The Linear Logical Framework

LLF

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(*joint work with Frank Pfenning*)

Contents

Overview

A **Logical Framework** is a formalism designed to represent and reason about deductive systems

Aim:

- identify the principles underlying logics and programming languages [Harper,Honsell,Plotkin'87; Pfenning'92; Michaylov,Pfenning'91; Shankar'94; Pfenning'95]

Intended applications:

- design of new and better logics and programming languages
- program verification and certification [Necula'97; Paulson'96]

Limitations:

- ineffective with imperative formalisms [Pfenning'94]

State

Till 2 years ago, **no** simple, general and effective treatment of the recurring notion of **state**

- store of an imperative programming language
- database
- communication among concurrent processes, ...

... Linear Logic [Girard'87]

- adequate for **representing** state and imperative computation [Chirimar'95; Hodas,Miller'94; Wadler'90]
- ineffective for **reasoning** about them

Achievements

- Design of a formalism, *LLF*, that combines
  - the meta-reasoning power of traditional logical frameworks
  - the possibility of linear logic of handling state
- Based on a *linear type theory*
- Conservative over *LF* [Harper,Honsell,Plotkin'93]
- Used to represent
  - imperative programming languages
  - substructural logics
  - games, ...

  and to *reason* about them

Logical Frameworks

Formalisms specially designed to provide effective meta-representations of formal systems

formal system

programming languages, logics, ...

meta-representation

represent language constructs, model their semantics, encode properties and their proofs

effectiveness

immediacy and executability

Logical framework = meta-language + representation methodology

An Example: *LF* (Meta-Language)

- **Syntax**

$$
\begin{array}{rrcl}
\textit{Kinds} & K & ::= & \texttt{type} \mid \Pi x\!:\!A.\, K \\
\textit{Type families} & P & ::= & a \mid P\, M \\
\textit{Types} & A & ::= & P \mid \Pi x\!:\!A.\, B \\
\textit{Objects} & M & ::= & x \mid c \mid \lambda x\!:\!A.\, M \mid M\, N
\end{array}
$$

- **Typing judgment**

$$\Gamma \vdash_{\Sigma} M : A$$

"M has type A in Γ and Σ"

Context
$x\!:\!A, \ldots$

Signature
$a\!:\!K, \ldots, c\!:\!A, \ldots$

An Example: *LF* (Meta-Language—Cont'd)

$$\frac{\Gamma, x:A \vdash_\Sigma M : B}{\Gamma \vdash_\Sigma \lambda x:A.\, M : \Pi x:A.\, B}\ \mathbf{lam} \qquad \frac{\Gamma \vdash_\Sigma M : \Pi x:A.\, B \quad \Gamma \vdash_\Sigma N : A}{\Gamma \vdash_\Sigma M\, N : [N/x]B}\ \mathbf{app}$$

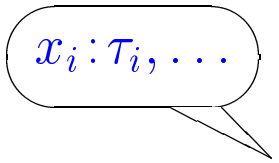
- **Main properties**
  - is strongly normalizing
  - admits unique canonical forms
  - type checking is decidable
  - can be implemented as a logic programming language (*Elf* [Pfenning'94])

An Example: *LF* (Representation Methodology)

Judgments-as-Types / Derivations-as-Objects

- Each object judgment is represented as a base type
- The context of an object judgment is encoded in the context of the meta-language
- Object-level inference rules are represented as constants that map derivations of their premisses to a derivation of their conclusion
- Derivations of an object judgment are represented as canonical terms of the corresponding base type

An Example: *LF* (Representation Methodology—Cont'd)



$$\ulcorner \quad \mathcal{T} \quad \urcorner = M$$
$$\Omega \vdash e : \tau$$

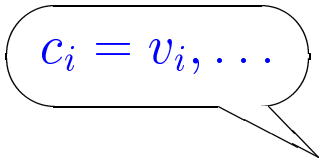
$$\ulcorner \Omega \urcorner \vdash_\Sigma M : \texttt{has\_type}\ \ulcorner e \urcorner\ \ulcorner \tau \urcorner$$

where for each $x_i : \tau_i$ in Ω,

$$\ulcorner x_i : \tau_i \urcorner = x_i : \texttt{exp},\ t_i : \texttt{has\_type}\ x_i\ \ulcorner \tau_i \urcorner$$

- context operations reduce to meta-level primitives
- meta-theoretic properties are inherited from the meta-language

Problem!



$$\ulcorner \quad \mathcal{E} \quad \urcorner = M$$
$$S \rhd K \vdash e \hookrightarrow a$$

$$\ulcorner S \urcorner \vdash_\Sigma M : \texttt{eval} \ulcorner K \urcorner \ulcorner e \urcorner \ulcorner a \urcorner$$

This does not work!

- S is subject to *destructive operations* (e.g. assignment)
- traditional log. frameworks do not allow removing assumptions from the context

A way out ...

$$\cdot \vdash_\Sigma M : \texttt{eval} \ulcorner S \urcorner \ulcorner K \urcorner \ulcorner e \urcorner \ulcorner a \urcorner$$

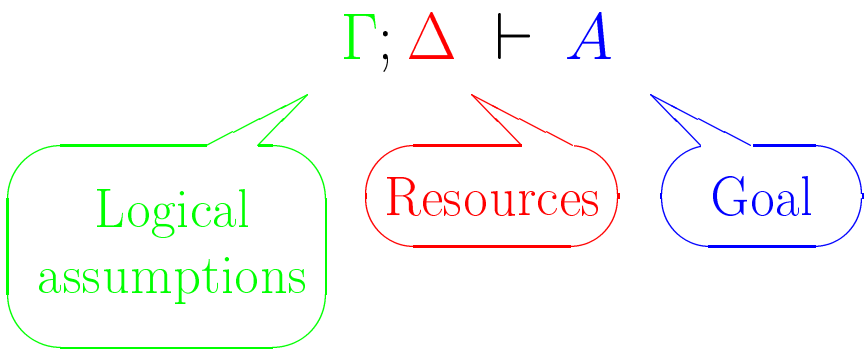
... but, we must encode *explicitly*

- context operations (lookup, insertion, ...)
- context-related properties (weakening, exchange, ...)

LLF

- **Meta-language**: $\lambda^{\Pi \multimap \& \top}$, a type theory based on Π, $\multimap$, & and $\top$
- **Representation methodology**: judgments-as-types, but provides direct encoding of state in the linear context
- **Range of applicability**: declarative and imperative formalisms

Linear Logic in Brief



Accessing a resource **consumes** it

Main resource operators

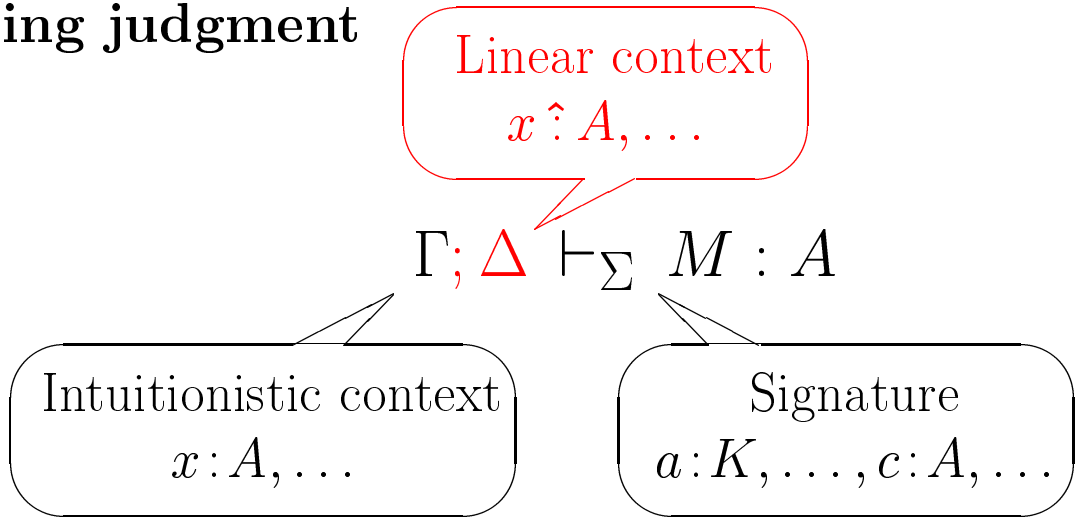
- $A \otimes B$ = "A and B simultaneously"
- $A \,\&\, B$ = "A and B alternatively"
- $\top$ = "resource sink"
- $A \multimap B$ = "B assuming A as a resource"
- $A \to B$ = "B assuming A as a logical hypothesis"

$\lambda^{\Pi\multimap\&\top}$, the Meta-Language of *LLF*

- **Syntax**

$$
\begin{array}{rrcl}
\textit{Kinds} & K & ::= & \mathtt{type} \mid \Pi x{:}A.\,K \\
\textit{Type families} & P & ::= & a \mid P\,M \\
\textit{Types} & A & ::= & P \mid \Pi x{:}A.\,B \\
 & & & \mid A \multimap B \mid A \,\&\, B \mid \top \\
\textit{Objects} & M & ::= & x \mid c \mid \lambda x{:}A.\,M \mid M\,N \\
 & & & \mid \hat{\lambda} x{:}A.\,M \mid M \mathbin{\hat{}} N \mid \langle M, N\rangle \mid \textsc{fst}\,M \mid \textsc{snd}\,M \mid \langle\rangle
\end{array}
$$

- **Typing judgment**



"M has type A in Γ, Δ and Σ"

LLF, Main Properties

- Church-Rosser property
- strongly normalizing
- unique canonical forms
- decidability of type checking
- abstract logic programming language
- conservative over *LF*

Immediacy in *LLF*

Direct correlation between an object system and its encoding

LLF gives direct support to recurrent representation patterns

- binding constructs via λ-abstraction
- derivations as proof-terms
- state manipulation via linear constructs

Case Study: *MLR*

MLR is a fragment of *ML* with

- references
- value polymorphism
- recursion

$$\begin{array}{rrcl}
\textit{Types} & \tau & ::= & \ldots \mid \mathbf{1} \mid \tau_1 \to \tau_2 \mid \tau \ \mathbf{ref} \\
\textit{Expressions} & e & ::= & x \\
 & & \mid & \langle\rangle \\
 & & \mid & \mathbf{lam}\ x.\, e \\
 & & \mid & e_1\, e_2 \\
 & & \mid & \ldots \\
 & & \mid & c \\
 & & \mid & \mathbf{ref}\ e \\
 & & \mid & !e \\
 & & \mid & e_1 := e_2 \\
\textit{Store} & S & ::= & \cdot \mid S, c = v
\end{array}$$

Expressions

```
exp   : type.
cell  : type.

unit   : exp.
lam    : (exp -> exp) -> exp.
app    : exp -> exp -> exp.
       ...
loc    : cell -> exp.
ref    : exp -> exp.
deref  : exp -> exp.
assign : exp -> exp -> exp.
```

MLR: Typing

$\Omega \vdash e : \tau$ "e has type τ in Ω"

Context $x_i : \tau_i, \ldots, c_j : \sigma_j, \ldots$ | Expression | Type

Representation: $\ulcorner\Omega\urcorner \vdash_\Sigma \ulcorner\mathcal{T}\urcorner : \texttt{exp\_type}\ \ulcorner e\urcorner\ \ulcorner\tau\urcorner$

$x_i : \texttt{exp}, \quad t_i : \texttt{exp\_type}\ x_i\ \ulcorner\tau_i\urcorner, \quad \ldots$
$c_j : \texttt{cell}, \ l_j : \texttt{cell\_type}\ c_j\ \ulcorner\sigma_j\urcorner, \ \ldots$

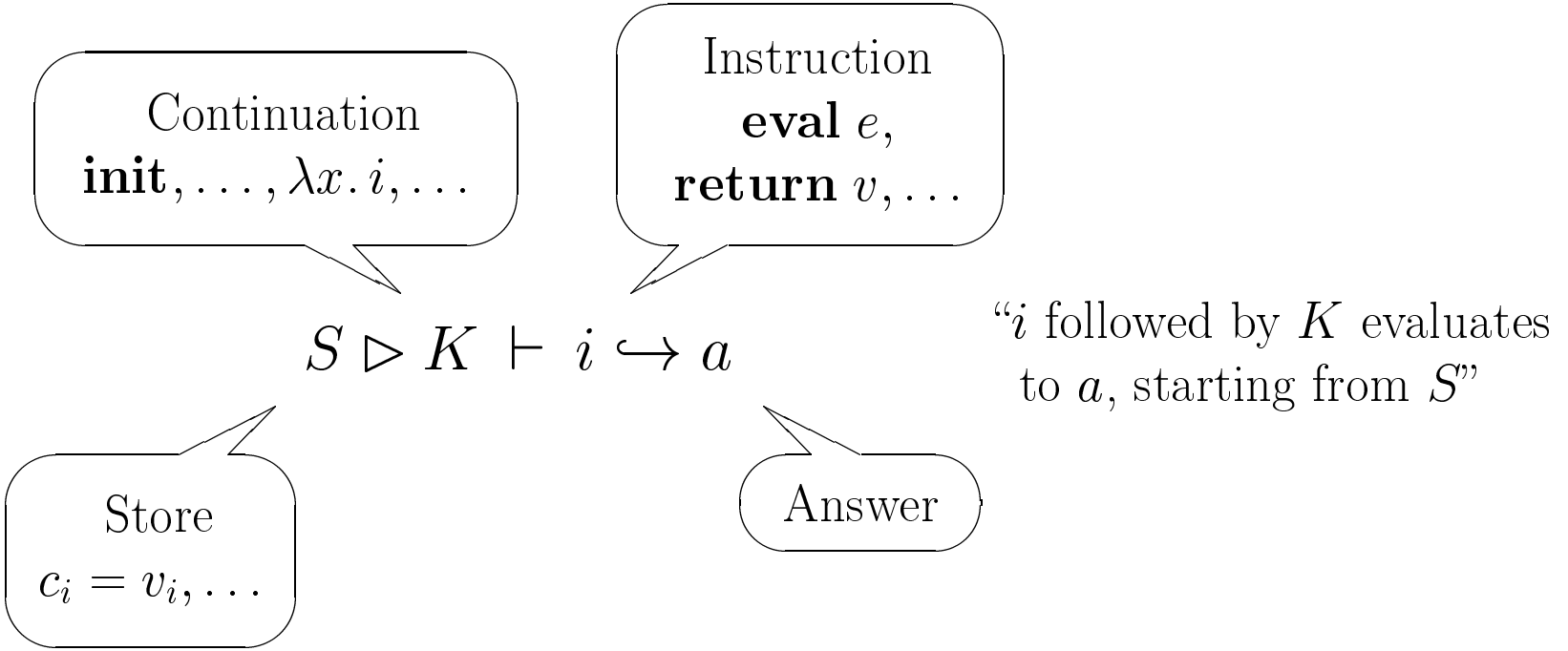
$$\frac{\Omega \vdash e_1 : \tau\ \mathbf{ref} \quad \Omega \vdash e_2 : \tau}{\Omega \vdash e_1 := e_2 : \mathbf{1}}\ \mathbf{et\_assign}$$

```
et_assign :  exp_type E1 (rf T)
          -> exp_type E2 T
          -> exp_type (assign E1 E2) 1.
```

$$\frac{\Omega \vdash e : \tau\ \mathbf{ref}}{\Omega \vdash\ !e : \tau}\ \mathbf{et\_deref}$$

```
et_deref  :  exp_type E (rf T)
          -> exp_type (deref E) T.
```

MLR: Evaluation



"i followed by K evaluates to a, starting from S"

Representation: $\ulcorner S \urcorner \vdash_\Sigma \ulcorner \mathcal{E} \urcorner : \texttt{eval}\ \ulcorner K \urcorner\ \ulcorner i \urcorner\ \ulcorner a \urcorner$

$c_i : \texttt{cell},\ h_i \mathbin{\hat{:}} \texttt{contains}\ c_i\ \ulcorner v_i \urcorner,\ \ldots$

MLR: Some Imperative Rules

$$\frac{S', c = v, S'' \rhd K \vdash \mathbf{return}\ \langle\rangle \hookrightarrow a}{S', c = v', S'' \rhd K \vdash c := v \hookrightarrow a}\ \mathbf{ev\_assign}$$

```
ev_assign :    (contains C V   -o  eval K (return unit) A)
            -o (contains C V'  -o  eval K (assign2 (loc C) V) A).
```

$$\frac{S', c = v, S'' \rhd K \vdash \mathbf{return}\ v \hookrightarrow a}{S', c = v, S'' \rhd K \vdash\ !c \hookrightarrow a}\ \mathbf{ev\_deref}$$

```
ev_deref  :  read C V
             & eval K (return V) A
            -o eval K (ref1 (loc C)) A.

rd :  contains C V
   -o <T>
   -o read C V.
```

MLR: Adequacy

Adequacy theorem (*Evaluation*)

Given a store $S = (c_1 = v_1, \ldots, c_n = v_n)$, a continuation K, an instruction i and an answer a, all closed, there is a bijection between derivations $\mathcal{E}$ of

$$S \rhd K \vdash i \hookrightarrow a$$

and canonical LLF objects M such that

$$\ulcorner S \urcorner \vdash_\Sigma M : \mathtt{eval}\ \ulcorner K \urcorner\ \ulcorner i \urcorner\ \ulcorner a \urcorner$$

is derivable, where

$$\ulcorner S \urcorner = \left[\begin{array}{c} c_1 : \mathtt{cell},\ h_1 \hat{:} \mathtt{contains}\ c_1\ \ulcorner v_1 \urcorner \\ \ldots \\ c_n : \mathtt{cell},\ h_n \hat{:} \mathtt{contains}\ c_n\ \ulcorner v_n \urcorner \end{array}\right]$$

MLR: Type Preservation

- Functional core: implemented in *LF* [Michaylov,Pfenning'91]
- References [Tofte'90; Harper'94]: implemented in *LLF* [Cervesato'96]

Theorem (*type preservation*)

If $S \rhd K \vdash i \hookrightarrow a$, with $\Omega \vdash i : \tau$, $\Omega \vdash K : \tau \Rightarrow \sigma$ and $\Omega \vdash S : \Omega$, then $\Omega \vdash a : \sigma$

Proof: by induction on the evaluation derivation

The high level of abstraction of the representation permits **transcribing** this proof into an *LLF* specification capturing its computational contents

- each case yields one declaration
- the meta-reasoning is itself *linear*

Representation

```
tpev : eval K I A -> cont_type K T S -> instr_type I T -> ans_type A S -> type.
```

Implementation

LLF is implemented as part of the *Twelf* project

- **Twelf, a successor to *Elf*** [Pfenning'94]
  - higher-order constraint logic programming language based on *LF* and *LLF*
  - automated theorem prover in a meta-logic for *LF* [Schürmann,Pfenning'98]
  - internals: explicit substitutions, spine calculus, compilation

- **Linear aspects**
  - linearity check
  - resource management [Hodas,Miller'94; Cervesato,Pfenning'96]
  - linear unification [Cervesato,Pfenning'97]

LLF, Summary

- combines the meta-reasoning power of logical frameworks with the ability of handling state of linear logic
- conservative extension of the logical framework LF
- implemented as a linear logic programming language
- used for the representation of
  - imperative programming languages
  - substructural and modal logics
  - puzzles and solitaires
  - planning
  - imperative graph search

Future Work

- Specification and verification of
  - "real" programming languages (e.g. *SML'97*, *Java*)
  - communication protocols
  - logics
- Proof-Carrying Code [Necula'97]
- Computer-assisted development environments for logics and programming languages (meta-logical frameworks)
- Type theoretic extensions of *LLF* (e.g. dependent linear types [Ishtiaq,Pym'97], non-commutativity [Pfenning'98])