overline{u}'_{1,2i}$ *dominate* d_{2i-1}*. The resulting dominance graph is denoted by* $(B_i, \succ_i^B)$*.*

4. *Connect the* m *dominance graphs* $(B_i, \succ_i^B)$*,* $1 \leq i \leq m$*, as follows: For each* i*,* $2 \leq i \leq m$*,* d_{2i-1} *dominates all alternatives in* A_{2i-2}*.*

---

that for any two graphs G and H, the size of a maximum clique in $G \bowtie H$ equals the sum of the sizes of a maximum clique in G and one in H. This operation is quite simple: Just connect every vertex of G with every vertex of H. In contrast, since minimality for minimal upward covering sets is defined in terms of set inclusion, it is not at all obvious how to define a similarly simple operation on dominance graphs such that the minimal upward covering sets in the given graphs are related to the minimal upward covering sets in the connected graph in a similarly useful way.

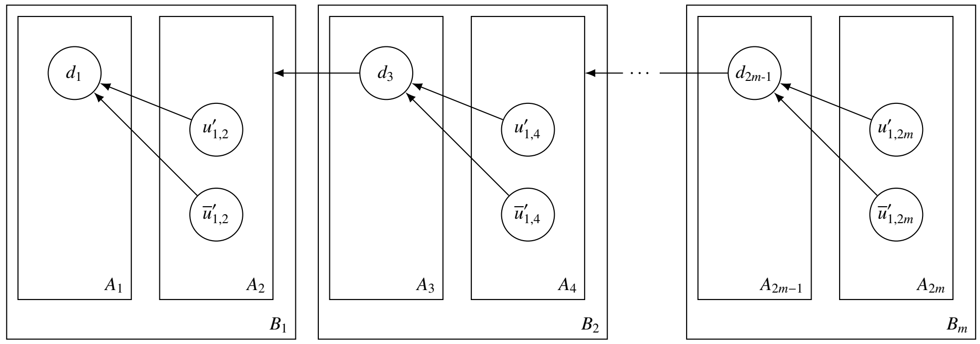


Figure 5: Dominance graph from Construction 4.7. Most alternatives, and all edges between pairs of alternatives, in A_j, $1 \leq j \leq 2m$, have been omitted. All edges between alternatives in A_i and alternatives in A_j for $i \neq j$ are shown. An edge incident to a set of alternatives represents an edge incident to *each* alternative in the set.

The dominance graph $(A, \succ)$ is sketched in Figure 5. Clearly, $(A, \succ)$ is computable in polynomial time.

Before we use this construction to obtain Θ_2^p-hardness results for some of our upward covering set problems in Section 5, we again show some useful properties of the dominance graph constructed, and we first consider the dominance graph $(B_i, \succ_i^B)$ (see Step 3 in Construction 4.7) separately,[4] for any fixed i with $1 \leq i \leq m$. Doing so will simplify our argument for the whole dominance graph $(A, \succ)$. Recall that $(B_i, \succ_i^B)$ results from the formulas φ_{2i-1} and φ_{2i}.

Claim 4.8. *Consider Construction 4.7. Alternative d_{2i-1} is contained in some minimal upward covering set for $(B_i, \succ_i^B)$ if and only if φ_{2i-1} is satisfiable and φ_{2i} is not satisfiable.*

Proof. Distinguish the following three cases.

Case 1: $\varphi_{2i-1} \in$ SAT and $\varphi_{2i} \in$ SAT. Since φ_{2i} is satisfiable, it follows from the proof of Claim 4.4 that for each minimal upward covering set M for $(B_i, \succ_i^B)$, either $\{u_{1,2i}, u'_{1,2i}\} \subseteq M$ or $\{\overline{u}_{1,2i}, \overline{u}'_{1,2i}\} \subseteq M$, but not both, and that none of the $e_{j,2i}$ and $e'_{j,2i}$ is in M. If $\overline{u}'_{1,2i} \in M$ but $u'_{1,2i} \notin M$, then $d_{2i-1} \notin \mathrm{UC}_u(M)$, since $\overline{u}'_{1,2i}$ upward covers d_{2i-1} within M. If $u'_{1,2i} \in M$ but $\overline{u}_{1,2i} \notin M$, then $d_{2i-1} \notin \mathrm{UC}_u(M)$, since $u'_{1,2i}$ upward covers d_{2i-1} within M. Hence, by internal stability, d_{2i-1} is not contained in M.

Case 2: $\varphi_{2i-1} \notin$ SAT and $\varphi_{2i} \notin$ SAT. Since $\varphi_{2i-1} \notin$ SAT, it follows from the proof of Theorem 4.1 that each minimal upward covering set M for $(B_i, \succ_i^B)$ contains at least one alternative $y_{j,2i-1}$ (corresponding to some clause of φ_{2i-1}) that upward covers d_{2i-1}. Thus d_{2i-1} cannot be in M, again by internal stability.

[4]Our argument about $(B_i, \succ_i^B)$ can be used to show, in effect, DP-hardness of upward covering set problems, where DP is the class of differences of any two NP sets [35]. Note that DP is the second level of the boolean hierarchy over NP (see Cai et al. [36, 37]), and it holds that NP $\cup$ coNP $\subseteq$ DP $\subseteq \Theta_2^p$. Wagner [14] proved appropriate analogs of Lemma 4.6 for each level of the boolean hierarchy. In particular, the analogous criterion for DP-hardness is obtained by using the wording of Lemma 4.6 except with the value of $m = 1$ being fixed.

Case 3: $\varphi_{2i-1} \in \mathrm{SAT}$ and $\varphi_{2i} \notin \mathrm{SAT}$. Since $\varphi_{2i-1} \in \mathrm{SAT}$, it follows from the proof of Theorem 4.1 that there exists a minimal upward covering set M' for $(A_{2i-1}, \succ_{2i-1})$ that corresponds to a satisfying truth assignment for φ_{2i-1}. In particular, none of the $y_{j,2i-1}$ is in M'. On the other hand, since $\varphi_{2i} \notin \mathrm{SAT}$, it follows from Claim 4.5 that A_{2i} is the only minimal upward covering set for $(A_{2i}, \succ_{2i})$. Define $M = M' \cup A_{2i}$. It is easy to see that M is a minimal upward covering set for $(B_i, \succ_i^B)$, since the only edges between A_{2i-1} and A_{2i} are those from $\overline{u}'_{1,2i}$ and $u'_{1,2i}$ to d_{2i-1}, and both $\overline{u}'_{1,2i}$ and $u'_{1,2i}$ are dominated by elements in M not dominating d_{2i-1}.

We now show that $d_{2i-1} \in M$. Note that $\overline{u}'_{1,2i}$, $u'_{1,2i}$, and the $y_{j,2i-1}$ are the only alternatives in B_i that dominate d_{2i-1}. Since none of the $y_{j,2i-1}$ is in M, they do not upward cover d_{2i-1}. Also, $u'_{1,2i}$ doesn't upward cover d_{2i-1}, since $\overline{u}_{1,2i} \in M$ and $\overline{u}_{1,2i}$ dominates $u'_{1,2i}$ but not d_{2i-1}. On the other hand, by our assumption that the first variable of φ_{2i} does not occur in all clauses, there exist alternatives $e_{j,2i}$ and $e'_{j,2i}$ in M that dominate $\overline{u}'_{1,2i}$ but not d_{2i-1}, so $\overline{u}'_{1,2i}$ doesn't upward cover d_{2i-1} either. Thus $d_{2i-1} \in M$.

Note that, by our assumption on how the formulas are ordered, the fourth case (i.e., $\varphi_{2i-1} \notin \mathrm{SAT}$ and $\varphi_{2i} \in \mathrm{SAT}$) cannot occur. Thus, the proof is complete. ❑

Claim 4.9. *Consider Construction 4.7. For each i, $1 \leq i \leq m$, let M_i be a minimal upward covering set for $(B_i, \succ_i^B)$ according to the cases in the proof of Claim 4.8. Then each of the sets M_i must be contained in every minimal upward covering set for $(A, \succ)$.*

Proof. The minimal upward covering set M_m for $(B_m, \succ_m^B)$ must be contained in every minimal upward covering set for $(A, \succ)$, since no alternative in $A - B_m$ dominates any alternative in B_m. On the other hand, for each i, $1 \leq i < m$, no alternative in B_i can be upward covered by d_{2i+1} (which is the only element in $A - B_i$ that dominates any of the elements of B_i), since d_{2i+1} is dominated within every minimal upward covering set for B_{i+1} (and, in particular, within M_{i+1}). Thus, each of the sets M_i, $1 \leq i \leq m$, must be contained in every minimal upward covering set for $(A, \succ)$. ❑

Claim 4.10. *Consider Construction 4.7. It holds that*

$$\|\{i \mid \varphi_i \in \mathrm{SAT}\}\| \textit{ is odd} \iff d_1 \textit{ is contained in some minimal upward covering set } M \textit{ for } A. \tag{4.2}$$

Proof. To show (4.2) from left to right, suppose $\|\{i \mid \varphi_i \in \mathrm{SAT}\}\|$ is odd. Recall that for each j, $1 < j \leq 2m$, if φ_j is satisfiable then so is φ_{j-1}. Thus, there exists some i, $1 \leq i \leq m$, such that $\varphi_1, \ldots, \varphi_{2i-1} \in \mathrm{SAT}$ and $\varphi_{2i}, \ldots, \varphi_{2m} \notin \mathrm{SAT}$. In Case 3 in the proof of Claim 4.8 we have seen that there is some minimal upward covering set for $(B_i, \succ_i^B)$—call it M_i—that corresponds to a satisfying assignment of φ_{2i-1} and that contains all alternatives of A_{2i}. Note that, M_i contains d_{2i-1}. For each $j \neq i$, $1 \leq j \leq m$, let M_j be some minimal upward covering set for $(B_j, \succ_j^B)$ according to Case 1 (if $j < i$) and Case 2 (if $j > i$) in the proof of Claim 4.8.

In Case 1 in the proof of Claim 4.8 we have seen that d_{2i-3} is upward covered either by $\overline{u}'_{1,2i-3}$ or by $u'_{1,2i-3}$. This is no longer the case, since d_{2i-1} is in M_i and it dominates all alternatives in A_{2i-2} but not d_{2i-3}. By assumption, φ_{2i-3} is satisfiable, so there exists a minimal upward covering set, which contains d_{2i-3} as well. Thus, setting

$$M = \{d_1, d_3, \ldots, d_{2i-1}\} \cup \bigcup_{1 \le j \le m} M_j,$$

it follows that M is a minimal upward covering set for $(A, \succ)$ containing d_1.

To show (4.2) from right to left, suppose that $\|\{i \mid \varphi_i \in \mathrm{SAT}\}\|$ is even. For a contradiction, suppose that there exists some minimal upward covering set M for $(A, \succ)$ that contains d_1. If $\varphi_1 \notin \mathrm{SAT}$ then we immediately obtain a contradiction by the argument in the proof of Theorem 4.1. On the other hand, if $\varphi_1 \in \mathrm{SAT}$ then our assumption that $\|\{i \mid \varphi_i \in \mathrm{SAT}\}\|$ is even implies that $\varphi_2 \in \mathrm{SAT}$. It follows from the proof of Claim 4.3 that every minimal upward covering set for $(A, \succ)$ (thus, in particular, M) contains either $\{u_{1,2i}, u'_{1,2i}\}$ or $\{\overline{u}_{1,2i}, \overline{u}'_{1,2i}\}$, but not both, and that none of the $e_{j,2i}$ and $e'_{j,2i}$ is in M. By the argument presented in Case 3 in the proof of Claim 4.8, the only way to prevent d_1 from being upward covered by an element of M, either $u'_{1,2}$ or $\overline{u}'_{1,2}$, is to include d_3 in M as well.[5] By applying the same argument $m - 1$ times, we will eventually reach a contradiction, since $d_{2m-1} \in M$ can no longer be prevented from being upward covered by an element of M, either $u'_{1,2m}$ or $\overline{u}'_{1,2m}$. Thus, no minimal upward covering set M for $(A, \succ)$ contains d_1, which completes the proof of (4.2). ❑

Furthermore, it holds that $\|\{i \mid \varphi_i \in \mathrm{SAT}\}\|$ is odd if and only if d_1 is contained in all minimum-size upward covering sets for A. This is true since the minimal upward covering sets for A that contain d_1 are those that correspond to some satisfying assignment for all satisfiable formulas φ_i, and as we have seen in the analysis of Construction 4.2 and the proof sketch of Theorem 4.1, these are the minimum-size upward covering sets for A.

4.2. Minimal and Minimum-Size Downward Covering Sets

Turning now to the constructions used to show complexity results about minimal/minimum-size downward covering sets, we again start by giving a proof sketch of a result due to Brandt and Fischer [1], since the following constructions and proofs are based on their construction and proof.

Theorem 4.11 (Brandt and Fischer [1]). *Deciding whether a designated alternative is contained in some minimal downward covering set for a given dominance graph is* NP*-hard (i.e.,* $\mathrm{MC_d}$-Member *is* NP*-hard), even if a downward covering set is guaranteed to exist.*

Proof Sketch. NP-hardness of $\mathrm{MC_d}$-Member is again shown by a reduction from SAT. Given a boolean formula in conjunctive normal form, $\varphi(v_1, v_2, \ldots, v_n) = c_1 \wedge c_2 \wedge \cdots \wedge c_r$, over the set $V = \{v_1, v_2, \ldots, v_n\}$ of variables, construct a dominance graph $(A, \succ)$ as follows. The set of alternatives is

$$A = \{x_i, \overline{x}_i, x'_i, \overline{x}'_i, x''_1, \overline{x}''_i \mid v_i \in V\} \cup \{y_j, z_j \mid c_j \text{ is a clause in } \varphi\} \cup \{d\},$$

where the membership of alternative d in a minimal downward covering set is to be decided. The dominance relation $\succ$ is defined as follows:

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[5] This implies that d_1 is not upward covered by either $u'_{1,2}$ or $\overline{u}'_{1,2}$, since d_3 dominates them both but not d_1.

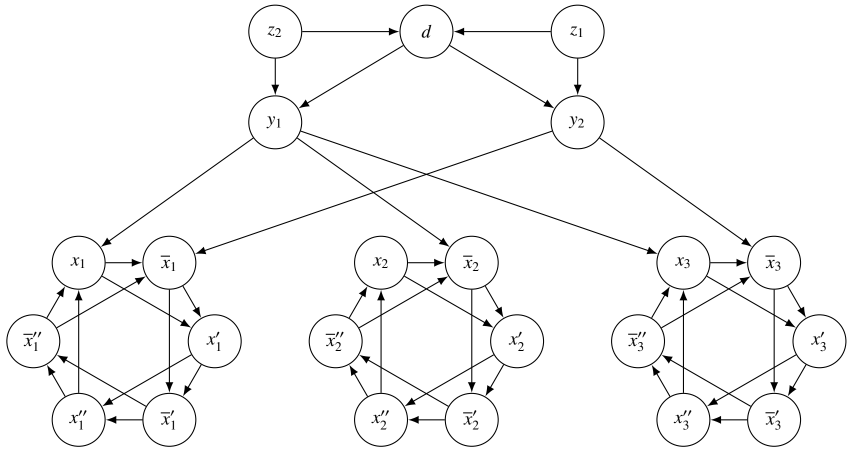

Figure 6: Dominance graph for Theorem 4.11, example for the formula $(v_1 \vee \neg v_2 \vee v_3) \wedge (\neg v_1 \vee \neg v_3)$.

- For each i, $1 \leq i \leq n$, there is a cycle $x_i \succ \overline{x}_i \succ x'_i \succ \overline{x}'_i \succ x''_i \succ \overline{x}''_i \succ x_i$ with two nested three-cycles, $x_i \succ x'_i \succ x''_i \succ x_i$ and $\overline{x}_i \succ \overline{x}'_i \succ \overline{x}''_i \succ \overline{x}_i$;
- if variable v_i occurs in clause c_j as a positive literal, then $y_j \succ x_i$;
- if variable v_i occurs in clause c_j as a negative literal, then $y_j \succ \overline{x}_i$;
- for each j, $1 \leq j \leq r$, we have $d \succ y_j$ and $z_j \succ d$; and
- for each i and j with $1 \leq i, j \leq r$ and $i \neq j$, we have $z_i \succ y_j$.

Brandt and Fischer [1] showed that there is a minimal downward covering set containing d if and only if φ is satisfiable. An example of this reduction is shown in Figure 6 for the boolean formula $(v_1 \vee \neg v_2 \vee v_3) \wedge (\neg v_1 \vee \neg v_3)$. The set $\{x_1, x'_1, x''_1, x_2, x'_2, x''_2, \overline{x}_3, \overline{x}'_3, \overline{x}''_3, y_1, y_2, z_1, z_2, d\}$ is a minimal downward covering set for the dominance graph shown in Figure 6. This set corresponds to the truth assignment that sets v_1 and v_2 to true and v_3 to false, and it contains the designated alternative d. ❑

Regarding their construction sketched above, Brandt and Fischer [1] showed that every minimal downward covering set for A must contain exactly three alternatives for every variable v_i (either x_i, x'_i, and x''_i, or $\overline{x}_i$, $\overline{x}'_i$, and $\overline{x}''_i$), and the undominated alternatives $z_1, \ldots, z_r$. Thus, each minimal downward covering set for A consists of at least $3n + r$ alternatives and induces a truth assignment α for φ. The number of alternatives contained in any minimal downward covering set for A corresponding to an assignment α is $3n + r + k$, where k is the number of clauses that are satisfied if α is an assignment not satisfying φ, and where $k = r + 1$ if α is a satisfying assignment for φ. As a consequence, minimum-size downward covering sets for A correspond to those assignments for φ that satisfy the least possible number of clauses of φ.[6]

[6]This is different from the case of minimum-size *upward* covering sets for the dominance graph constructed in the proof sketch of Theorem 4.1. The construction in the proof sketch of Theorem 4.11 cannot be used to obtain

Next, we provide a different construction to transform a given boolean formula into a dominance graph. This construction will later be merged with the construction from the proof sketch of Theorem 4.11 so as to apply Lemma 4.6 to downward covering set problems.

Construction 4.12 (for NP- and coNP-hardness of downward covering set problems). *Given a boolean formula in conjunctive normal form, $\varphi(w_1, w_2, \ldots, w_k) = f_1 \wedge f_2 \wedge \cdots \wedge f_\ell$, over the set $W = \{w_1, w_2, \ldots, w_k\}$ of variables, we construct a dominance graph $(A, \succ)$. The set of alternatives is*

$$A = A_1 \cup A_2 \cup \{\widehat{a} \mid a \in A_1 \cup A_2\} \cup \{b, c, d\}$$

with $A_1 = \{x_i, x_i', x_i'', \overline{x}_i, \overline{x}_i', \overline{x}_i'', z_i, z_i', z_i'' \mid w_i \in W\}$ and $A_2 = \{y_j \mid f_j \text{ is a clause in } \varphi\}$, and the dominance relation $\succ$ is defined by:

- *For each i, $1 \leq i \leq k$, there is, similarly to the construction in the proof of Theorem 4.11, a cycle $x_i \succ \overline{x}_i \succ x_i' \succ \overline{x}_i' \succ x_i'' \succ \overline{x}_i'' \succ x_i$ with two nested three-cycles, $x_i \succ x_i' \succ x_i'' \succ x_i$ and $\overline{x}_i \succ \overline{x}_i' \succ \overline{x}_i'' \succ \overline{x}_i$, and additionally we have $z_i' \succ z_i \succ x_i$, $z_i'' \succ z_i \succ \overline{x}_i$, $z_i' \succ x_i$, $z_i'' \succ \overline{x}_i$, and $d \succ z_i$;*
- *if variable w_i occurs in clause f_j as a positive literal, then $x_i \succ y_j$;*
- *if variable w_i occurs in clause f_j as a negative literal, then $\overline{x}_i \succ y_j$;*
- *for each $a \in A_1 \cup A_2$, we have $b \succ \widehat{a}$, $a \succ \widehat{a}$, and $\widehat{a} \succ d$;*
- *for each j, $1 \leq j \leq \ell$, we have $d \succ y_j$; and*
- *$c \succ d$.*

An example of this construction is shown in Figure 7 for the boolean formula $(\neg w_1 \vee w_2 \vee w_3) \wedge (\neg w_2 \vee \neg w_3)$, which can be satisfied by setting for example each of w_1, w_2, and w_3 to false. A minimal downward covering set corresponding to this assignment is $M = \{b, c\} \cup \{\overline{x}_i, \overline{x}_i', \overline{x}_i'', z_i', z_i'' \mid 1 \leq i \leq 3\}$. Obviously, the undominated alternatives b, c, z_i', and z_i'', $1 \leq i \leq 3$, are contained in every minimal downward covering set for the dominance graph constructed. The alternative d, however, is not contained in any minimal downward covering set for A. This can be seen as follows. If d were contained in some minimal downward covering set M' for A then none of the alternatives $\widehat{a}$ with $a \in A_1 \cup A_2$ would be downward covered. Hence, all alternatives in $A_1 \cup A_2$ would necessarily be in M', since they all dominate a different alternative in M'. But then M' is no minimal downward covering set for A, since the minimal downward covering set M for A is a strict subset of M'.

We now show some properties of Construction 4.12 in general.

Claim 4.13. *Minimal downward covering sets are guaranteed to exist for the dominance graph defined in Construction 4.12.*

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complexity results for minimum-size downward covering sets in the same way as the construction in the proof sketch of Theorem 4.1 was used to obtain complexity results for minimum-size upward covering sets.

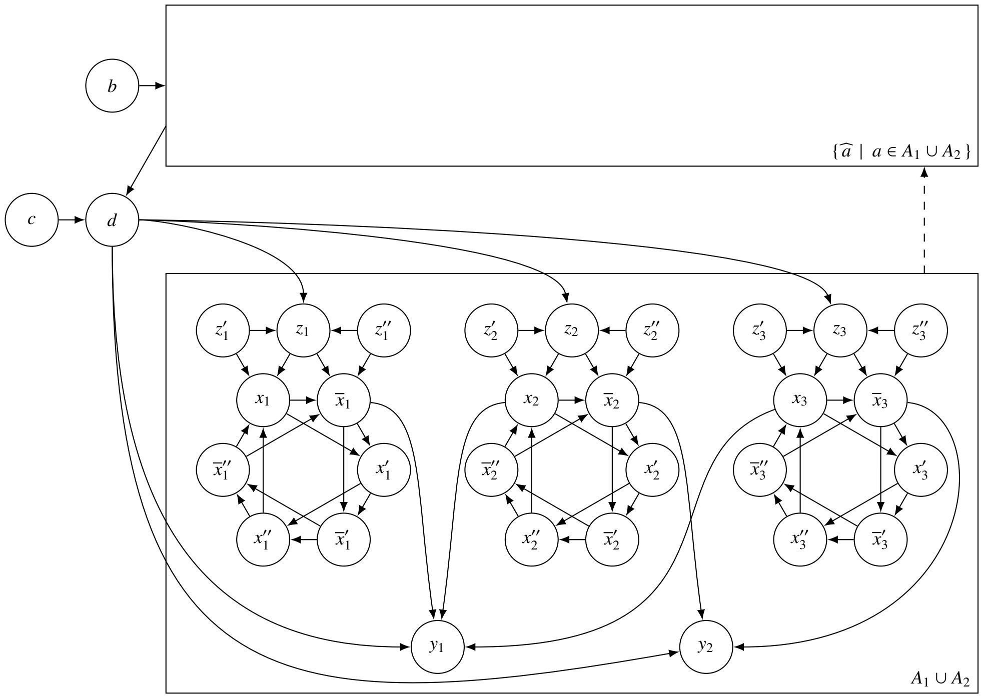

Figure 7: Dominance graph resulting from the formula $(\neg w_1 \vee w_2 \vee w_3) \wedge (\neg w_2 \vee \neg w_3)$ according to Construction 4.12. An edge incident to a set of alternatives represents an edge incident to *each* alternative in the set. The dashed edge indicates that $a \succ \widehat{a}$ for each $a \in A_1 \cup A_2$.

Proof. The set A of all alternatives is a downward covering set for itself. Hence, there always exists a minimal downward covering set for the dominance graph defined in Construction 4.12. ❑

Claim 4.14. *Consider the dominance graph $(A, \succ)$ created by Construction 4.12. For each minimal downward covering set M for A, if M contains the alternative d then all other alternatives are contained in M as well (i.e., $A = M$).*

Proof. If d is contained in some minimal downward covering set M for A, then $\{a, \widehat{a}\} \subseteq M$ for every $a \in A_1 \cup A_2$. To see this, observe that for an arbitrary $a \in A_1 \cup A_2$ there is no $a' \in A$ with $a' \succ \widehat{a}$ and $a' \succ d$ or with $a' \succ a$ and $a' \succ \widehat{a}$. Since the alternatives c and b are undominated, they are also in M, so $M = A$. ❑

Claim 4.15. *Consider Construction 4.12. The boolean formula φ is satisfiable if and only if there is no minimal downward covering set for A that contains d.*

Proof. For the direction from left to right, consider a satisfying assignment $\alpha : W \to \{0,1\}$ for φ, and define the set

$$B_\alpha = \{b, c\} \cup \{x_i, x_i', x_i'' \mid \alpha(w_i) = 1\} \cup \{\overline{x}_i, \overline{x}_i', \overline{x}_i'' \mid \alpha(w_i) = 0\} \cup \{z_i', z_i'' \mid 1 \le i \le k\}.$$

It is not hard to verify that B_α is a minimal downward covering set for A. Thus, there exists a minimal downward covering set for A that does not contain d. If there were a minimal downward covering set M for A that contains d, Claim 4.14 would imply that $M = A$. However, since $B_\alpha \subset A = M$, this contradicts minimality, so no minimal downward covering set for A can contain d.

For the direction from right to left, assume that no minimal downward covering set for A contains d. Since by Claim 4.13 minimal downward covering sets are guaranteed to exist for the dominance graph defined in Construction 4.12, there exists a minimal downward covering set B for A that does not contain d, so $B \neq A$. It holds that $\{z_i \mid w_i \text{ is a variable in } \varphi\} \cap B = \emptyset$ and $\{y_j \mid f_j \text{ is a clause in } \varphi\} \cap B = \emptyset$, for otherwise a contradiction would follow by observing that there is no $a \in A$ with $a \succ d$ and $a \succ z_i$, $1 \le i \le k$, or with $a \succ d$ and $a \succ y_j$, $1 \le j \le \ell$. Furthermore, we have $x_i \notin B$ or $\overline{x}_i \notin B$, for each variable $w_i \in W$. By external stability, for each clause f_j there must exist an alternative $a \in B$ with $a \succ y_j$. By construction and since $d \notin B$, we must have either $a = x_i$ for some variable w_i that occurs in f_j as a positive literal, or $a = \overline{x}_i$ for some variable w_i that occurs in f_j as a negative literal. Now define $\alpha : W \to \{0,1\}$ such that $\alpha(w_i) = 1$ if $x_i \in B$, and $\alpha(w_i) = 0$ otherwise. It is readily appreciated that α is a satisfying assignment for φ. ❑

Claim 4.16. *Consider Construction 4.12. The boolean formula φ is not satisfiable if and only if there is a unique minimal downward covering set for A.*

Proof. We again assume that if φ is satisfiable, it has at least two satisfying assignments. If φ is not satisfiable, there must be a minimal downward covering set for A that contains d by Claim 4.15, and by Claim 4.14 there must be a minimal downward covering set for A containing all alternatives. Hence, there is a unique minimal downward covering set for A. Conversely, if there is a unique minimal downward covering set for A, φ cannot be satisfiable, since otherwise there would be at least two distinct minimal downward covering sets for A, corresponding to the distinct truth assignments for φ, which would yield a contradiction. ❑

In the dominance graph created by Construction 4.12, the minimal downward covering sets for A coincide with the minimum-size downward covering sets for A. If φ is not satisfiable, there is only one minimal downward covering set for A, so this is also the only minimum-size downward covering set for A, and if φ is satisfiable, the minimal downward covering sets for A correspond to the satisfying assignments of φ. As we have seen in the proof of Claim 4.15, these minimal downward covering sets for A always consist of $5k + 2$ alternatives. Thus, they each are also minimum-size downward covering sets for A.

Merging the construction from the proof sketch of Theorem 4.11 with Construction 4.12, we again provide a reduction applying Lemma 4.6, this time to downward covering set problems.

Construction 4.17 (for applying Lemma 4.6 to downward covering set problems). *We again apply Wagner's Lemma with the* NP*-complete problem* $S = \mathrm{SAT}$ *and construct a dominance graph. Fix an arbitrary* $m \geq 1$ *and let* $\varphi_1, \varphi_2, \ldots, \varphi_{2m}$ *be* $2m$ *boolean formulas in conjunctive normal form such that the satisfiability of* φ_j *implies the satisfiability of* φ_{j-1}*, for each* $j \in \{2, \ldots, 2m\}$*. Without loss of generality, we assume that for each* j*,* $1 \leq j \leq 2m$*,* φ_j *has at least two satisfying assignments, if* φ *is satisfiable.*

We now define a polynomial-time computable function f*, which maps the given* $2m$ *boolean formulas to a dominance graph* $(A, \succ)$ *that has useful properties for our downward covering set problems. The set of alternatives is*

$$A = \left(\bigcup_{i=1}^{2m} A_i\right) \cup \left(\bigcup_{i=1}^{m} \{r_i, s_i, t_i\}\right) \cup \{c^*, d^*\},$$

and the dominance relation $\succ$ *on* A *is defined by*

$$\left(\bigcup_{i=1}^{2m} \succ_i\right) \cup \left(\bigcup_{i=1}^{m} \{(r_i, d_{2i-1}), (r_i, d_{2i}), (s_i, r_i), (s_i, d_{2i-1}), (t_i, r_i), (t_i, d_{2i})\}\right) \cup \left(\bigcup_{i=1}^{k} \{(d^*, r_i)\}\right) \cup \{(c^*, d^*)\},$$

where we use the following notation:

1. *For each* i*,* $1 \leq i \leq m$*, let* $(A_{2i-1}, \succ_{2i-1})$ *be the dominance graph that results from the formula* φ_{2i-1} *according to Brandt and Fischer's construction given in the proof sketch of Theorem 4.11. We again use the same names for the alternatives in* A_{2i-1} *as in that proof sketch, except that we attach the subscript* $2i - 1$*.*

2. *For each* i*,* $1 \leq i \leq m$*, let* $(A_{2i}, \succ_{2i})$ *be the dominance graph that results from the formula* φ_{2i} *according to Construction 4.12. We again use the same names for the alternatives in* A_{2i} *as in that construction, except that we attach the subscript* $2i$*.*

3. *For each* i*,* $1 \leq i \leq m$*, the dominance graphs* $(A_{2i-1}, \succ_{2i-1})$ *and* $(A_{2i}, \succ_{2i})$ *are connected by the alternatives* s_i*,* t_i*, and* r_i *(which play a similar role as the alternatives* z_i*,* z_i'*, and* z_i'' *for each variable in Construction 4.12). The resulting dominance graph is denoted by* $(B_i, \succ_i^B)$*.*

4. *Connect the* m *dominance graphs* $(B_i, \succ_i^B)$*,* $1 \leq i \leq m$ *(again similarly as in Construction 4.12). The alternative* c^* *dominates* d^**, and* d^* *dominates the* m *alternatives* r_i*,* $1 \leq i \leq m$*.*

This construction is illustrated in Figure 8. Clearly, $(A, \succ)$ is computable in polynomial time.

Claim 4.18. *Consider Construction 4.17. For each* i*,* $1 \leq i \leq 2m$*, let* M_i *be a minimal downward covering set for* $(A_i, \succ_i)$*. Then each of the sets* M_i *must be contained in every minimal downward covering set for* $(A, \succ)$*.*

Proof. For each i, $1 \leq i \leq 2m$, the only alternative in A_i dominated from outside A_i is d_i. Since d_i is also dominated by the undominated alternative $z_{1,i} \in A_i$ for odd i, and by the undominated alternative $c_i \in A_i$ for even i, it is readily appreciated that internal and external

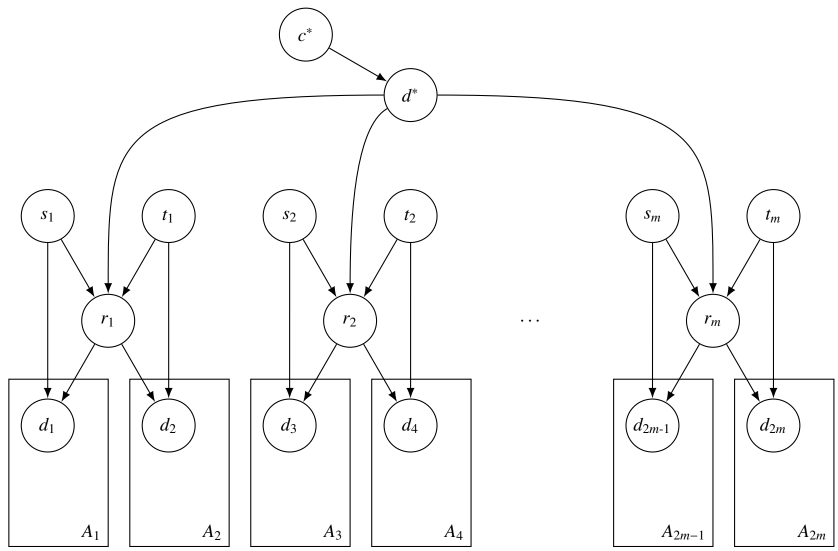

Figure 8: Dominance graph from Construction 4.17.

stability with respect to elements of A_i only depends on the restriction of the dominance graph to A_i. ❑

Claim 4.19. *Consider Construction 4.17. It holds that*

$$\|\{i \mid \varphi_i \in \mathrm{SAT}\}\| \textit{ is odd}$$
$$\iff d^* \textit{ is contained in some minimal downward covering set } M \textit{ for } A. \tag{4.3}$$

Proof. For the direction from left to right in (4.3), assume that $\|\{i \mid \varphi_i \in \mathrm{SAT}\}\|$ is odd. Thus, there is some $j \in \{1, \ldots, m\}$ such that $\varphi_1, \varphi_2, \ldots, \varphi_{2j-1}$ are each satisfiable and $\varphi_{2j}, \varphi_{2j+1}, \ldots, \varphi_{2m}$ are each not. Define

$$M = \left(\bigcup_{i=1}^{2m} M_i\right) \cup \left(\bigcup_{i=1}^{m} \{s_i, t_i\}\right) \cup \left\{r_j, c^*, d^*\right\},$$

where for each i, $1 \le i \le 2m$, M_i is some minimal downward covering set of the restriction of the dominance graph to A_i, satisfying that $d_i \in M_i$ if and only if

1. i is odd and φ_i is satisfiable, or

2. i is even and φ_i is not satisfiable.

Such sets M_i exist by the proof sketch of Theorem 4.11 and by Claim 4.15. In particular, φ_{2j-1} is satisfiable and φ_{2j} is not, so $\{d_{2j-1}, d_{2j}\} \subseteq M$. There is no alternative that dominates d_{2j-1}, d_{2j}, and r_j. Thus, r_j must be in M. The other alternatives r_i, $1 \le i \le m$ and $i \ne j$, are downward covered by either s_i if $d_{2i-i} \notin M$, or t_i if $d_{2i} \notin M$. Finally, d^* cannot be downward covered, because $d^* > r_j$ and no alternative dominates both d^* and r_j. Internal and external stability with respect to the elements of M_i, as well as minimality of $\bigcup_{i=1}^{2k} M_i$, follow from the proofs of Theorem 4.11 and Claim 4.15. All other elements of M are undominated and thus contained in every downward covering set. We conclude that M is a minimal downward covering set for A that contains d^*.

For the direction from right to left in (4.3), assume that there exists a minimal downward covering set M for A with $d^* \in M$. By internal stability, there must exist some j, $1 \leq j \leq k$, such that $r_j \in M$. Thus, d_{2j-1} and d_{2j} must be in M, too. It then follows from the proof sketch of Theorem 4.11 and Claim 4.15 that φ_{2j-1} is satisfiable and φ_{2j} is not. Hence, $\|\{i \mid \varphi_i \in \mathrm{SAT}\}\|$ is odd. ❑

By the remark made after Theorem 4.11, Construction 4.17 cannot be used straightforwardly to obtain complexity results for minimum-size downward covering sets.

5. Proof of Theorem 3.1

In this section, we prove Theorem 3.1 by applying the constructions and the properties of the resulting dominance graphs presented in Section 4. We start with the results on minimal and minimum-size upward covering sets.

5.1. Minimal and Minimum-Size Upward Covering Sets

Theorem 5.1. *It is* NP*-complete to decide, given a dominance graph* $(A, >)$ *and a positive integer k, whether there exists a minimal/minimum-size upward covering set for A of size at most k. That is, both* $\mathrm{MC_u}$-Size *and* $\mathrm{MSC_u}$-Size *are* NP*-complete.*

Proof. This result can be proven by using the construction of Theorem 4.1. Let φ be a given boolean formula in conjunctive normal form, and let n be the number of variables occurring in φ. Setting the bound k for the size of a minimal/minimum-size upward covering set to $2n+1$ proves that both problems are hard for NP. Indeed, as we have seen in the paragraph after the proof sketch of Theorem 4.1, there is a size $2n + 1$ minimal upward covering set (and hence a minimum-size upward covering set) for A if and only if φ is satisfiable. Both problems are NP-complete, since they can obviously be decided in nondeterministic polynomial time. ❑

Theorem 5.2. *Deciding whether a designated alternative is contained in some minimal upward covering set for a given dominance graph is hard for* Θ_2^p *and in* Σ_2^p*. That is,* $\mathrm{MC_u}$-Member *is hard for* Θ_2^p *and in* Σ_2^p*.*

Proof. Θ_2^p-hardness follows directly from Claim 4.10. For the upper bound, let $(A, >)$ be a dominance graph and d a designated alternative in A. First, observe that we can verify in polynomial time whether a subset of A is an upward covering set for A, simply by checking whether it satisfies internal and external stability. Now, we can guess an upward covering set $B \subseteq A$ with $d \in B$ in nondeterministic polynomial time and verify its minimality by checking that none of its subsets is an upward covering set for A. This places the problem in $\mathrm{NP}^{\mathrm{coNP}}$ and consequently in Σ_2^p. ❑

Theorem 5.3. *1. It is Θ_2^p-complete to decide whether a designated alternative is contained in some minimum-size upward covering set for a given dominance graph. That is,* $\mathrm{MSC_u}$-MEMBER *is Θ_2^p-complete.*

2. It is Θ_2^p-complete to decide whether a designated alternative is contained in all minimum-size upward covering sets for a given dominance graph. That is, $\mathrm{MSC_u}$-MEMBER-ALL *is Θ_2^p-complete.*

Proof. By the remark made after Claim 4.10, both problems are hard for Θ_2^p.

To see that $\mathrm{MSC_u}$-MEMBER is contained in Θ_2^p, let $(A, \succ)$ be a dominance graph and d a designated alternative in A. Obviously, in nondeterministic polynomial time we can decide, given $(A, \succ)$, $x \in A$, and some positive integer $\ell \leq \|A\|$, whether there exists some upward covering set B for A such that $\|B\| \leq \ell$ and $x \in B$. Using this problem as an NP oracle, in Θ_2^p we can decide, given $(A, \succ)$ and $d \in A$, whether there exists a minimum-size upward covering set for A containing d as follows. The oracle is asked whether for each pair (x, ℓ), where $x \in A$ and $1 \leq \ell \leq \|A\|$, there exists an upward covering set for A of size bounded by ℓ that contains the alternative x. The number of queries is polynomial (more specifically in $O(\|A\|^2)$), and all queries can be asked in parallel. Having all the answers, determine the size k of a minimum-size upward covering set for A, and accept if the oracle answer to (d, k) was yes, otherwise reject.

To show that $\mathrm{MSC_u}$-MEMBER-ALL is in Θ_2^p, let $(A, \succ)$ be a dominance graph and d a designated alternative in A. We now use as our oracle the set of all (x, ℓ), where $x \in A$ is an alternative, and $\ell \leq \|A\|$ a positive integer, such that there exists some upward covering set B for A with $\|B\| \leq \ell$ and $x \notin B$. Clearly, this problem is also in NP, and the size k of a minimum-size upward covering set for A can again be determined by asking $O(\|A\|^2)$ queries in parallel (if all oracle answers are no, it holds that $k = \|A\|$). Now, the Θ_2^p machine accepts its input $((A, \succ), d)$ if the oracle answer for the pair (d, k) is no, and otherwise it rejects. ❑

Theorem 5.4. *1. (Brandt and Fischer [1]) It is* coNP*-complete to decide whether a designated alternative is contained in all minimal upward covering sets for a given dominance graph. That is,* $\mathrm{MC_u}$-MEMBER-ALL *is* coNP*-complete.*

2. It is coNP*-complete to decide whether a given subset of the alternatives is a minimal upward covering set for a given dominance graph. That is,* $\mathrm{MC_u}$-TEST *is* coNP*-complete.*

3. It is coNP*-hard and in Σ_2^p to decide whether there is a unique minimal upward covering set for a given dominance graph. That is,* $\mathrm{MC_u}$-UNIQUE *is* coNP*-hard and in Σ_2^p.*

Proof. It follows from Claim 4.5 that φ is not satisfiable if and only if the entire set of alternatives A is a (unique) minimal upward covering set for A. Furthermore, if φ is satisfiable, there exists more than one minimal upward covering set for A and none of them contains e_1 (provided that φ has more than one satisfying assignment, which can be ensured, if needed, by adding a dummy variable such that the satisfiability of the formula is not affected). This proves coNP-hardness for all three problems. $\mathrm{MC_u}$-MEMBER-ALL and $\mathrm{MC_u}$-TEST are also *contained* in coNP, as they can be decided in the positive by checking whether there does *not* exist an upward covering set that satisfies certain properties related to the problem at hand, so they both are coNP-complete. $\mathrm{MC_u}$-Unique can be decided in the positive by checking whether there exists an upward covering set M such that all sets that are not strict supersets of M are *not* upward covering sets for the set of all alternatives. Thus, $\mathrm{MC_u}$-Unique is in Σ_2^p. ❑

The first statement of Theorem 5.4 was already shown by Brandt and Fischer [1]. However, their proof—which uses essentially the reduction from the proof of Theorem 4.1, except that they start from the coNP-complete problem Validity (which asks whether a given formula is valid, i.e., true under every assignment [21])—does not yield any of the other coNP-hardness results in Theorem 5.4.

Theorem 5.5. *It is* coNP*-complete to decide whether a given subset of the alternatives is a minimum-size upward covering set for a given dominance graph. That is,* $\mathrm{MSC_u}$-Test *is* coNP*-complete.*

Proof. This problem is in coNP, since it can be decided in the positive by checking whether the given subset M of alternatives is an upward covering set for the set A of all alternatives (which is easy) and all sets of smaller size than M are not upward covering sets for A (which is a coNP predicate), and coNP-hardness follows directly from Claim 4.5. ❑

Theorem 5.6. *Deciding whether there exists a unique minimum-size upward covering set for a given dominance graph is hard for* coNP *and in* Θ_2^p*. That is,* $\mathrm{MSC_u}$-Unique *is* coNP*-hard and in* Θ_2^p*.*

Proof. It is easy to see that coNP-hardness follows directly from the coNP-hardness of $\mathrm{MC_u}$-Unique (see Theorem 5.4). Membership in Θ_2^p can be proven by using the same oracle as in the proof of the first part of Theorem 5.3. We ask for all pairs (x, ℓ), where $x \in A$ and $1 \leq \ell \leq \|A\|$, whether there is an upward covering set B for A such that $\|B\| \leq \ell$ and $x \in B$. Having all the answers, determine the minimum size k of a minimum-size upward covering set for A. Accept if there are exactly k distinct alternatives $x_1, \ldots, x_k$ for which the answer for (x_i, k), $1 \leq i \leq k$, was yes, otherwise reject. ❑

An important consequence of the proofs of Theorems 5.4 and 5.6 (and of Construction 4.2 that underpins these proofs) regards the hardness of the search problems $\mathrm{MC_u}$-Find and $\mathrm{MSC_u}$-Find.

Theorem 5.7. *Assuming* $\mathrm{P} \neq \mathrm{NP}$*, neither minimal upward covering sets nor minimum-size upward covering sets can be found in polynomial time. That is, neither* $\mathrm{MC_u}$-Find *nor* $\mathrm{MSC_u}$-Find *are polynomial-time computable unless* $\mathrm{P} = \mathrm{NP}$.

Proof. Consider the problem of deciding whether there exists a *nontrivial* minimal/minimum-size upward covering set, i.e., one that does *not* contain all alternatives. By Construction 4.2 that is applied in proving Theorems 5.4 and 5.6, there exists a trivial minimal/minimum-size upward covering set for A (i.e., one containing all alternatives in A) if and only if this set is the only minimal/minimum-size upward covering set for A. Thus, the coNP-hardness proof for the problem of deciding whether there is a unique minimal/minimum-size upward covering set for A (see the proofs of Theorems 5.4 and 5.6) immediately implies that the problem of deciding whether there is a nontrivial minimal/minimum-size upward covering set for A is NP-hard. However, since the latter problem can easily be reduced to the search problem (because the search problem, when used as a function oracle, yields the set of all alternatives if and only if this set is the only minimal/minimum-size upward covering set for A), it follows that the search problem cannot be solved in polynomial time unless P = NP. ❑

5.2. Minimal and Minimum-Size Downward Covering Sets

Theorem 5.8. *It is* NP-*complete to decide, given a dominance graph* $(A, \succ)$ *and a positive integer k, whether there exists a minimal/minimum-size downward covering set for A of size at most k. That is,* $\mathrm{MC_d}$-Size *and* $\mathrm{MSC_d}$-Size *are both* NP-*complete.*

Proof. Membership in NP is obvious, since we can nondeterministically guess a subset $M \subseteq A$ of the alternatives with $\|M\| \leq k$ and can then check in polynomial time whether M is a downward covering set for A. NP-hardness of $\mathrm{MC_d}$-Size and $\mathrm{MSC_d}$-Size follows from Construction 4.12, the proof of Claim 4.15, and the comments made after Claim 4.16: If φ is a given formula with n variables, then there exists a minimal/minimum-size downward covering set of size $5n + 2$ if and only if φ is satisfiable. ❑

Theorem 5.9. $\mathrm{MSC_d}$-Member*,* $\mathrm{MSC_d}$-Member-All*, and* $\mathrm{MSC_d}$-Unique *are* coNP-*hard and in* Θ_2^p*.*

Proof. It follows from Claim 4.16 that φ is not satisfiable if and only if the entire set A of all alternatives is the unique minimum-size downward covering set for itself. Moreover, assuming that φ has at least two satisfying assignments, if φ is satisfiable, there are at least two distinct minimum-size downward covering sets for A. This shows that each of $\mathrm{MSC_d}$-Member, $\mathrm{MSC_d}$-Member-All, and $\mathrm{MSC_d}$-Unique is coNP-hard. For all three problems, membership in Θ_2^p is shown similarly to the proofs of the corresponding minimum-size upward covering set problems. However, since downward covering sets may fail to exist, the proofs must be slightly adapted. For $\mathrm{MSC_d}$-Member and $\mathrm{MSC_d}$-Unique, the machine rejects the input if the size k of a mininum-size downward covering set cannot be computed (simply because there doesn't exist any such set). For $\mathrm{MSC_d}$-Member-All, if all oracle answers are no, it must be checked whether the set of all alternatives is a downward covering set for itself. If so, the machine accepts the input, otherwise it rejects. ❑

Theorem 5.10. *It is* coNP-*complete to decide whether a given subset is a minimum-size downward covering set for a given dominance graph. That is,* $\mathrm{MSC_d}$-Test *is* coNP-*complete.*

Proof. This problem is in coNP, since its complement (i.e., the problem of deciding whether a given subset of the set A of alternatives is not a minimum-size downward covering set for A) can be decided in nondeterministic polynomial time. Hardness for coNP follows directly from Claim 4.16. ❑

Theorem 5.11. *Deciding whether a designated alternative is contained in some minimal downward covering set for a given dominance graph is hard for* Θ_2^p *and in* Σ_2^p*. That is,* MC$_{\mathrm{d}}$-MEMBER *is hard for* Θ_2^p *and in* Σ_2^p*.*

Proof. Membership in Σ_2^p can be shown analogously to the proof of Theorem 5.2, and Θ_2^p-hardness follows directly from Claim 4.19. ❑

Theorem 5.12.

1. *(Brandt and Fischer [1]) It is* coNP*-complete to decide whether a designated alternative is contained in all minimal downward covering sets for a given dominance graph. That is,* MC$_{\mathrm{d}}$-MEMBER-ALL *is* coNP*-complete.*
2. *It is* coNP*-complete to decide whether a given subset of the alternatives is a minimal downward covering set for a given dominance graph. That is,* MC$_{\mathrm{d}}$-TEST *is* coNP*-complete.*
3. *It is* coNP*-hard and in* Σ_2^p *to decide whether there is a unique minimal downward covering set for a given dominance graph. That is,* MC$_{\mathrm{d}}$-UNIQUE *is* coNP*-hard and in* Σ_2^p*.*

Proof. It follows from Claim 4.16 that φ is not satisfiable if and only if the entire set of alternatives A is a unique minimal downward covering set for A. Furthermore, if φ is satisfiable, there exists more than one minimal downward covering set for A and none of them contains d (provided that φ has more than one satisfying assignment, which can be ensured, if needed, by adding a dummy variable such that the satisfiability of the formula is not affected). This proves coNP-hardness for all three problems. MC$_{\mathrm{d}}$-MEMBER-ALL and MC$_{\mathrm{d}}$-TEST are also contained in coNP, because they can be decided in the positive by checking whether there does not exist a downward covering set that satisfies certain properties related to the problem at hand. Thus, they are both coNP-complete. MC$_{\mathrm{d}}$-UNIQUE can be decided in the positive by checking whether there exists a downward covering set M such that all sets that are not strict supersets of M are *not* downward covering sets for the set of all alternatives. This shows that MC$_{\mathrm{d}}$-UNIQUE is in Σ_2^p. ❑

The first statement of Theorem 5.12 was already shown by Brandt and Fischer [1]. However, their proof—which uses essentially the reduction from the proof of Theorem 4.11, except that they start from the coNP-complete problem VALIDITY—does not yield any of the other coNP-hardness results in Theorem 5.12.

An important consequence of the proofs of Theorems 5.9 and 5.12 regards the hardness of the search problems MC$_{\mathrm{d}}$-FIND and MSC$_{\mathrm{d}}$-FIND. (Note that the hardness of MC$_{\mathrm{d}}$-FIND also follows from a result by Brandt and Fischer [1, Thm. 9], see the discussion in Section 3.)

Theorem 5.13. *Assuming* $\mathrm{P} \neq \mathrm{NP}$*, neither minimal downward covering sets nor minimum-size downward covering sets can be found in polynomial time (i.e., neither* MC$_{\mathrm{d}}$-FIND *nor* MSC$_{\mathrm{d}}$-FIND *are polynomial-time computable unless* $\mathrm{P} = \mathrm{NP}$*), even when the existence of a downward covering set is guaranteed.*

Proof. Consider the problem of deciding whether there exists a *nontrivial* minimal/minimum-size downward covering set, i.e., one that does *not* contain all alternatives. By Construction 4.12 that is applied in proving Theorems 5.9 and 5.12, there exists a trivial minimal/minimum-size downward covering set for A (i.e., one containing all alternatives in A) if and only if this set is the only minimal/minimum-size downward covering set for A. Thus, the coNP-hardness proof for the problem of deciding whether there is a unique minimal/minimum-size downward covering set for A (see the proofs of Theorems 5.9 and 5.12) immediately implies that the problem of deciding whether there is a nontrivial minimal/minimum-size downward covering set for A is NP-hard. However, since the latter problem can easily be reduced to the search problem (because the search problem, when used as a function oracle, yields the set of all alternatives if and only if this set is the only minimal/minimum-size downward covering set for A), it follows that the search problem cannot be solved in polynomial time unless P = NP. ❑

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