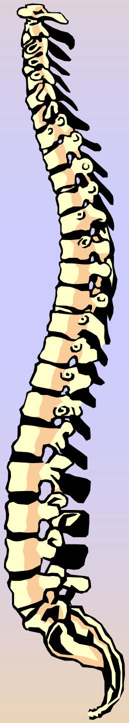
A (Linear) Spine Calculus



Iliano Cervesato iliano@itd.nrl.navy.mil

ITT Industries, Inc @ NRL - Washington DC

http://www.cs.stanford.edu/~iliano/

Joint work with Frank Pfenning

Our Motivations

- Speed up term manipulation
  - (Linear) higher-order unification
  - (Weak-head) normalization
- Implementation of Twelf, LLF
  - Type reconstruction
  - Logic programming
  - Theorem proving

Other People's Motivations

- Study the λ-calculus
  - Barendregt
- Proof-terms for sequent calculi
  - Herbelin
  - Schwitchtenberg
- Speed-up HOU
  - Nadathur

Roadmap

1. The λ-calculus $\lambda^{\rightarrow \multimap \& \top}$
2. The spine calculus $S^{\rightarrow \multimap \& \top}$
3. Translations
4. NIL transitions
5. Uniform provability

Part I

The λ-Calculus

$$\lambda^{\rightarrow \multimap \& \top}$$

$\lambda^{\to \multimap \& \top}$

- Terms

  x, c

  $\lambda x{:}A.\,M$, $\lambda x\hat{}A.\,M$, $\langle M,N \rangle$, $\langle\rangle$

  $M\;N$, $M\hat{}N$, $\mathsf{fst}\;M$, $\mathsf{snd}\;M$

- Types

  $a,\quad A \to B,\quad \underbrace{A \multimap B,\quad A \mathbin{\&} B,\quad \top}_{\text{Generally skipped}}$

$\lambda^{\rightarrow \multimap \& \top}$ (cont'd)

- Simply-typed fragment of LLF
- Core of
  - λProlog
  - Lolli
- Biggest fragment of ILL that admits unique normal forms

η-long Forms ??

- Simplifies unification / normalization
  - Efficiency
    - Simpler code
    - No types needed
  - Clarity
- Preserved by β-reduction
- Easy to build from type declarations
  - η-rule needed for type reconstruction

Examples

$$c : (a \to a) \multimap a \,\&\, a \to a$$

c ✗

λf^a→a. λp:a&a. c^f p ✗

λf^a→a. λp:a&a. c ^ (λx:a.f x) <fst p, snd p> ✓

Pre-canonical Typing

$\Gamma; \Delta \vdash M \Uparrow A$

M is *pre-canonical* of type *A* in Γ and Δ

$\Gamma; \Delta \vdash M \downarrow A$

M is *pre-atomic* of type *A* in Γ and Δ

Term Formation Rules

$$\frac{\Gamma, x{:}A \;|\!-\; M \Uparrow B}{\Gamma \;|\!-\; \lambda x{:}A.\, M \Uparrow A \to B}$$

$$\frac{\Gamma \;|\!-\; M \downarrow B \to A \qquad \Gamma \;|\!-\; N \Uparrow B}{\Gamma \;|\!-\; M\, N \downarrow A}$$

$$\frac{}{\Gamma, x{:}A \;|\!-\; x \downarrow A}$$

Tying it Together

$$\frac{\Gamma \mid\!\!- \; M \downarrow a}{\Gamma \mid\!\!- \; M \Uparrow a}$$

Forces extensionality

$$\frac{\Gamma \mid\!\!- \; M \Uparrow A}{\Gamma \mid\!\!- \; M \downarrow A}$$

Allows β-redices

Remove to force
β-normal terms

Examples

- Not η-long: ✗ $\underbrace{f}_{\downarrow a \to a} \Uparrow a \to a$
- η-long ✓ $\underbrace{\lambda x{:}a.\ f\ x}_{\Uparrow a \to a} \Uparrow a \to a$
- β-redex ✓ $\underbrace{(\lambda x{:}a.\ f\ x)\ c}_{\Uparrow a} \Uparrow a$

Reductions

$$(\lambda x{:}A.\ M)\ N \Rightarrow_\beta [N/x]M$$

$$(\lambda x\hat{}A.\ M)\ \hat{}\ N \Rightarrow_\beta [N/x]M$$

$$\mathsf{fst}\ \langle M,N\rangle \Rightarrow_\beta M$$

$$\mathsf{snd}\ \langle M,N\rangle \Rightarrow_\beta N$$

+ congruences

Properties

- Church-Rosser
- Subject reduction
- Strong normalization
- Uniqueness of normal forms

Part II

The Spine Calculus

$S^{\rightarrow \multimap \& \top}$

Unification

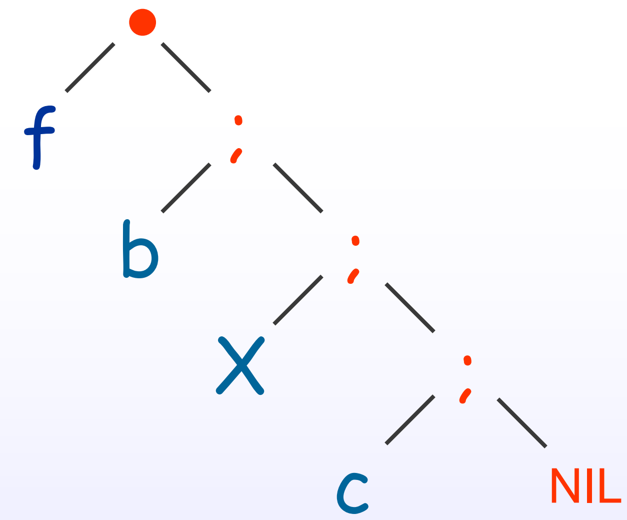
- X: *logical variables* in terms
- Find subst. σ such that $M^{\sigma} =_{\beta} N^{\sigma}$
- Algorithms work on the *head* of M, N
  - Pre-unification
  - Higher-order patterns

$h\ M_1 \ldots M_n$

First Order vs. Higher Order

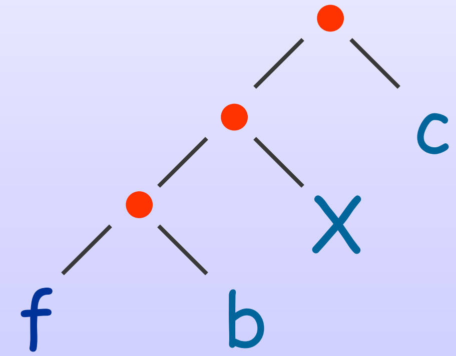
- First Order:

  $f(X, g(a), c) = f(b, Y, c)$



- Higher-Order:

  $(((f\ X)\ (g\ a))\ c) = (((f\ b)\ Y)\ c)$



Objective

Devise representation of λ-terms that

- allows efficient head access
  - does not penalize F.O. problems
- allows simple recursive descriptions
- is not ad-hoc

fst ((snd c) ^ x y)

$c: A \mathbin{\&} (B \rightarrow C \multimap D \mathbin{\&} E)$

Idea

end of spine

$$h \quad M_1 \quad \dots \quad M_n \qquad\qquad h \bullet \overbrace{(M_1 ; \dots M_n ; \mathrm{NIL})}^{\text{spine}}$$

$\xleftarrow{\text{Assoc.}}$ $\qquad\qquad$ $\xrightarrow{\text{Assoc.}}$

$$\mathrm{fst}\ ((\mathrm{snd}\ c)\ \hat{}\ x\ \ y)$$

$$c \bullet (\pi_2\ x\ ;\hat{}\ y\ ;\ \pi_1\ \mathrm{NIL})$$

The Spine Calculus

- Terms

  $x \bullet S$, $\lambda x{:}A.\ U$, $\lambda x\hat{\ }A.\ U$, $\langle U,V\rangle$, $\langle\rangle$

- Spines

  NIL, $V;S$, $V;\hat{\ }S$, $\pi_1\ S$, $\pi_2\ S$

- Heads

  x, U, c

Typing Judgments

$\Gamma; \Delta \vdash U : A$

Term U has

type A in Γ and Δ

$\Gamma; \Delta \vdash S : A > a$

S is a *spine*

from heads of type A to terms of type a

in Γ and Δ

Rules for Terms

$$\frac{\Gamma, x{:}A \vdash U : B}{\Gamma \vdash \lambda x{:}A.\ U : A \to B}$$

$$\frac{\Gamma, x{:}A \vdash S : A > a}{\Gamma, x{:}A \vdash x \bullet S : a}$$

$$\frac{\Gamma \vdash U : A \qquad \Gamma \vdash S : A > a}{\Gamma \vdash U \bullet S : a}$$

Rules for Spines

$$\frac{\Gamma \mid\!- U : B \qquad\qquad \Gamma \mid\!- S : A > a}{\Gamma \mid\!- U; S : B \rightarrow A > a}$$

$$\frac{}{\Gamma \mid\!- \mathrm{NIL} : a > a}$$

β-reductions

$$(\lambda x{:}A.\ U) \bullet (V;S) \Rightarrow_\beta [V/x]U \bullet S$$

$$(\lambda x\hat{}A.\ U) \bullet (V;\hat{}S) \Rightarrow_\beta [V/x]U \bullet S$$

$$\langle U,V\rangle \bullet (\pi_1\ S) \Rightarrow_\beta U \bullet S$$

$$\langle U,V\rangle \bullet (\pi_2\ S) \Rightarrow_\beta V \bullet S$$

\+ congruences

NIL-reduction

$$
\begin{tikzcd}
(\lambda x.\ f\ x)\ c \arrow[r] \arrow[ddd, "\beta"] & (\lambda x.\ f \bullet (x \bullet \mathrm{NIL};\ \mathrm{NIL})) \bullet (c \bullet \mathrm{NIL};\ \mathrm{NIL}) \arrow[d, "\beta"] \\
 & (f \bullet ((c \bullet \mathrm{NIL}) \bullet \mathrm{NIL};\ \mathrm{NIL})) \bullet \mathrm{NIL} \arrow[d, "\mathrm{NIL}"] \\
 & f \bullet ((c \bullet \mathrm{NIL}) \bullet \mathrm{NIL};\ \mathrm{NIL}) \arrow[d, "\mathrm{NIL}"] \\
f\ c \arrow[r] & f \bullet (c \bullet \mathrm{NIL};\ \mathrm{NIL})
\end{tikzcd}
$$

$$\boxed{(U \bullet S) \bullet \mathrm{NIL} \quad \Rightarrow_{\mathrm{NIL}} U \bullet S}$$

Properties of NIL-reduction

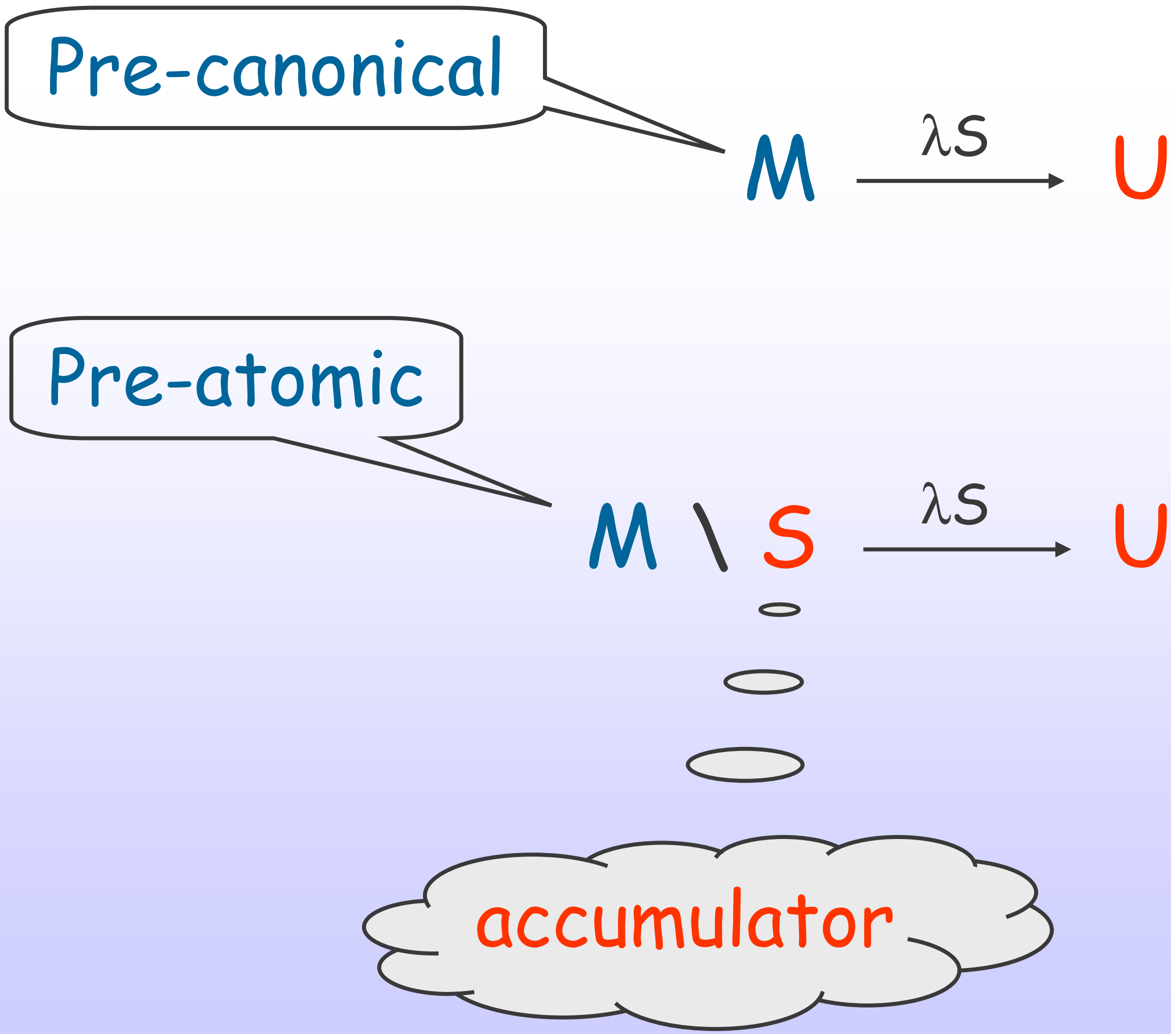
- Preserves typing (both directions)
- Strong NIL-normalization
- Uniqueness of NIL-normal forms
- Permutes past β-reductions

$$\begin{tikzcd} U \arrow[r, "\mathrm{NIL}"'] \arrow[d, dotted, "\beta"] & V' \arrow[d, "\beta"] \\ U' \arrow[r, dotted, "\mathrm{NIL}"'] & V \end{tikzcd}$$

Part III

Translations

Translating to Spine Notation



Some Rules

$$\frac{M \xrightarrow{\lambda S} U}{\lambda x{:}A.\,M \xrightarrow{\lambda S} \lambda x{:}A.\,U}$$

destructor

$$\frac{M \setminus \mathrm{NIL} \xrightarrow{\lambda S} H \bullet S}{M \xrightarrow{\lambda S} H \bullet S}$$

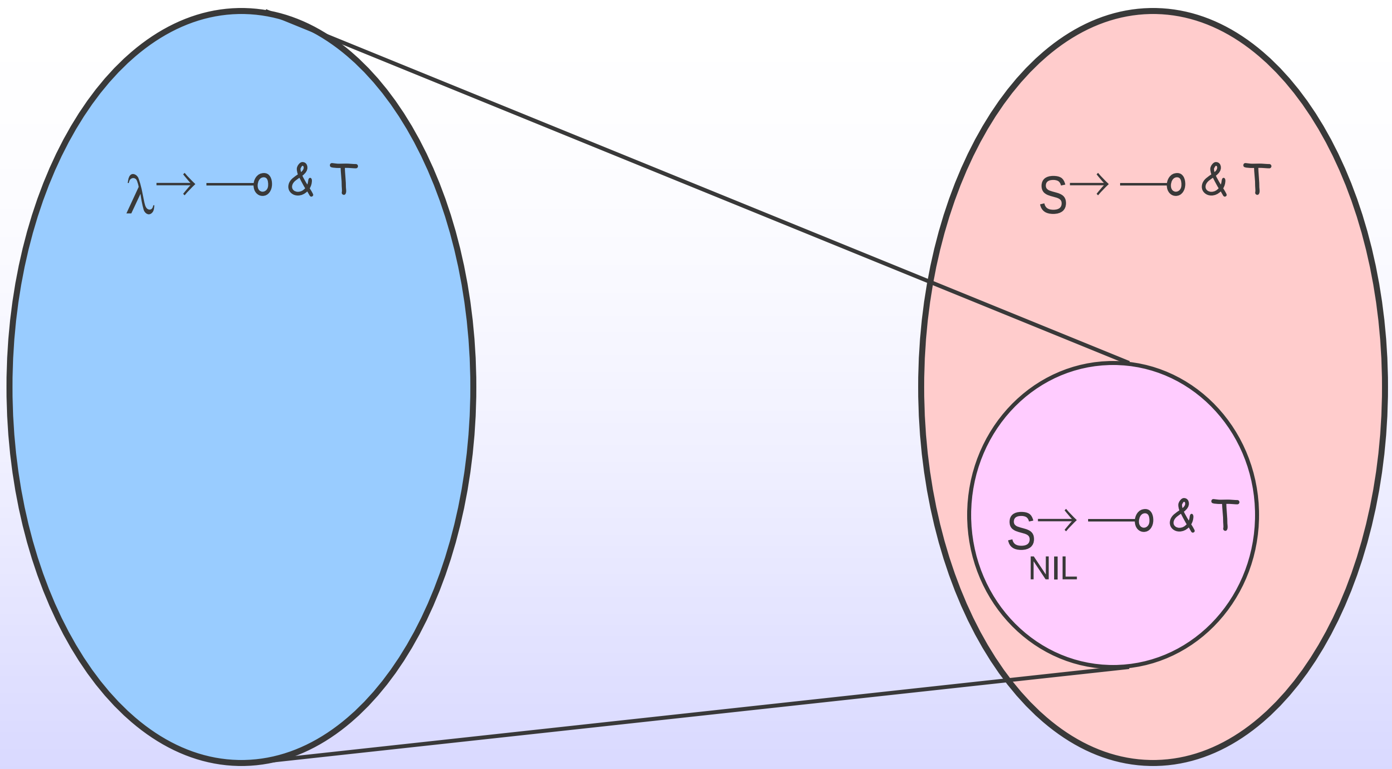
constructor

$$\frac{}{x \setminus S \xrightarrow{\lambda S} x \bullet S}$$

$$\frac{M \xrightarrow{\lambda S} H}{M \setminus S \xrightarrow{\lambda S} U \bullet S}$$

$$\frac{N \xrightarrow{\lambda S} U \qquad M \setminus U;S \xrightarrow{\lambda S} V}{M\,N \setminus S \xrightarrow{\lambda S} V}$$

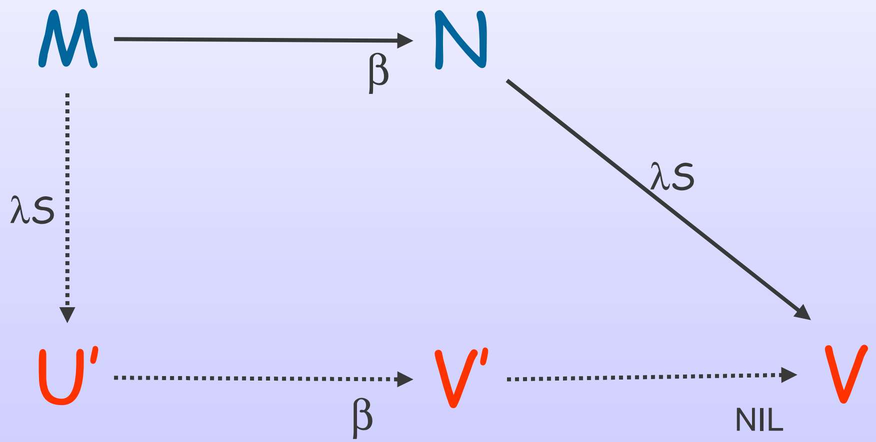
Schematically



Always produces NIL-normal forms

Properties of the Translation

- Sound w.r.t. typing
- Sound w.r.t. reducibility, but



Translating from Spine Notation

$$U \xrightarrow{S\lambda} M$$

Pre-canonical

$$S \setminus N \xrightarrow{S\lambda} M$$

Pre-atomic

accumulator

Some Rules

$$\frac{U \xrightarrow{S\lambda} M}{\lambda x{:}A.\ U \xrightarrow{S\lambda} \lambda x{:}A.\ M}$$

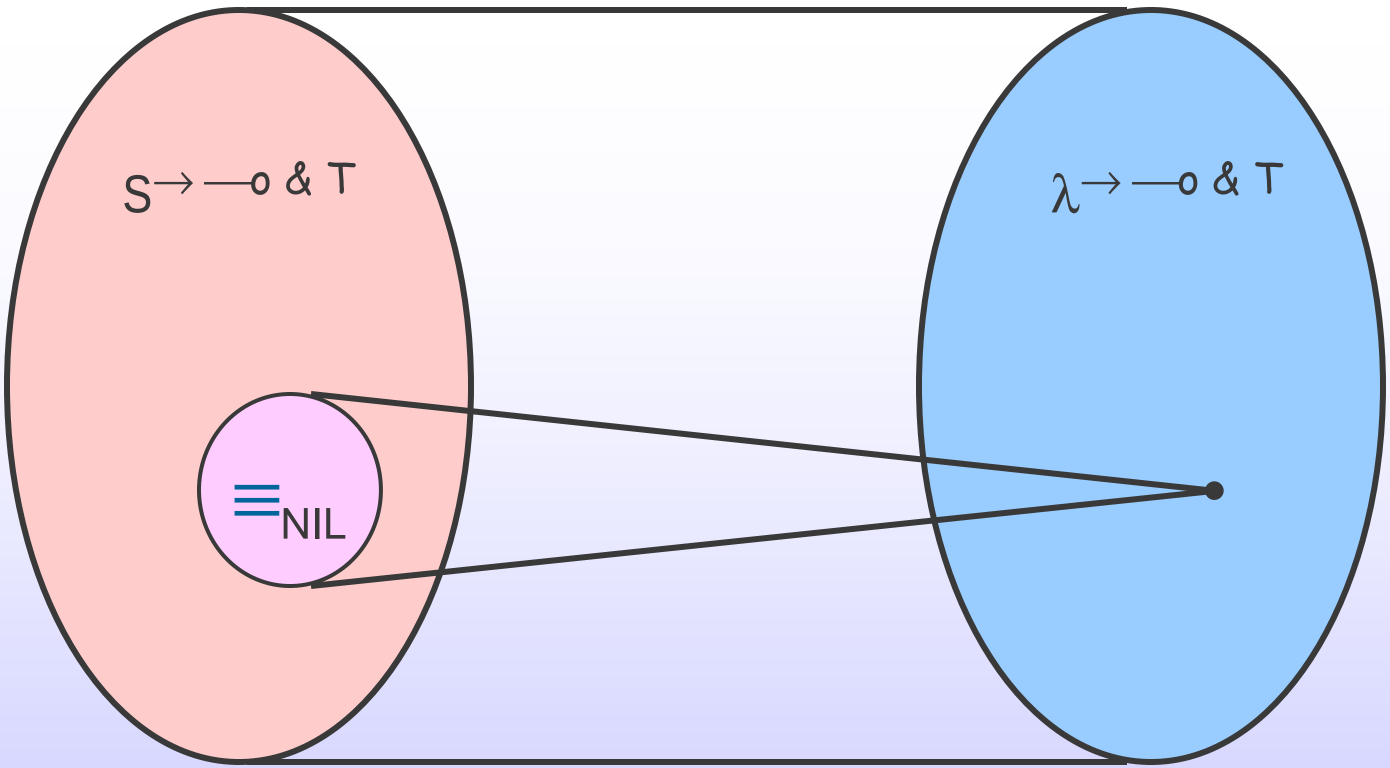
$$\frac{S \setminus x \xrightarrow{S\lambda} M}{x \bullet S \xrightarrow{S\lambda} M}$$

$$\frac{U \xrightarrow{S\lambda} N \qquad S \setminus N \xrightarrow{S\lambda} M}{U \bullet S \xrightarrow{S\lambda} M}$$

$$\frac{}{\mathrm{NIL} \setminus M \xrightarrow{S\lambda} M}$$

$$\frac{U \xrightarrow{S\lambda} M' \qquad S \setminus N\ M' \xrightarrow{S\lambda} M}{U;\ S \setminus N \xrightarrow{S\lambda} M}$$

Schematically



Identifies NIL-equivalent terms

Properties of the Translation

- Weak inverse of λS
  - $\lambda^{\rightarrow \multimap \& \top} \quad \leftrightarrow \quad S^{\rightarrow \multimap \& \top}_{\mathrm{NIL}}$
- Sound w.r.t. typing
  - Weak completeness of λS and $S\lambda$
- Sound w.r.t. reductions
  - Weak completeness of λS and $S\lambda$

Properties of the Spine Calculus

- Church-Rosser
- Subject reduction
- Strong normalization
- Uniqueness of normal forms

β and NIL reductions

Down to Earth

- Empirical evaluation
  - Efficiency improvement ?
- Logical status of spine NIL-reduction
- Logical status of spine calculus

The Twelf / LLF Experience

- Noticeable efficiency gains (not only spines)
- Extends to dependent types
- Extends to linearity
- Supports weak normal forms
- Mixes with explicit substitutions
- Great for logic programming / compil.

Part IV

NIL Transitions

A glimpse back to $\lambda^{\rightarrow \multimap \& \top}$

Mismatch between

- $\lambda^{\rightarrow \multimap \& \top}$
- Pre-canonical derivations

$$\frac{\Gamma \mid\!- M \downarrow a}{\Gamma \mid\!- M \Uparrow a} \qquad\qquad \frac{\Gamma \mid\!- M \Uparrow A}{\Gamma \mid\!- M \downarrow A}$$

No trace of rule application

Explicit Mode Switching

$$M ::= \quad \ldots \quad | \quad \mathrm{ac}\ M \quad | \quad \mathrm{ca}\ M$$

$$\frac{\Gamma \mid\!\!- M \downarrow a}{\Gamma \mid\!\!- \mathrm{ac}\ M \Uparrow a} \qquad \frac{\Gamma \mid\!\!- M \Uparrow A}{\Gamma \mid\!\!- \mathrm{ca}\ M \downarrow A}$$

Witness of rule application

Reduction Rules

$$\left.\begin{array}{rcl}
(\mathrm{ca}\ (\lambda x{:}A.\ M))\ N & \Rightarrow_\beta & (\mathrm{ca}\ [\mathrm{ca}\ N/x]M) \\
(\mathrm{ca}\ (\lambda x\hat{}A.\ M))\ \hat{}\ N & \Rightarrow_\beta & (\mathrm{ca}\ [\mathrm{ca}\ N/x]M) \\
\mathrm{fst}\ (\mathrm{ca}\ \langle M,N\rangle) & \Rightarrow_\beta & (\mathrm{ca}\ M) \\
\mathrm{snd}\ (\mathrm{ca}\ \langle M,N\rangle) & \Rightarrow_\beta & (\mathrm{ca}\ N)
\end{array}\right\}\ \beta$$

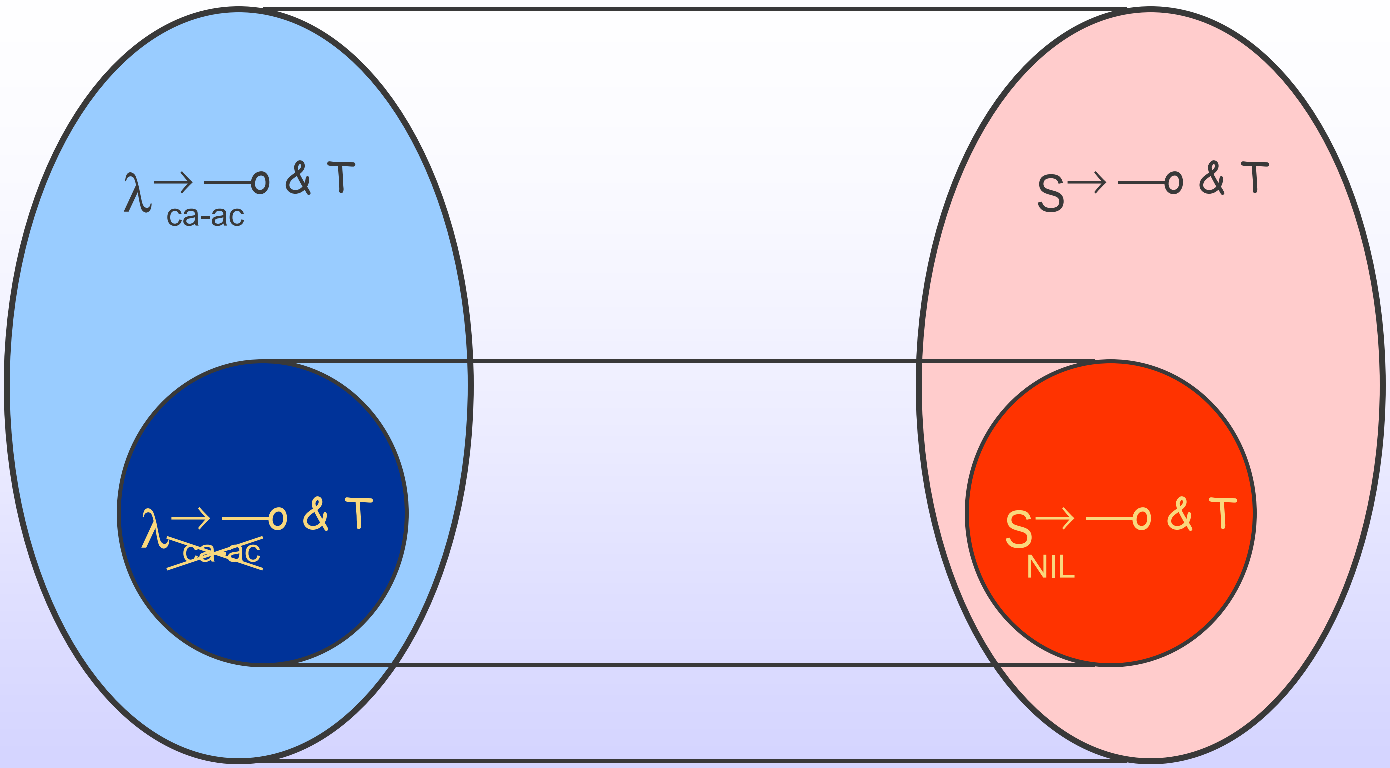
$$\mathrm{ca}\ (\mathrm{ac}\ M) \Rightarrow M \qquad \text{(coercion)}$$

Updating the translation

- Coercion mapped to NIL-reduction
- The translations are now exact

$$\lambda^{\rightarrow \multimap \& \top}_{\text{ca-ac}} \leftrightarrow S^{\rightarrow \multimap \& \top}_{\text{NIL}}$$

Schematically



Part V

Uniform Proofs

Uniform Sequent Derivations

1. Apply right rules till RHS a is atomic
2. Chose focus A from lhs
3. Apply left rules till A matches a
4. Repeat

Abstract logic programing languages
are complete w.r.t.
uniform provability

Intuitionistic Implicational Logic

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \qquad \frac{\Gamma, A \xRightarrow{A} a}{\Gamma, A \Rightarrow a}$$

$$\frac{\Gamma \Rightarrow B \qquad \Gamma \xRightarrow{A} a}{\Gamma \xRightarrow{B \rightarrow A} a} \qquad \frac{}{\Gamma \xRightarrow{a} a}$$

$$\left(\frac{\Gamma \Rightarrow A \qquad \Gamma \xRightarrow{A} a}{\Gamma \Rightarrow a}\right)$$

Recovering the Spine Calculus

$$\frac{\Gamma, x{:}A \mid\!- U : B}{\Gamma \mid\!- \lambda x{:}A.\, U : A \to B} \qquad \frac{\Gamma, x{:}A \mid\!- S : A > a}{\Gamma, x{:}A \mid\!- x \bullet S : a}$$

$$\frac{\Gamma \mid\!- U : B \quad \Gamma \mid\!- S : A > a}{\Gamma \mid\!- U;\, S : B \to A > a} \qquad \frac{}{\Gamma \mid\!- \mathrm{NIL} : a > a}$$

$$\left(\frac{\Gamma \mid\!- U : A \quad \Gamma \mid\!- S : A > a}{\Gamma \mid\!- U \bullet S : a} \right)$$

Extending the Curry-Howard Iso.

"Uniform logic"

Term calculi

$$
\begin{tikzcd}
ND^{\cdots} \arrow[rr, leftrightarrow] \arrow[d, leftrightarrow, "\text{Curry-Howard}"'] & & SC^{\cdots} \arrow[d, leftrightarrow, "\text{Curry-Howard}"] \\
\lambda^{\cdots} \arrow[rr, leftrightarrow, "\lambda S", "S\lambda"'] & & S^{\cdots}
\end{tikzcd}
$$

Beyond Uniform Provability

- Howard ['69]
  - Indirect reference
- Huet ['89]
  - "The Constructive Machine"
- Herbelin ['95]
  - LJT $\leftrightarrow$ $\underline{\lambda}$: implic. int. logic with focus
  - No η-long forms: "spines" concatenation
  - Interprets *cut* as explicit substitutions

Beyond ... (cont'd)

- Dyckhoff & Pinto ['98,'99,'01]
  - Refines Herbelin's work
  - Explicit substitutions
- Schwichtenberg ['97]
  - Also $\forall$ and $\wedge$
  - No focus: commuting conversions
  - No concatenation / substitutions

Details

- Tech. Report

  www.cs.stanford.edu/~iliano/byYear/#1997

- Forthcoming paper
  - 2001 ?