Metric, Algorithmic, and Logical Characterisations of Probabilistic Bisimulation

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Outline

1. Probabilistic bisimulation
2. Justifying the lifting of relations
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   (b) Justification by network flow
3. Metric characterisation of probabilistic bisimulation
4. Algorithmic characterisation of probabilistic bisimulation
5. Logical characterisation of probabilistic bisimulation

Probabilistic bisimulation

Probability distributions

- A (discrete) probability distribution over a countable set S is a function $\Delta : S \to [0,1]$ s.t. $\sum_{s \in S} \Delta(s) = 1$
- The support of Δ: $\lceil \Delta \rceil := \{s \in S | \Delta(s) > 0\}$
- $\mathcal{D}(S)$: the set of all distributions over S
- $\overline{s}$: the point distribution $\overline{s}(s) = 1$
- Given distributions $\Delta_1, ..., \Delta_n$, we form their linear combination $\sum_{i \in 1..n} p_i \cdot \Delta_i$, where $\forall i : p_i > 0$ and $\sum_{i \in 1..n} p_i = 1$.

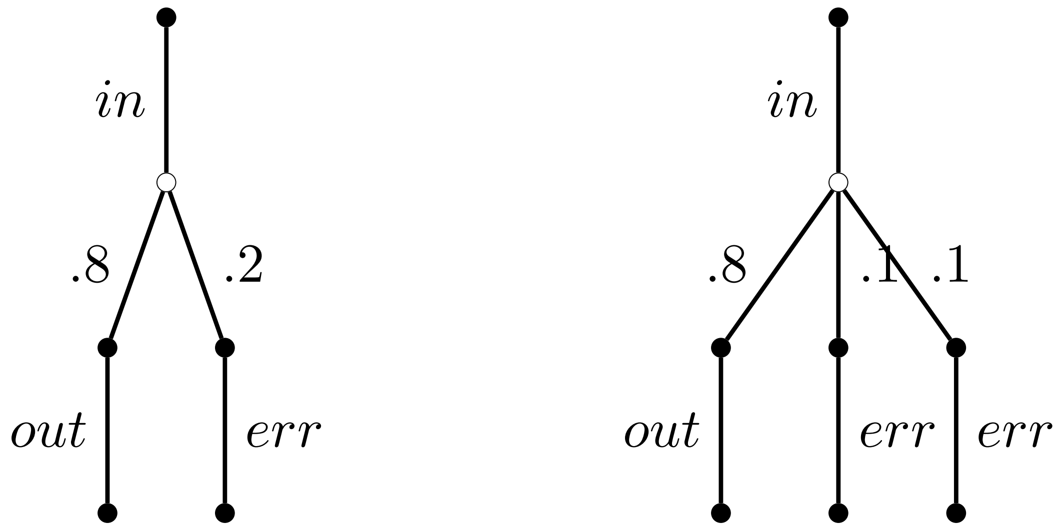
Probabilistic labelled transition systems

Def. A *probabilistic labelled transition system* (pLTS) is a triple $\langle S, Act, \rightarrow\rangle$, where

1. S is a set of states
2. Act is a set of actions
3. $\rightarrow \;\subseteq\; S \times Act \times \mathcal{D}(S)$.

We usually write $s \xrightarrow{\alpha} \Delta$ in place of $(s, \alpha, \Delta) \in \rightarrow$. An LTS may be viewed as a degenerate pLTS that only uses point distributions.

Example



Lifting relations

Def. Let $\mathcal{R} \subseteq S \times T$ be a relation between sets S and T. Then $\mathcal{R}^\dagger \subseteq \mathcal{D}(S) \times \mathcal{D}(T)$ is the smallest relation that satisfies:

1. $s \, \mathcal{R} \, t$ implies $\overline{s} \, \mathcal{R}^\dagger \, \overline{t}$
2. $\Delta_i \, \mathcal{R}^\dagger \, \Theta_i$ implies $(\sum_{i \in I} p_i \cdot \Delta_i) \, \mathcal{R}^\dagger \, (\sum_{i \in I} p_i \cdot \Theta_i)$ for any $p_i \in [0,1]$ with $\sum_{i \in I} p_i = 1$.

More discussion about the lifting operation later.

Bisimulation

Def. A binary relation $\mathcal{R} \subseteq S \times S$ is a simulation if whenever $s \mathrel{\mathcal{R}} t$:

- if $s \xrightarrow{a} \Delta$, there exists some Θ such that $t \xrightarrow{a} \Theta$ and $\Delta \mathrel{\mathcal{R}^\dagger} \Theta$.

The relation $\mathcal{R}$ is a bisimulation if both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are simulations. Bisimilarity, written $\sim$, is the union of all bisimulations.

Justifying the lifting of relations

Alternative ways of lifting (1/2)

Prop. $\Delta \, \mathcal{R}^\dagger \, \Theta$ if and only if

1. $\Delta = \sum_{i \in I} p_i \cdot \overline{s_i}$, where I is a countable index set and $\sum_{i \in I} p_i = 1$
2. For each $i \in I$ there is a state t_i such that $s_i \, \mathcal{R} \, t_i$
3. $\Theta = \sum_{i \in I} p_i \cdot \overline{t_i}$.

Alternative ways of lifting (2/2)

Prop.

1. Let Δ, Θ be distributions over S and $\mathcal{R}$ be an equivalence relation. Then
$$\Delta\ \mathcal{R}^\dagger\ \Theta \quad \text{iff} \quad \forall C \in S/\mathcal{R} : \Delta(C) = \Theta(C)$$
where $\Delta(C) = \sum_{s \in C} \Delta(s)$.

2. Let Δ and Θ be distributions over S and T, respectively. Then $\Delta\ \mathcal{R}^\dagger\ \Theta$ iff there exists a weight function $w : S \times T \to [0,1]$ such that
   (a) $\forall s \in S : \sum_{t \in T} w(s,t) = \Delta(s)$
   (b) $\forall t \in T : \sum_{s \in S} w(s,t) = \Theta(t)$
   (c) $\forall (s,t) \in S \times T : w(s,t) > 0 \Rightarrow s\ \mathcal{R}\ t$.

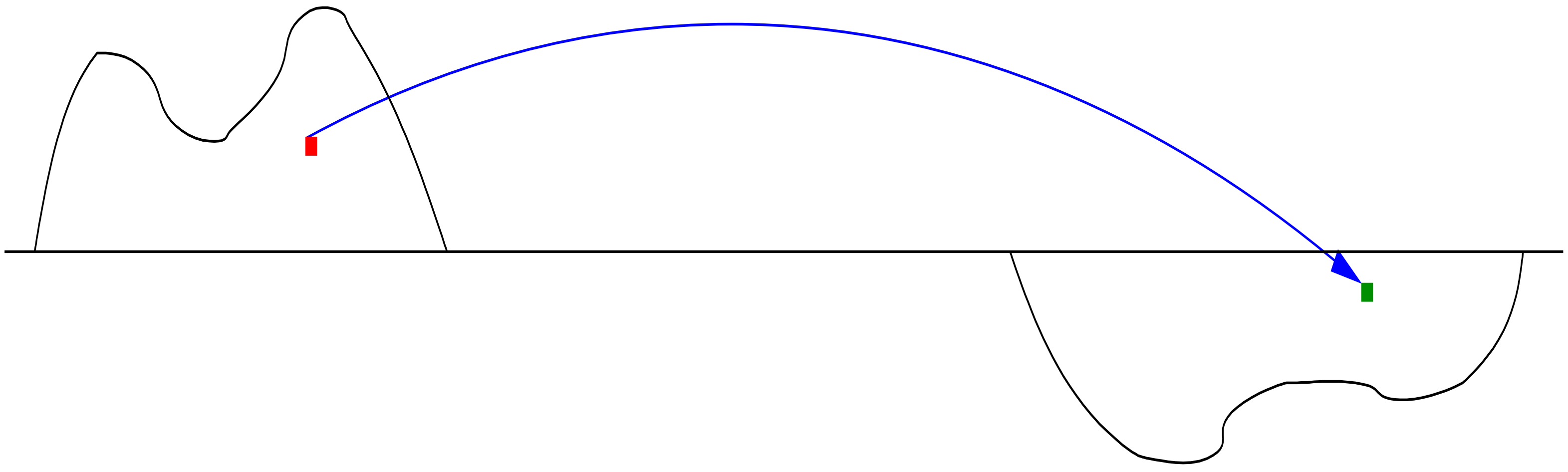
Relating the lifting operation with Kantorovich metric

The Kantorovich metric was motivated by the transportation problem.

The transportation problem

The original transportation problem (formulated by the French mathematician Gaspard Monge in 1781):

What's an optimal way of shovelling a pile of sand into a hole of the same volume?



Kantorovich metric

Def. Let (S, m) be a separable metric space. For any two Borel probability measures Δ and Θ on S, the *Kantorovich distance* between Δ and Θ is defined by

$$\hat{m}(\Delta, \Theta) = \sup \left\{ \left| \int f d\Delta - \int f d\Theta \right| : ||f|| \leq 1 \right\}.$$

where $||\cdot||$ is the *Lipschitz semi-norm* defined by $||f|| = \sup_{x \neq y} \frac{|f(x) - f(y)|}{m(x,y)}$ for a function $f : S \to \mathbb{R}$ with $\mathbb{R}$ being the set of all real numbers.

Kantorovich-Rubinstein Theorem

Write $M(\Delta, \Theta)$ for the set of all Borel probability measures on the product space $S \times S$ with marginal measures Δ and Θ, i.e. if $\Gamma \in M(\Delta, \Theta)$ then $\int_{y \in S} d\Gamma(x, y) = d\Delta(x)$ and $\int_{x \in S} d\Gamma(x, y) = d\Theta(y)$ hold.

Thm. If (S, m) is a separable metric space then for any two distributions $\Delta, \Theta \in \mathcal{D}(S)$ we have

$$\hat{m}(\Delta, \Theta) = \inf \left\{ \int m(x, y) d\Gamma(x, y) : \Gamma \in M(\Delta, \Theta) \right\}.$$

Interpretation of Kantorovich metric

Intuitively, a probability measure $\Gamma \in M(\Delta, \Theta)$ can be understood as a *transportation* from one unit mass distribution Δ to another unit mass distribution Θ. If the distance $m(x, y)$ represents the cost of moving one unit of mass from location x to location y then $\hat{m}(\Delta, \Theta)$ gives the optimal total cost of transporting the mass of Δ to Θ.

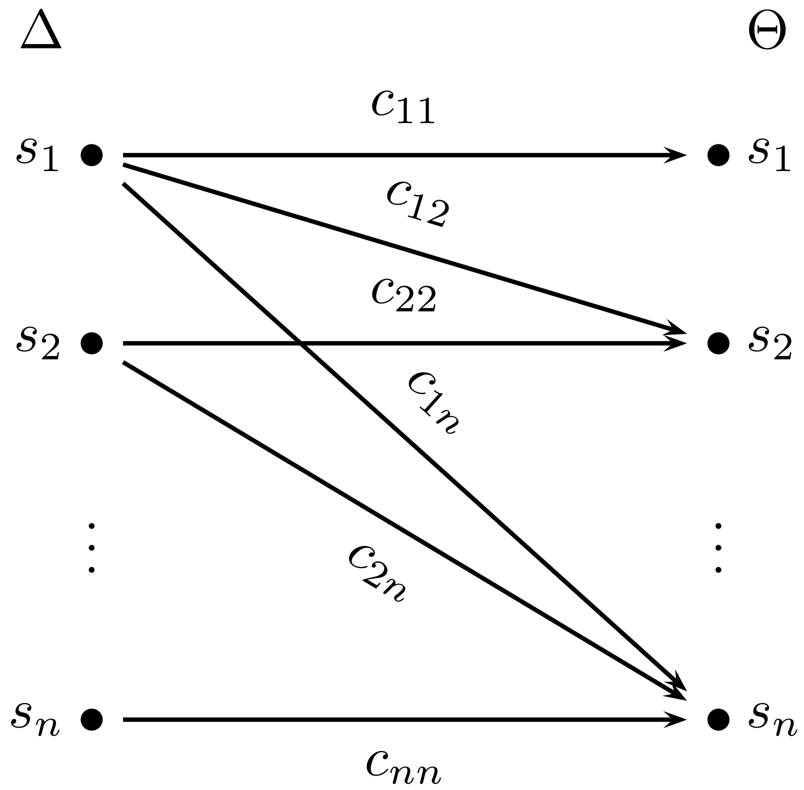
Discrete transportation problem

For two discrete distributions Δ and Θ with finite supports $\{x_1, ..., x_n\}$ and $\{y_1, ..., y_l\}$, respectively, minimizing the total cost of a discretized version of the transportation problem reduces to the following linear programming problem:

$$
\begin{array}{ll}
\text{minimize} & \sum_{i=1}^{n} \sum_{j=1}^{l} \Gamma(x_i, y_j) m(x_i, y_j) \\
\text{subject to} & \bullet\ \forall 1 \leq i \leq n : \sum_{j=1}^{l} \Gamma(x_i, y_j) = \Delta(x_i) \\
& \bullet\ \forall 1 \leq j \leq l : \sum_{i=1}^{n} \Gamma(x_i, y_j) = \Theta(y_j) \\
& \bullet\ \forall 1 \leq i \leq n, 1 \leq j \leq l : \Gamma(x_i, y_j) \geq 0.
\end{array}
\tag{1}
$$

i.e. $\hat{m}(\Delta, \Theta)$ is the minimum value of problem (1).

Discrete transportation problem



c_{ij} stands for $m(s_i, s_j)$, for all i, j

Lifting relations vs. lifting metrics

Prop. Let R be a binary relation and m a pseudometric on a state space S satisfying

$$s\ R\ t \quad \text{iff} \quad m(s,t) = 0$$

for any $s, t \in S$. Then it holds that

$$\Delta \ \ R^{\dagger} \ \ \Theta \qquad \text{iff} \qquad \hat{m}(\Delta, \Theta) = 0$$

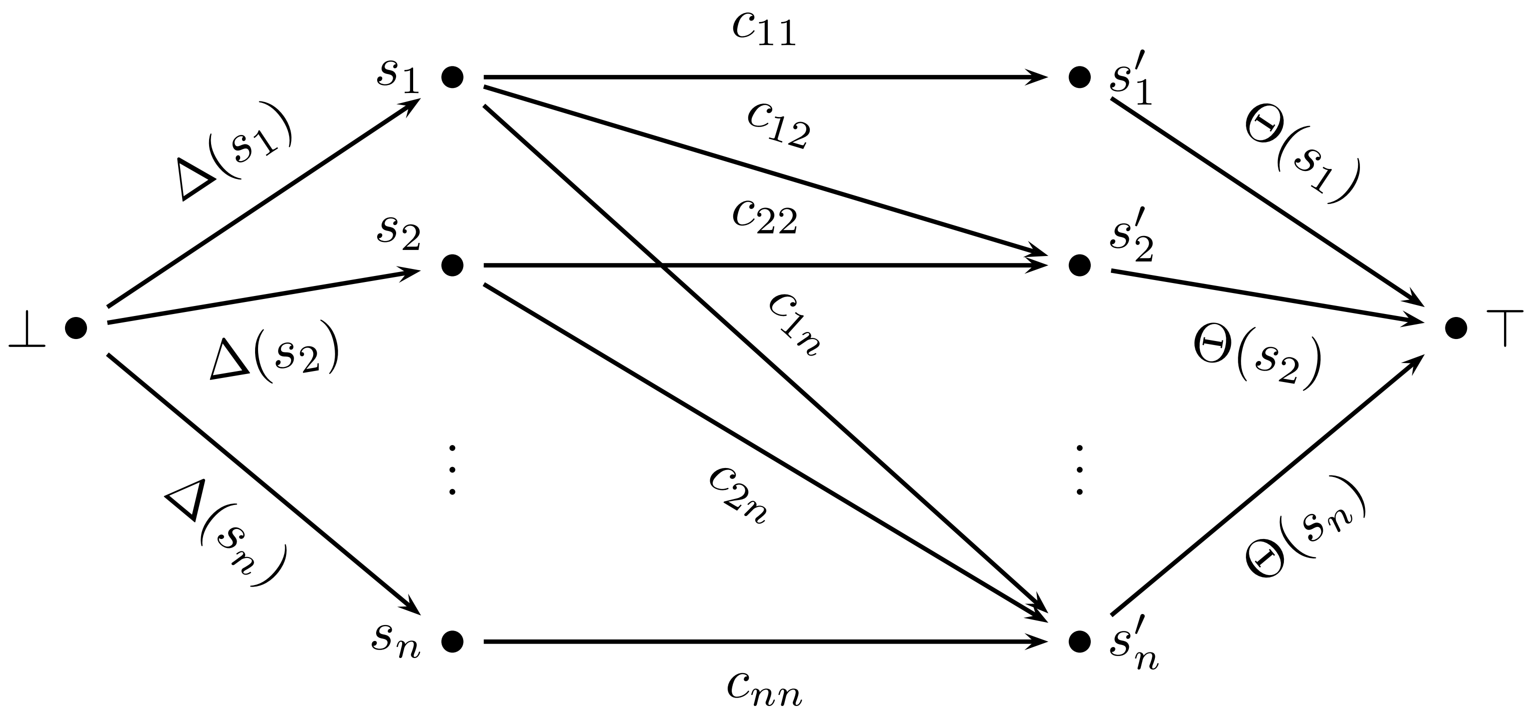
for any distributions $\Delta, \Theta \in \mathcal{D}(S)$.

Network

Def. A network is a tuple $\mathcal{N} = (N, E, \bot, \top, c)$ where

- (N, E) is a finite directed graph (i.e. N is a set of nodes and $E \subseteq N \times N$ is a set of edges)
- $\bot$ and $\top$ are the source and sink nodes respectively
- c is a capability function that assigns to each edge $(v, w) \in E$ a non-negative number $c(v, w)$.

Example



$c_{ij} = 1$ for all i, j

Flow function

Def. A flow function f for $\mathcal{N}$ is a function that assigns to each edge e a real number $f(e)$ such that

- $0 \leq f(e) \leq c(e)$ for all edges e.
- For each node $v \in N \backslash \{\perp, \top\}$,

$$\sum_{e \in in(v)} f(e) = \sum_{e \in out(v)} f(e)$$

where $in(v)$ is the set of incoming edges to node v;
$out(v)$ the set of outgoing edges from node v.

Maximum flow

Def. The flow $F(f)$ of f is given by

$$F(f) \; = \sum_{e \in out(\bot)} f(e) - \sum_{e \in in(\bot)} f(e).$$

The maximum flow in $\mathcal{N}$ is the supremum (maximum) over the flows $F(f)$, where f is a flow function in $\mathcal{N}$.

The network $\mathcal{N}(\Delta, \Theta, \mathcal{R})$

Def. Let $S' = \{s' \mid s \in S\}$ and $\bot, \top$ are two new states with $\bot, \top \notin S \cup S'$. For any $\Delta, \Theta \in \mathcal{D}(S)$ and $\mathcal{R} \subseteq S \times S$, we construct the following network $\mathcal{N}(\Delta, \Theta, \mathcal{R}) = (N, E, \bot, \top, c)$.

- $N = S \cup S' \cup \{\bot, \top\}$.
- $E = \{(s, t') \mid (s, t) \in \mathcal{R}\} \cup \{(\bot, s) \mid s \in S\} \cup \{(s', \top) \mid s \in S\}$.
- c is defined by $c(\bot, s) = \Delta(s)$, $c(t', \top) = \Theta(t)$ and $c(s, t') = 1$ for all $s, t \in S$.

Relating the Lifting operation with network flow

Lem. [Baier et al., 2000] The following statements are equivalent.

1. There exists a weight function w for (Δ, Θ) with respect to $\mathcal{R}$.
2. The maximum flow in $\mathcal{N}(\Delta, \Theta, \mathcal{R})$ is 1.

Cor. $\Delta \; \mathcal{R}^{\dagger} \; \Theta$ iff the maximum flow in $\mathcal{N}(\Delta, \Theta, \mathcal{R})$ is 1.

Metric characterisation of bisimulation

Bisimilarity might be too strong

Problem: "Quantitative" (e.g. probabilistic, timed) bisimulation might not be robust.

$$
\begin{aligned}
P &= c.([0.5]\ a.\mathbf{0} \quad \oplus \quad [0.5]\ b.\mathbf{0}) \\
Q &= c.([0.499]\ a.\mathbf{0} \quad \oplus \quad [0.501]\ b.\mathbf{0})
\end{aligned}
$$

$$P \not\sim Q$$

Metric

Problem: "Quantitative" (e.g. probabilistic, timed) bisimulation might not be robust.

$$P = c.([0.5]\, a.\mathbf{0} \oplus [0.5]\, b.\mathbf{0})$$
$$Q = c.([0.499]\, a.\mathbf{0} \oplus [0.501]\, b.\mathbf{0})$$

$$P \not\sim Q$$

Solution: Using metrics to measure the distance between two processes.

$$\mathrm{m(P,Q)} = 0.01$$

1-bounded (pseudo)metrics

Def. 1-bounded (pseudo)metric $m : S \times S \mapsto [0,1]$

1. $m(s,s) = 0$;
2. $m(s,t) = m(t,s)$;
3. $m(s,t) \leq m(s,u) + m(u,t)$.

Def. $\mathcal{M}$ is the class of 1-bounded metrics with the ordering

$$m_1 \preceq m_2 \text{ if } \forall s,t : m_1(s,t) \geq m_2(s,t).$$

Lem. $(\mathcal{M}, \preceq)$ is a complete lattice.

State-metrics

Def. For each $m \in \mathcal{M}$, we lift it to be a metric on distributions $\hat{m}$ using the Kantorovich metric.

Def. $m \in \mathcal{M}$ is a *state-metric* if, for all $\epsilon \in [0, 1)$, $m(s, t) \leq \epsilon$ implies:

- if $s \xrightarrow{\alpha} \Delta$ then there exists some Δ' such that $t \xrightarrow{\alpha} \Delta'$ and $\hat{m}(\Delta, \Delta') \leq \epsilon$.

The greatest state-metric

$$m_{max} = \bigsqcup \{m \in \mathcal{M} \mid m \text{ is a state-metric}\}.$$

State-metric as post-fixed point (1/2)

Def. Given a 1-bounded metric m on Z, the Hausdorff distance between two subsets X, Y of Z is defined as follows:

$$H_m(X,Y) = \max\{\sup_{x \in X} \inf_{y \in Y} m(x,y), \sup_{y \in Y} \inf_{x \in X} m(y,x)\}$$

where $\inf\ \emptyset = 1$ and $\sup\ \emptyset = 0$.

Def. Let $der(s,a) = \{\Delta \mid s \xrightarrow{a} \Delta\}$. $F(m)$ is a pseudometric given by:

$$F(m)(s,t) = \max_{a \in Act}\{H_{\hat{m}}(der(s,a), der(t,a))\}.$$

State-metric as post-fixed point (2/2)

Thm. There is a monotonic function on $\mathcal{M}$ such that

1. m is a state-metric iff $m \preceq F(m)$;
2. m_{max} is the greatest fixed point of F.

Bisimulation vs. state-metric

Thm. Given a binary relation $\mathcal{R}$ and a pseudometric $m \in \mathcal{M}$ such that

$$s \,\mathcal{R}\, t \;\text{ iff }\; m(s,t) = 0.$$

Then

$$\mathcal{R} \text{ is a bisimulation} \quad \text{iff} \quad m \text{ is a state-metric.}$$

Cor. $s \sim t$ iff $m_{max}(s,t) = 0$.

Algorithmic characterisation of bisimulation

Algorithms for checking probabilistic bisimilarity

For finite-state and finitely branching pLTSs,

- computing behavioural pseudometrics [vBW06]
- "partition-refinement" algorithm [BEM00]
- "on-the-fly" algorithm [DD09]
- ...

A two-stage "partition-refinement" algorithm (1/2)

$\mathcal{B} := S/\sim_A$ where $s \sim_A t$ iff $\{a \mid s \xrightarrow{a}\} = \{a \mid t \xrightarrow{a}\}$

$\mathcal{M} := \{(a, M_a) \mid a \in A\}$ where $M_a = \bigcup_{s \in S}\{\Delta \mid s \xrightarrow{a} \Delta\}$

As long as $\mathcal{B}$ or $\mathcal{M}$ can be modified perform one of the following steps:

- either choose some $B \in \mathcal{B}$ and put $\mathcal{M} := \bigcup_{(a,M)\in\mathcal{M}} \textbf{Split}_1(B, (a, M))$
- or choose some $(a, M) \in \mathcal{M}$ and put $\mathcal{B} := \bigcup_{B\in\mathcal{B}} \textbf{Split}_2((a, M), B)$

Return $\mathcal{B}$

1. $\textbf{Split}_1(B, (a, M)) = \{(a, M') \mid M' \in M/\sim_B\}$, where $\Delta \sim_B \Theta$ iff $\Delta(B) = \Theta(B)$.

2. $\textbf{Split}_2((a, M), B) = \{B_{(a,M)}, B \backslash B_{(a,M)}\}$ and $B_{(a,M)} = \{s \in B \mid s \xrightarrow{a} M\}$.

A two-stage "partition-refinement" algorithm (2/2)

Thm. [BEM00] The bisimulation equivalence classes of a pLTS with n states and m transitions can be decided in time $\mathcal{O}(mn(\log m + \log n))$ and space $\mathcal{O}(mn)$.

An "on-the-fly" algorithm (1/3)

...

MatchAction$(s, t, a) =$

for all $s \xrightarrow{a} \Delta_i$

for all $t \xrightarrow{a} \Theta_j$

$b_{ij} =$ **MatchDistribution**(Δ_i, Θ_j)

end for

end for

return $(\bigwedge_i (\bigvee_j b_{ij})) \underline{\wedge (\bigwedge_j (\bigvee_i b_{ij}))}$

MatchDistribution(Δ, Θ)=

Assume $\lceil \Delta \rceil = \{s_1, ..., s_n\}$ and $\lceil \Theta \rceil = \{t_1, ..., t_m\}$

$\mathcal{R}$:= $\{(s_i, t_j) \mid$ **Close**$(s_i, t_j) = \mathit{true}\}$

return **Check**$(\Delta, \Theta, \mathcal{R})$

...

An "on-the-fly" algorithm (2/3)

Check$(\Delta, \Theta, \mathcal{R}) =$

Input: A nonempty finite set S, distributions

$\Delta, \Theta \in \mathcal{D}(S)$ and relation $\mathcal{R} \subseteq S \times S$

Output: If $\Delta \; \mathcal{R}^\dagger \; \Theta$ then "yes" else "no"

Method:

Construct the network $\mathcal{N}(\Delta, \Theta, \mathcal{R})$

Compute the maximum flow F in $\mathcal{N}(\Delta, \Theta, \mathcal{R})$

If $F < 1$ then return "no" else "yes".

Lem. The test whether $\Delta \; \mathcal{R}^\dagger \; \Theta$ can be done in time $O(n^3/\log n)$ and space $O(n^2)$.

An "on-the-fly" algorithm (3/3)

Thm. Let s and t be two states in a pLTS with n states in total. Checking if s is bisimilar to t can be done in time $O(n^7/\log n)$ and space $O(n^2)$.

Logical characterisation of bisimulation

Adequacy and expressivity

Let $\mathcal{L}$ be a logic. The set of formulae that state s satisfies is denoted by $\mathcal{L}(s)$. Then $s =^{\mathcal{L}} t$ iff $\mathcal{L}(s) = \mathcal{L}(t)$.

- The logic $\mathcal{L}$ is adequate w.r.t. $\sim$ on a pLTS if for any states s and t,

$$s =^{\mathcal{L}} t \ \text{ iff } \ s \sim t.$$

- The logic $\mathcal{L}$ is expressive w.r.t. $\sim$ on a pLTS if for each state s there exists a characteristic formula $\varphi_s \in \mathcal{L}$ such that, for any states s and t,

$$t \models \varphi_s \ \text{ iff } \ s \sim t.$$

An adequate logic

$$\begin{array}{rcl} \varphi & := & \top \mid \varphi_1 \wedge \varphi_2 \mid \langle a \rangle \psi \mid \neg \varphi \\ \psi & := & \bigoplus_{i \in I} p_i \cdot \varphi_i \end{array}$$

- $s \models \top$ for all $s \in S$.
- $s \models \varphi_1 \wedge \varphi_2$ if $s \models \varphi_i$ for $i = 1, 2$.
- $s \models \langle a \rangle \psi$ if for some $\Delta \in \mathcal{D}(S)$, $s \xrightarrow{a} \Delta$ and $\Delta \models \psi$.
- $s \models \neg \varphi$ if it is not the case that $s \models \varphi$.
- $\Delta \models \bigoplus_{i \in I} p_i \cdot \varphi_i$ if there are $\Delta_i \in \mathcal{D}(S)$, for all $i \in I, t \in \lceil \Delta_i \rceil$, with $t \models \varphi_i$, such that $\Delta = \sum_{i \in I} p_i \cdot \Delta_i$.

Thm. $s \sim t$ iff $s =^{\mathcal{L}} t$.

Probabilistic modal μ-calculus (1/2)

Let Var be a set of variables. We define a set $\mathcal{L}_\mu$ of modal formulae in positive normal form:

$$\begin{aligned} \varphi &::= \langle a \rangle \varphi \mid [a]\varphi \mid \bigwedge_{i \in I} \varphi_i \mid \bigvee_{i \in I} \varphi_i \mid X \mid \mu X.\varphi \mid \nu X.\varphi \\ \psi &::= \bigoplus_{i \in I} p_i \cdot \varphi_i \end{aligned}$$

where $a \in Act$, I is an finite index set and $\sum_{i \in I} p_i = 1$. Let $\bigwedge_{i \in \emptyset} \varphi_i = \top$ and $\bigvee_{i \in \emptyset} \varphi_i = \bot$.

Probabilistic modal μ-calculus (2/2)

Let $Env = \{ \rho \mid \rho : Var \to \mathcal{P}(S) \}$

$[\![\ ]\!] : \mathcal{L}_\mu \to Env \to \mathcal{P}(S)$

$$
\begin{array}{rcl}
[\![\top]\!]_\rho & = & S \\
[\![\bot]\!]_\rho & = & \emptyset \\
[\![\bigwedge_{i\in I} \varphi_i]\!]_\rho & = & \bigcap_{i\in I} [\![\varphi_i]\!]_\rho \\
[\![\bigvee_{i\in I} \varphi_i]\!]_\rho & = & \bigcup_{i\in I} [\![\varphi_i]\!]_\rho \\
[\![\langle a\rangle\psi]\!]_\rho & = & \{ s \in S \mid \exists\Delta : s \xrightarrow{a} \Delta \ \wedge \ \Delta \in [\![\psi]\!]_\rho \} \\
[\![[a]\varphi]\!]_\rho & = & \{ s \in S \mid \forall\Delta : s \xrightarrow{a} \Delta \ \Rightarrow \ \Delta \in [\![\psi]\!]_\rho \} \\
[\![X]\!]_\rho & = & \rho(X) \\
[\![\mu X.\varphi]\!]_\rho & = & \bigcap\{ V \subseteq S \mid [\![\varphi]\!]_{\rho[X\mapsto V]} \subseteq V \} \\
[\![\nu X.\varphi]\!]_\rho & = & \bigcup\{ V \subseteq S \mid [\![\varphi]\!]_{\rho[X\mapsto V]} \supseteq V \} \\
[\![\bigoplus_{i\in I} p_i \cdot \varphi_i]\!]_\rho & = & \{ \Delta \in \mathcal{D}(S) \mid \Delta = \bigoplus_{i\in I} p_i \cdot \Delta_i \ \wedge \ \forall i \in I, \forall t \in \lceil\Delta_i\rceil : t \in [\![\varphi_i]\!]_\rho \}
\end{array}
$$

Equation system of formulae

Let E be a closed equation systems of formulae.

$$E : X_1 = \varphi_1$$
$$\vdots$$
$$X_n = \varphi_n$$

E viewed as a function $E : Var \to \mathcal{L}_\mu$ defined by $E(X_i) = \varphi_i$ for $i = 1, ..., n$ and $E(Y) = Y$ for other variables $Y \in Var$.

Def. An environment ρ is a solution of E if $\forall i : [\![X_i]\!]_\rho = [\![\varphi_i]\!]_\rho$.

Existence of solutions

1. The set *Env* with the partial order $\leq$ given by

$$\rho \leq \rho' \text{ iff } \forall X \in \mathit{Var} : \rho(X) \subseteq \rho'(X)$$

   forms a complete lattice.

2. The equation functional $\mathcal{E} : \mathit{Env} \to \mathit{Env}$ given by

$$\mathcal{E} := \lambda\rho.\lambda X.[\![E(X)]\!]_\rho$$

   is monotonic.

3. The Knaster-Tarski fixpoint theorem guarantees existence of solutions, and the largest solution

$$\rho_E := \bigsqcup\{\, \rho \mid \rho \leq \mathcal{E}(\rho) \,\}$$

Characteristic equation system

Def. Given a finite state pLTS, its characteristic equation system consists of one equation for each state $s_1, ..., s_n \in S$.

$$\begin{array}{rcl} E : X_{s_1} & = & \varphi_{s_1} \\ & \vdots & \\ X_{s_n} & = & \varphi_{s_n} \end{array}$$

where

$$\varphi_s := (\bigwedge_{s \xrightarrow{a} \Delta} \langle a \rangle X_\Delta) \wedge (\bigwedge_{a \in Act} [a] \bigvee_{s \xrightarrow{a} \Delta} X_\Delta)$$

with $X_\Delta := \bigoplus_{s \in \lceil \Delta \rceil} \Delta(s) \cdot X_s$.

Thm. If E is a characteristic equation system then $s \sim t$ iff $t \in \rho_E(X_s)$.

Characteristic formulae

- Rule 1: $E \to F$
- Rule 2: $E \to G$
- Rule 3: $E \to H$ if $X_n \notin fv(\varphi_1, ..., \varphi_n)$

$$
\begin{array}{rcl}
E : X_1 & = & \varphi_1 \\
& \vdots & \\
X_{n-1} & = & \varphi_{n-1} \\
X_n & = & \varphi_n
\end{array}
\qquad
\begin{array}{rcl}
F : X_1 & = & \varphi_1 \\
& \vdots & \\
X_{n-1} & = & \varphi_{n-1} \\
X_n & = & \nu X_n.\varphi_n
\end{array}
\qquad
\begin{array}{rcl}
G : X_1 & = & \varphi_1[\varphi_n/X_n] \\
& \vdots & \\
X_{n-1} & = & \varphi_{n-1}[\varphi_n/X_n] \\
X_n & = & \varphi_n
\end{array}
\qquad
\begin{array}{c}
H : X_1 = \varphi_1 \\
\vdots \\
X_{n-1} = \varphi_{n-1}
\end{array}
$$

Figure 1: Transformation rules

Thm. Given a characteristic equation system E, there is a characteristic formula φ_s such that $\rho_E(X_s) = [\![\varphi_s]\!]$ for any state s.

Summary

Summary

- A notion of probabilistic bisimulation based on a lifting operation
- The lifting is closely related to the Kantorovich metric and network flow problem
- Characterising probabilistic bisimulation via metrics, decision algorithms, and modal logics

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