Compressing Polarized Boxes

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linear logic and boxes

- **Proof nets**: the **graphical syntax** for linear logic.
- Brought new deep perspectives about **normalization**:
  1. Optimal reductions;
  2. Implicit computational complexity;
  3. Explicit substitutions;
  4. Strong normalization.
- **Key tool**: **boxes** for the promotion rule, the heart of the system.
- **This work**: a **new understanding of boxes**, via **polarity**.

Multiplicative Linear Logic (MLL)

Identity rules:

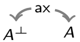
$$\frac{}{\vdash A^{\perp}, A}\ \mathsf{ax} \qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ \mathsf{cut}$$

Multiplicative rules:

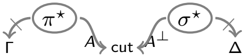
$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr$$

Proof nets for MLL

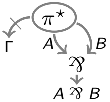
$$\frac{}{\vdash A^{\perp}, A}\ \mathsf{ax} \quad \rightsquigarrow$$



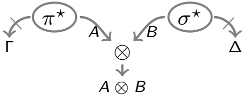
$$\frac{\begin{array}{c}\pi \\ \vdots \\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\sigma \\ \vdots \\ \vdash \Delta, A^{\perp}\end{array}}{\vdash \Gamma, \Delta}\ \mathsf{cut} \quad \rightsquigarrow$$



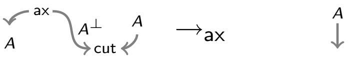
$$\frac{\begin{array}{c}\pi \\ \vdots \\ \vdash \Gamma, A, B\end{array}}{\vdash \Gamma, A \parr B}\ \parr \quad \rightsquigarrow$$

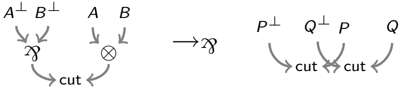


$$\frac{\begin{array}{c}\pi \\ \vdots \\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\sigma \\ \vdots \\ \vdash \Delta, B\end{array}}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \quad \rightsquigarrow$$



Cut-elimination for MLL





No duplication/erasure of subnets

$\Rightarrow$

Everything works **fine**

Multiplicative Exponential Linear Logic (MELL)

MLL

+

Exponential rules:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,\mathsf{d} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,!$$

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,\mathsf{c} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,\mathsf{w}$$

Exponential Cut-elimination

Consider the following cut with **contraction**:

$$
\dfrac{
\dfrac{\begin{array}{c}\rho\\ \vdots\\ \vdash ?\Delta, A\end{array}}{\vdash ?\Delta, !A}\;!
\qquad
\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash ?A^{\perp}, ?A^{\perp}, \Gamma\end{array}}{\vdash ?A^{\perp}, \Gamma}\;\mathsf{c}
}{\vdash ?\Delta, \Gamma}\;\mathsf{cut}
$$

Its elimination requires to **duplicate** ρ:

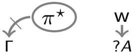
$$
\dfrac{
\dfrac{
\dfrac{\begin{array}{c}\rho\\ \vdots\\ \vdash ?\Delta, A\end{array}}{\vdash ?\Delta, !A}\;!
\qquad
\dfrac{
\dfrac{\begin{array}{c}\rho\\ \vdots\\ \vdash ?\Delta, A\end{array}}{\vdash ?\Delta, !A}\;!
\qquad
\begin{array}{c}\pi\\ \vdots\\ \vdash ?A^{\perp}, ?A^{\perp}, \Gamma\end{array}
}{\vdash ?\Delta, ?A^{\perp}, \Gamma}\;\mathsf{cut}
}{\vdash ?\Delta, ?\Delta, \Gamma}\;\mathsf{cut}
}{\begin{array}{c}\vdots\end{array}}\;\mathsf{c}
$$

$$
\dfrac{\vdots}{\vdash ?\Delta, \Gamma}\;\mathsf{c}
$$

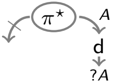
Similarly, **weakening** induces **erasure** of sub-proofs.

Naïve proof nets for MELL

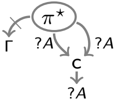
$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?A}\ \mathsf{w} \quad \rightsquigarrow$$



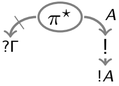
$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, A\end{array}}{\vdash \Gamma, ?A}\ \mathsf{d} \quad \rightsquigarrow$$



$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, ?A, ?A\end{array}}{\vdash \Gamma, ?A}\ \mathsf{c} \quad \rightsquigarrow$$



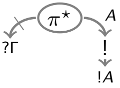
$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash ?\Gamma, A\end{array}}{\vdash ?\Gamma, !A}\ ! \quad \rightsquigarrow$$



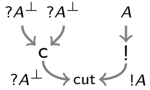
How to eliminate cuts?

Naïve translation of promotion:

$$\dfrac{\begin{array}{c}\pi \\ \vdots \\ \vdash ?\Gamma, A\end{array}}{\vdash ?\Gamma, !A}\,! \quad \rightsquigarrow$$



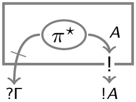
Given this cut in a generic net:



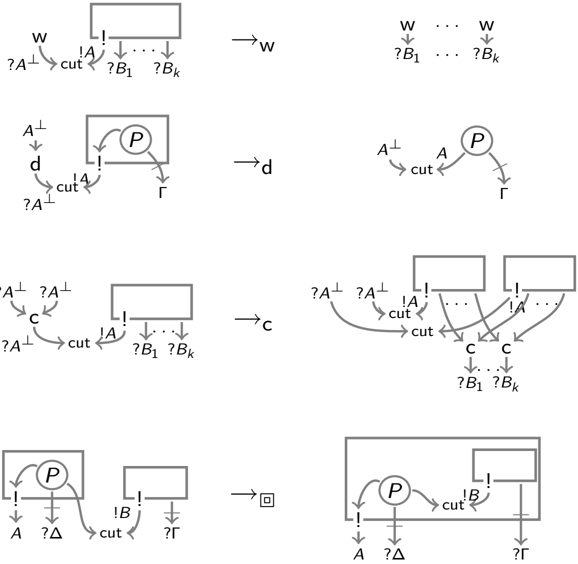
There is **no way** of recovering the **sub-proof to duplicate**.

Then **!-rules** are represented as **boxes**:

$$\dfrac{\begin{array}{c}\pi \\ \vdots \\ \vdash ?\Gamma, A\end{array}}{\vdash ?\Gamma, !A}\,! \quad \rightsquigarrow$$



Exponential cut elimination implemented using boxes



Boxes

- Boxes **solve the problem** of defining cut-elimination.
- However, the solution is **drastic**, equivalent to **give up**.
- Some fragments seem to have an **inherent notion of box**.
- Where does the problem lie?
- Is there a **logic feature** that **internalizes boxes**?

Last rule 1

- **Main problem**: in proof nets there is **no last rule**.
- Re-consider:

$$
\dfrac{\dfrac{\begin{array}{c}\rho\\ \vdots\\ \vdash ?\Delta, A\end{array}}{\vdash ?\Delta, !A}\,! \qquad \dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash ?A^{\perp}, ?A^{\perp}, \Gamma\end{array}}{\vdash ?A^{\perp}, \Gamma}\,\mathsf{c}}{\vdash ?\Delta, \Gamma}\,\mathsf{cut}
\quad\longrightarrow\quad
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\rho\\ \vdots\\ \vdash ?\Delta, A\end{array}}{\vdash ?\Delta, !A}\,! \qquad \dfrac{\dfrac{\begin{array}{c}\rho\\ \vdots\\ \vdash ?\Delta, A\end{array}}{\vdash ?\Delta, !A}\,! \qquad \begin{array}{c}\pi\\ \vdots\\ \vdash ?A^{\perp}, ?A^{\perp}, \Gamma\end{array}}{\vdash ?\Delta, ?A^{\perp}, \Gamma}\,\mathsf{cut}}{\vdash ?\Delta, ?\Delta, \Gamma}\,\mathsf{cut}}{\vdots}\,\mathsf{c}}{\vdash ?\Delta, \Gamma}\,\mathsf{c}
$$

- In sequent calculus:

  rule occurrence $r \mapsto$ **sub-proof** ending on r.

- **No such thing in proof nets**!

Last rule 2

- **Intuition**:

  Internalizing a notion of **last rule**

  will internalize **boxes**

- **Partially internalized boxes**: Olivier Laurent's **polarized MELL**.
- **Abstract last rule** = **last positive rule**.
- **This work**: **totally internalized boxes** for MELL**P**.
- **Expressiveness**: MELLP codes **classical logic**/$\lambda\mu$-calculus.

Outline

1. Polarized MELL
2. Compressing polarized boxes

Polarization

Formulas:

$$\begin{array}{llllllll} P,Q ::= & X & \mid & 1 & \mid & P \otimes Q & \mid & !N \\ N,M ::= & X^{\perp} & \mid & \perp & \mid & N \parr M & \mid & ?P \end{array}$$

Sequents:

$$\vdash \Gamma\,;P \quad \text{or} \quad \vdash \Gamma\,;\_$$

Multiplicative rules:

$$\frac{\vdash \Gamma; P \qquad \vdash \Delta, P^{\perp}; [Q]}{\vdash \Gamma, \Delta; [Q]}\ \text{cut} \qquad \frac{}{\vdash P^{\perp}; P}\ \text{ax}$$

$$\frac{\vdash \Gamma; [P]}{\vdash \Gamma, \perp; [P]}\ \perp \qquad \frac{}{\vdash; 1}\ 1$$

$$\frac{\vdash \Gamma, N, M; [P]}{\vdash \Gamma, N \parr M; [P]}\ \parr \qquad \frac{\vdash \Gamma; P \qquad \vdash \Delta; Q}{\vdash \Gamma, \Delta; P \otimes Q}\ \otimes$$

Laurent's MELLP: adding exponentials

- Exponential rules:

$$\frac{\vdash \Gamma; [P]}{\vdash \Gamma, N; [P]}\ \mathsf{w} \qquad \frac{\vdash \Gamma; P}{\vdash \Gamma, ?P; \_}\ \mathsf{d}$$

$$\frac{\vdash \Gamma, N, N; [P]}{\vdash \Gamma, N; [P]}\ \mathsf{c} \qquad \frac{\vdash \Gamma, N; \_}{\vdash \Gamma; !N}\ !$$

- **Difference with linear logic**:

  Promotion, contraction, and weakening **do not need** the ? modality.

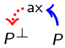
- **Important**:

  Only **positives** are duplicated/erased.

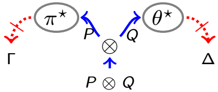
- Positives are last rules and **every positive** will have a **box**.

$$\frac{}{\vdash ; 1}\ 1 \quad\rightsquigarrow$$

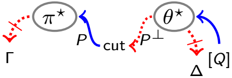
$$\frac{}{\vdash P^{\perp} ; P}\ \mathsf{ax} \quad\rightsquigarrow$$



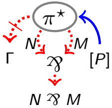
$$\frac{\vdash \Gamma ; P \qquad \vdash \Delta ; Q}{\vdash \Gamma, \Delta ; P \otimes Q}\ \otimes \quad\rightsquigarrow$$



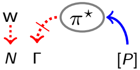
$$\frac{\vdash \Gamma ; P \qquad \vdash \Delta, P^{\perp} ; [Q]}{\vdash \Gamma, \Delta ; [Q]}\ \mathsf{cut} \quad\rightsquigarrow$$



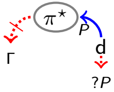
$$\frac{\vdash \Gamma, N, M ; [P]}{\vdash \Gamma, N \parr M ; [P]}\ \parr \quad\rightsquigarrow$$



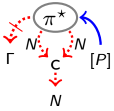
$$\frac{\vdash \Gamma; [P]}{\vdash \Gamma, N; [P]}\ \mathsf{w} \quad \rightsquigarrow$$



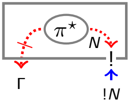
$$\frac{\vdash \Gamma; P}{\vdash \Gamma, ?P; \_}\ \mathsf{d} \quad \rightsquigarrow$$



$$\frac{\vdash \Gamma, N, N; [P]}{\vdash \Gamma, N; [P]}\ \mathsf{c} \quad \rightsquigarrow$$



$$\frac{\vdash \Gamma, N; \_}{\vdash \Gamma; !N}\ ! \quad \rightsquigarrow$$



Positive Trees

$$\frac{\vdash \Gamma; P \qquad \vdash \Delta, P^\perp; [Q]}{\vdash \Gamma, \Delta; [Q]}\ \text{cut} \qquad \frac{}{\vdash P^\perp; P}\ \text{ax}$$

$$\frac{\vdash \Gamma; [P]}{\vdash \Gamma, \perp; [P]}\ \perp \qquad \frac{}{\vdash ; 1}\ 1$$

$$\frac{\vdash \Gamma, N, M; [P]}{\vdash \Gamma, N \parr M; [P]}\ \parr \qquad \frac{\vdash \Gamma; P \qquad \vdash \Delta; Q}{\vdash \Gamma, \Delta; P \otimes Q}\ \otimes$$

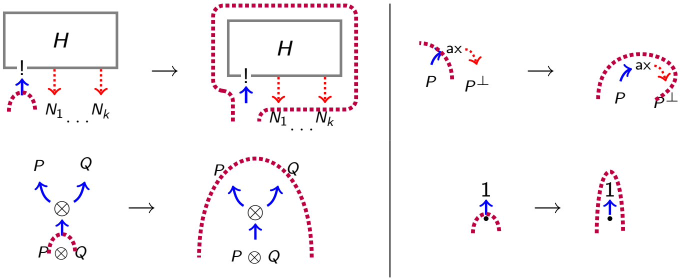
$$\frac{\vdash \Gamma; [P]}{\vdash \Gamma, N; [P]}\ \text{w} \qquad \frac{\vdash \Gamma; P}{\vdash \Gamma, ?P; \_}\ \text{d}$$

$$\frac{\vdash \Gamma, N, N; [P]}{\vdash \Gamma, N; [P]}\ \text{c} \qquad \frac{\vdash \Gamma, N; \_}{\vdash \Gamma; !N}\ !$$

Note: positives have a **forest structure**.

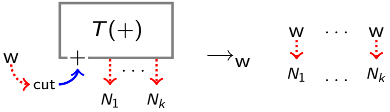
Positive Tree

- **Positive** connectives: $1, \otimes, !$.
- **Explicit** boxes for $!$ $\Rightarrow$ **induced box** for every positive:

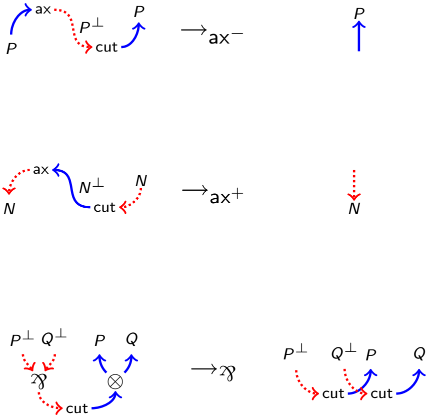


- **My contribution**: **explicit** boxes for **!** can be made **implicit**.

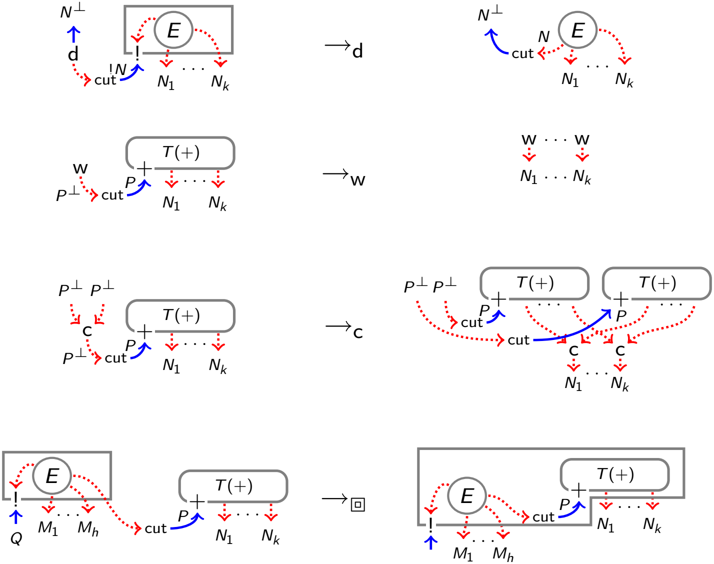
Laurent uses the **positive tree** to generalize box rules:



Polarized cut-elimination 1



Polarized cut-elimination 2



Outline

1. Polarized MELL
2. Compressing polarized boxes

Matching property

$$\frac{\vdash \Gamma; P \qquad \vdash \Delta, P^{\perp}; [Q]}{\vdash \Gamma, \Delta; [Q]}\ \mathsf{cut} \qquad \frac{}{\vdash P^{\perp}; P}\ \mathsf{ax}$$

$$\frac{\vdash \Gamma; [P]}{\vdash \Gamma, \perp; [P]}\ \perp \qquad \frac{}{\vdash ; 1}\ 1$$

$$\frac{\vdash \Gamma, N, M; [P]}{\vdash \Gamma, N \parr M; [P]}\ \parr \qquad \frac{\vdash \Gamma; P \qquad \vdash \Delta; Q}{\vdash \Gamma, \Delta; P \otimes Q}\ \otimes$$

$$\frac{\vdash \Gamma; [P]}{\vdash \Gamma, N; [P]}\ \mathsf{w} \qquad \frac{\vdash \Gamma; P}{\vdash \Gamma, ?P; \_}\ \mathsf{d}$$

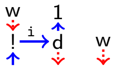
$$\frac{\vdash \Gamma, N, N; [P]}{\vdash \Gamma, N; [P]}\ \mathsf{c} \qquad \frac{\vdash \Gamma, N; \_}{\vdash \Gamma; !N}\ !$$

Matching property: every !-rule is **enabled** by a d-rule.

Materializing the matching property 1

- Consider:

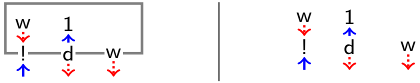
- **Problem**: without box the content is disconnected.
- **Idea**: let's materialize the matching property with an additional edge.



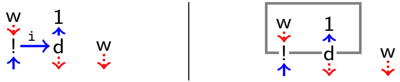
- The content and the positive sub-graphs are now **connected**.
- **The induced box**: the **positive tree** plus the **negative trees** on it.

Quotient and weakenings

- Let's do it again:



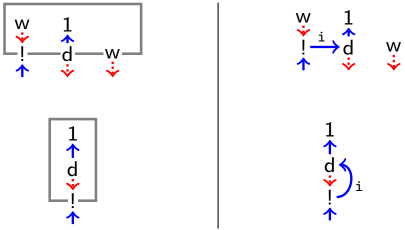
- We do not recover the original box:



- **Interpretation**: we are **quotienting** proof nets with explicit boxes.
- **Remark**: weakenings are not attached!
- ⇒ improvement over **Francois Lamarche's essential nets**.

Box borders

- Let's do it again:



- **Remark**: we are not attaching the border of the box.
- $\Rightarrow$ improvement over **Ian Mackie's interaction nets** technique.

Implicit boxes

Recipe:

- Take a **cut-free proof net**.
- **Matching**: every !-box has a **unique dereliction at level 0**.
- **Remove the explicit box** and **add the matching edge**.

Then:

- The **induced boxes** define a net with explicit boxes.
- Induced boxes are **locally reconstructable**.
- There is a simple **correctness criterion** (i.e. not *ad-hoc*).
- It is a **canonical** representation (i.e. no choice).

Summing up

In a **cut-free** proof net

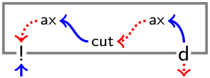
the **explicit box** of a !

can be replaced by a **single edge**

in a **canonical** and **more parallel** way.

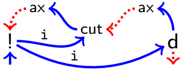
Cuts

Cuts introduce a **problem**:

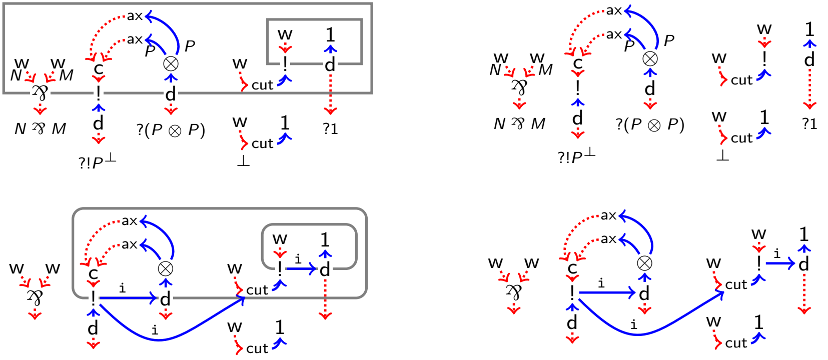


The **positive sub-graph** is **no longer connected**.

Let's iterate **the same idea**:

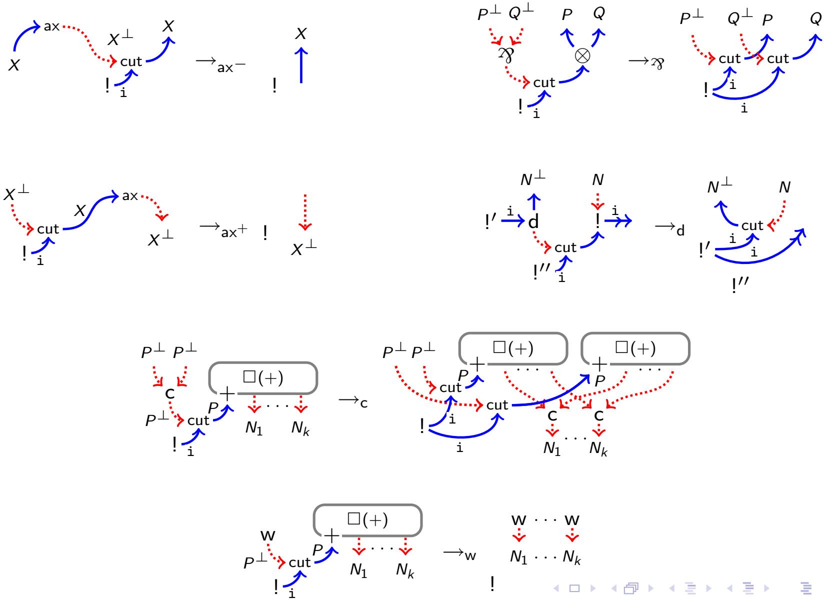


Example



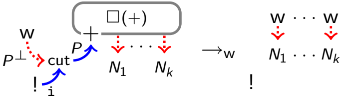
- **Implicit box**: one dereliction plus the cuts at level 0.
- **Induced box**: **positive tree** plus **negative sub-trees**.
- **Novelty**: $\parr$ commutes with box borders!

Cut elimination

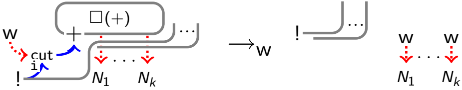


Side effects 1

- Cut elimination has '**side effects**'.
- Consider the weakening rule:



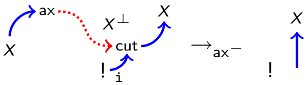
- It **automatically pushes the created weakenings out of boxes**:



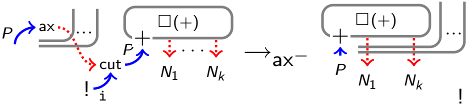
- Similarly for **contraction**.

No commutative cuts

- There is **no commutative rule**.
- It is included inside the axiom and dereliction rules.
- The axiom rule:

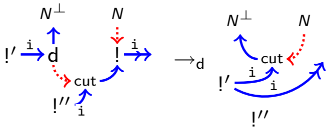


and its **action through box borders**:

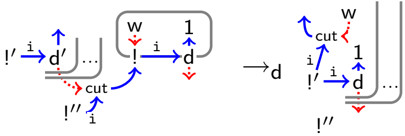


No commutative cuts 2

The dereliction rule:



and an example of its action:



Cut-elimination

On η-**expanded nets**:

- Cut elimination is **strongly normalizing** (SN).
- Confluence requires **assoc**., **comm**., and **neutrality** for contractions.
- Then: **Church-Rosser modulo** and **SN modulo** hold.

The general case:

- More and wilder '**side effects**'.
- **Difficult critical pairs** and known techniques do **not work**.
- **By-product**: new simpler proof of SN for linear logic (RTA 2013).

Conclusions

- An **alternative representation of boxes**:
  - Simple;
  - Canonical;
  - More parallel;
  - Provided of a correctness criterion;
  - A local reconstruction of boxes;
  - Results on the dynamics.
- **New perspective on polarity**.
- It already lead to **new understanding of SN** for linear logic.

THANKS!