Application of Algebra and Calculus in Euclidean Geometry

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Glossary

Abbreviations	
MO	Abbrev. for **Mathematical Olympiad**, the highest-level math contest for high school students in a region - for example, **VMO** (Vietnam), **USAMO** (America), **IMO** (International).
TST	Abbrev. for **Team Selection Test**, a test given to top performers in a national olympiad to select the nation's representative team for an international olympiad.

Mathematical notions	
a, b, c, r, R	Unless otherwise specified, in $\triangle ABC$, a, b, c denote the lengths of segments BC, CA, AB respectively. p is its semiperimeter, i.e., $p = \frac{1}{2}(a+b+c)$. r and R denote its inradius and circumradius respectively.
$S_{\triangle XYZ}$	Denote the area of $\triangle XYZ$.
(XYZ)	For any three non-collinear points X, Y, Z, denote the circumcircle of $\triangle XYZ$.
$(I; r)$ and (I)	Denote the circle with center I and radius r. When it is clear from the context that there is only one circle with center I, the parameter r can be omitted.
$\overline{AB}$	The algebraic length of segment AB. If $\overrightarrow{AB}$ points in the positive direction, $\overline{AB} = \lvert\overrightarrow{AB}\rvert$; otherwise, $\overline{AB} = -\lvert\overrightarrow{AB}\rvert$.
$(ABCD)$	For any four collinear points (A, B, C, D), denote the cross-ratio $$(ABCD) = \frac{\overline{AC}}{\overline{AD}} : \frac{\overline{BC}}{\overline{BD}}.$$ When $(ABCD) = -1$, A and B are harmonic conjugates of each other with respect to C and D.
(AB, CD)	The oriented angle starting at AB and ending at CD.

1 Introduction

We begin with an analysis of a simple geometry problem, proposed by Dr. Trinh Le, deputy delegate of the Vietnam IMO team.

Problem 0

In $\triangle ABC$ consider a point D on segment BC. Let I and J be the midpoints of DB and DC respectively. The perpendicular bisectors of DB and DC intersect AB at M and AC at N respectively. Let O be the circumcenter of $\triangle ABC$. Prove that $AMON$ is a cyclic quadrilateral.

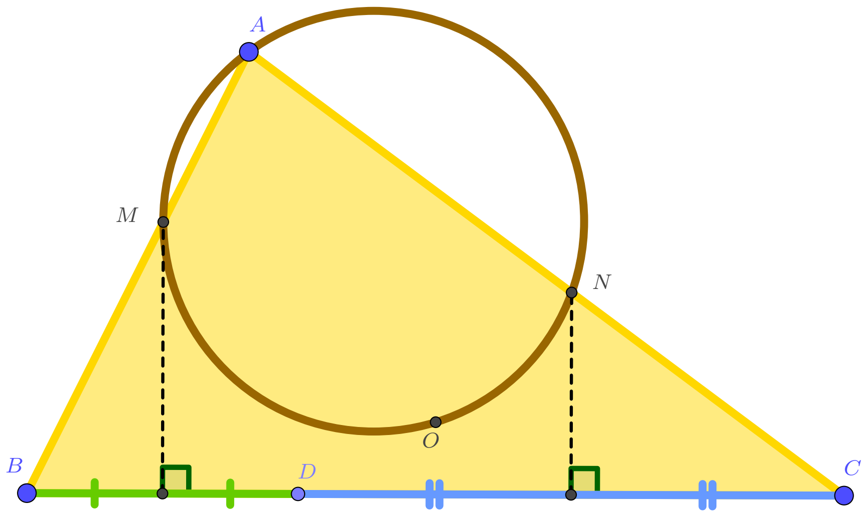
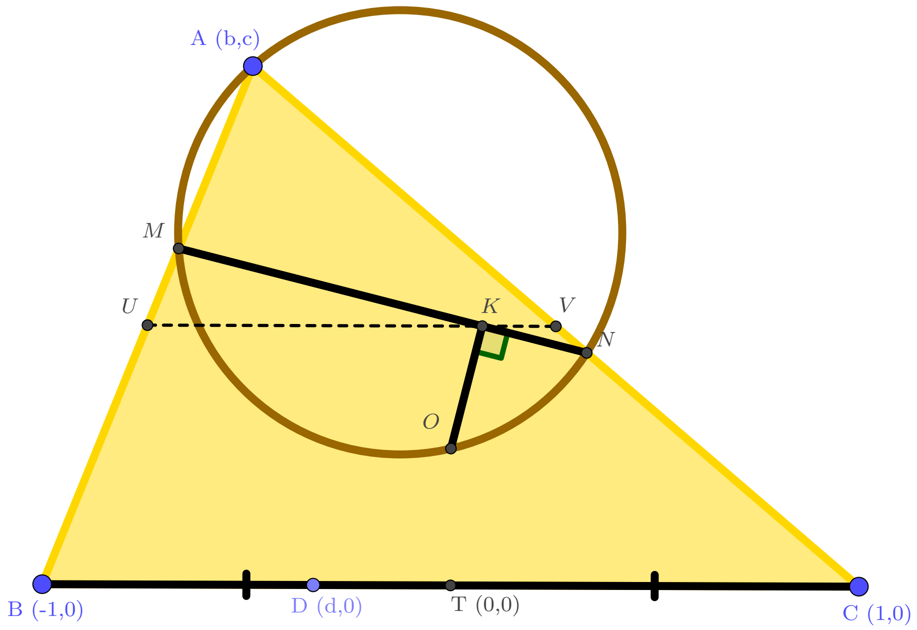


Figure 1: Prove that $AMON$ is a cyclic quadrilateral.

The following six solutions were devised by six students from the Ho Chi Minh City VMO team. Let us first take a look at their approaches and analyze the pros and cons of each.

Proof 1



Let T be the midpoint of BC. Consider the Cartesian coordinate system Txy where T is the origin, Tx is BC and Ty is the perpendicular bisector of BC.
Denote the following coordinate points: $B(-1,0), C(1,0), D(d,0)$ and $A(a,b)$.
Let U and V be the midpoints of AB and AC respectively, and let K be the projection of O on MN. Furthermore, for any line l, denote $\vec{n_l}$ as its normal vector.
We can then compute the followings:

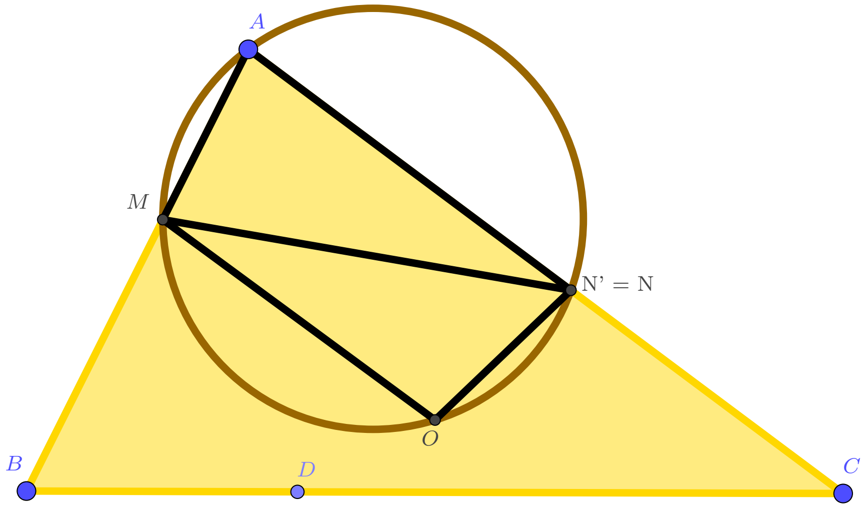
1. $(AB): y = \dfrac{b(x+1)}{a+1}, \quad (AC): y = \dfrac{b(x-1)}{a-1}.$
2. $U\left(\dfrac{a-1}{2}, \dfrac{b}{2}\right), V\left(\dfrac{a+1}{2}, \dfrac{b}{2}\right), M\left(\dfrac{d-1}{2}, \dfrac{b(d+1)}{2(a+1)}\right), N\left(\dfrac{d+1}{2}, \dfrac{b(d-1)}{2(a-1)}\right).$
3. $x_O = x_T = 0, \ \overrightarrow{OU} \parallel \overrightarrow{n_{AB}} \Rightarrow O\left(0; \dfrac{a^2+b^2-1}{2ab}\right).$
4. $(MN): \dfrac{b(a-d)}{a^2-1}\left(x - \dfrac{d-1}{2}\right) + y - \dfrac{b(d+1)}{2(a+1)} = 0.$
5. $\overrightarrow{OK} \parallel \overrightarrow{n_{MN}} \Rightarrow \dfrac{x_K}{y_K - \frac{a^2+b^2-1}{2b}} = \dfrac{b(a-d)}{a^2-1}.$

Observe that $y_U = y_v = \frac{b}{2}$, so it turns out the line UV is $(UV): y = \frac{b}{2}$. Furthermore, the point K is uniquely determined from (4) and (5), and we notice that $\left(x_K = \frac{d-a}{2}, y_K = \frac{b}{2}\right)$ satisfies both. Hence $K\left(\frac{d-a}{2}, \frac{b}{2}\right)$ and therefore $K \in UV$. By the converse Simson's theorem $AMON$ is cyclic. (QED)

Proof 2

Let $N' \neq A$ be the intersection point of (AMO) and AC. We prove $AN' = AN$, which would imply $N' \equiv N$. Denote $DB = x$, then

$$BM = \frac{x}{2\cos B}, \quad AM = c - \frac{x}{2\cos B}, \quad AN = b - \frac{a-x}{2\cos C}. \tag{1}$$



Since $AMON'$ is a cyclic quadrilateral, by Ptolemy's theorem:

$$AO \cdot MN' = AM \cdot ON' + AN' \cdot OM. \tag{2}$$

Further observe that $\angle OMN' = \angle OAC = 90^0 - B$ and $\angle ON'M = \angle OAB = 90^0 - C$. Hence, in $\triangle OMN'$, applying the sine Rule, we have

$$\frac{ON'}{OM} = \frac{\cos B}{\cos C}.$$

Multiplying both sides of (2) by $\frac{\cos C}{OM}$ we have

$$(AO\cos C) \cdot \frac{MN'}{OM} = (AM\cos C) \cdot \frac{ON'}{OM} + AN'\cos C.$$

which yields

$$\begin{aligned} AN'\cos C &= (AO\cos C) \cdot \frac{\sin\angle MON'}{\sin\angle ON'M} - (AM\cos C) \cdot \frac{\cos B}{cosC} \\ &= \left(\frac{a}{2\sin A} \cdot \cos C \cdot \frac{\sin A}{\cos C}\right) - \left(c - \frac{x}{2\cos B}\right)\cos B \\ &= \frac{a+x}{2} - c\cos B. \end{aligned}$$

From (1) we also have

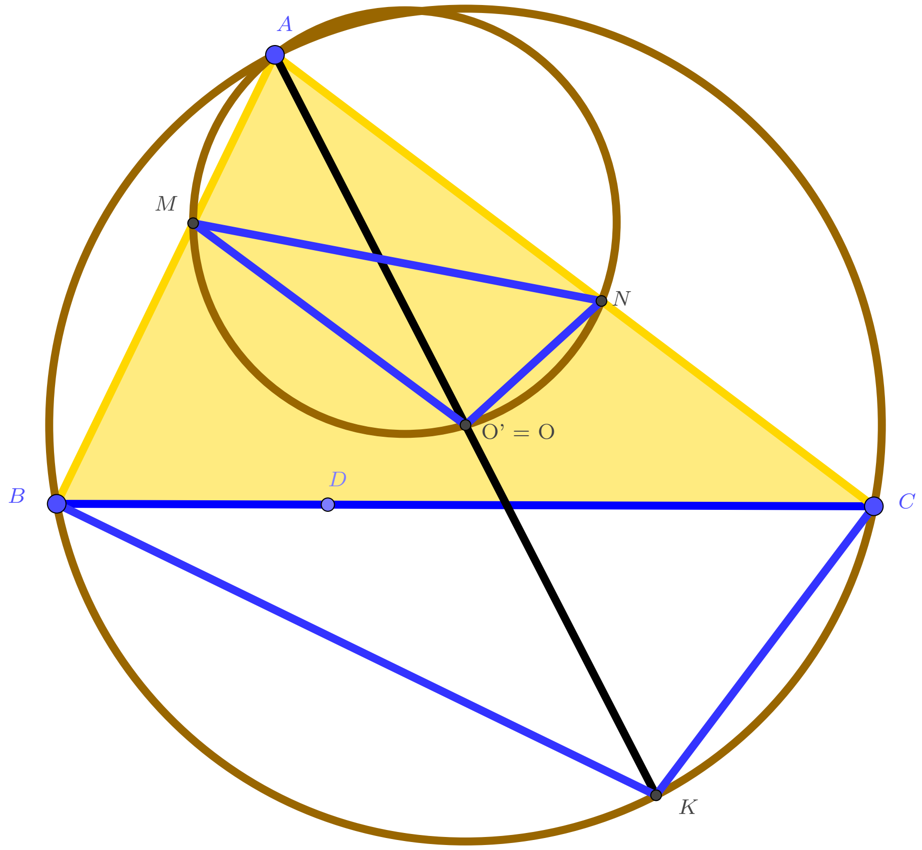
$$AN\cos C = b\cos C - \frac{a-x}{2},$$

hence $AN' = AN$ would be equivalent to

$$\frac{a+x}{2} - c\cos B = b\cos C - \frac{a-x}{2},$$

which can be rewritten as $a = b\cos C + c\cos B$, which is true. Thus $AN' = AN$ and therefore $N' \equiv N$. (QED)

Proof 3



(AMN) intersects the diameter AK of (ABC) at $O' \neq A$. We will prove that $AK = 2AO'$, which implies $O' \equiv O$.

Observe that $\triangle O'MN \sim \triangle KBC$ so

$$\frac{OM}{KB} = \frac{MN}{BC} = \frac{O'N}{KC} = k,$$

which means that

$$MN = kBC,\ O'M = kKB,\ O'N = kKC. \tag{3}$$

Applying Ptolemy's theorem to cyclic quadrilaterals $AMO'N$ and $ABKC$ we have

$$AO' \cdot MN = AM \cdot O'N + AN \cdot O'M. \tag{4}$$

$$AK \cdot BC = AB \cdot KC + AC \cdot KB. \tag{5}$$

Dividing both sides of (4) by k we have

$$AO \cdot BC = AM \cdot KC + AN \cdot KB. \tag{6}$$

From Solution 2, with $DB = x$, we also have

$$AM = c - \frac{x}{2\cos B}, AN = b - \frac{a-x}{2\cos C}.$$

Further note that that $KB = 2R\cos C$ and $KC = 2R\cos B$. Looking at (5) and (6), to show $AK = 2AO'$ we need to prove that

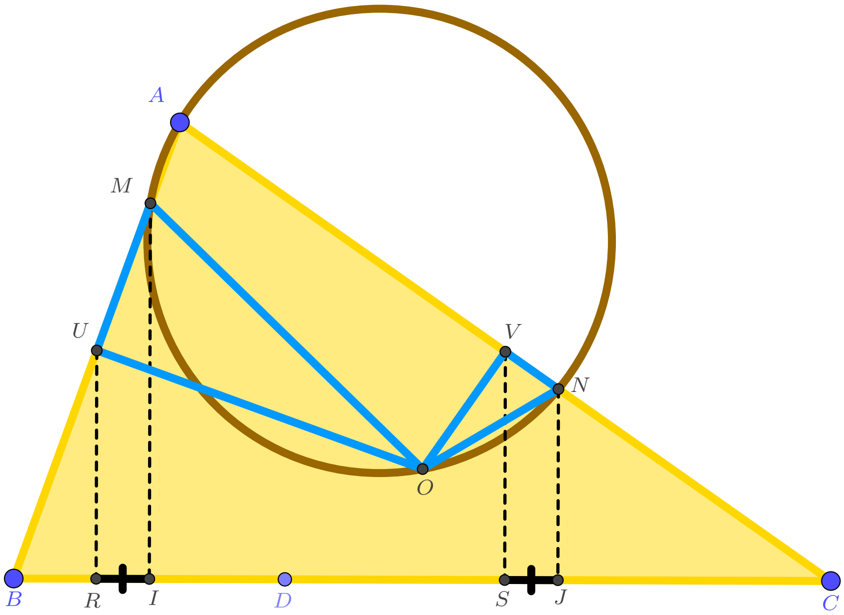
$$AB \cdot KC + AC \cdot KB = 2(AM \cdot KC + AN \cdot KB),$$

which is equivalent to

$$c\cos B + b\cos C = 2\cos B\left(c - \frac{x}{2\cos B}\right) + 2\cos C\left(b - \frac{a-x}{2\cos C}\right),$$

which reduces to $a = b\cos C + c\cos B$, which is true. Hence $O \equiv O' \in (AMN)$ and therefore $AMON$ is cyclic. (QED)

Proof 4



Let U, V be the midpoints of AB, AC respectively and R, S be the projections of U, V onto BC respectively. It is easy to see that $IR = JS$. Indeed, on axis BC, denote the direction of $\overrightarrow{BC}$ as the positive direction. We then have

$$\overline{IR} = \overline{BR} - \overline{BI} = BU\cos(BR, BU) - \frac{x}{2} = \frac{c\cos B - x}{2},$$
$$\overline{JS} = \overline{CS} - \overline{CJ} = CV\cos(CS, CA) + \frac{a-x}{2} = \frac{-b\cos C + a - x}{2}.$$

Since $a = b\cos C + c\cos B$, $\overline{IR} = \overline{JS}$ so $IR = JS$.

Applying Thales's theorem for $UR \parallel MI$ and $VS \parallel NJ$ we have

$$\frac{MU}{IR} = \frac{BU}{BR} = \frac{1}{\cos B}, \ \frac{NV}{SJ} = \frac{CN}{CJ} = \frac{1}{\cos C}.$$

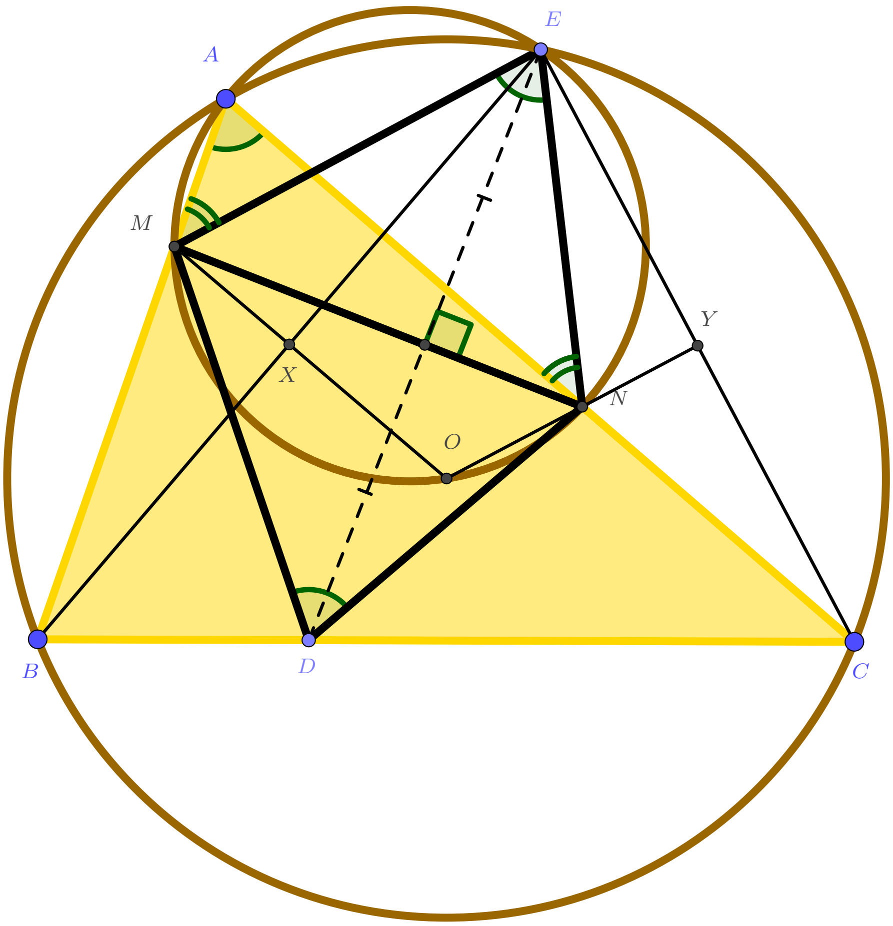
Hence

$$\frac{MU}{NV} = \frac{IR}{JS} \cdot \frac{\cos C}{\cos B} = \frac{\cos C}{\cos B} = \frac{OU}{OV}.$$

Observe that $\triangle OMU$ and $\triangle ONV$ are directionally similar. Let f be the spiral similarity with center O, angle (OU, OV) and ratio $k = \frac{OU}{OV}$, then $f(U) = V, f(M) = N$.

Hence $(MO, MU) \equiv (NO, NV) \pmod{\pi}$, so $AMON$ is a cyclic quadrilateral. (QED)

Proof 5



Let E be the reflection of D across MN. We then have

$$\angle MEN = \angle MDN = 180^0 - \angle MDB - \angle NDC = 180^0 - B - C = A.$$

Hence, $AENM$ is a cyclic quadrilateral, which implies $\angle AME = \angle ANE$. Note that $ME = MD = MB$ and $NE = ND = NC$, so $\triangle MBE$ and $\triangle NCE$ are isosceles. In other words, $\angle AME = 2\angle MBE$ and $\angle ANE = 2\angle NCE$. Therefore $\angle MBE = \angle NCE$, so $AECB$ is cylic and $E \in (ABC)$.

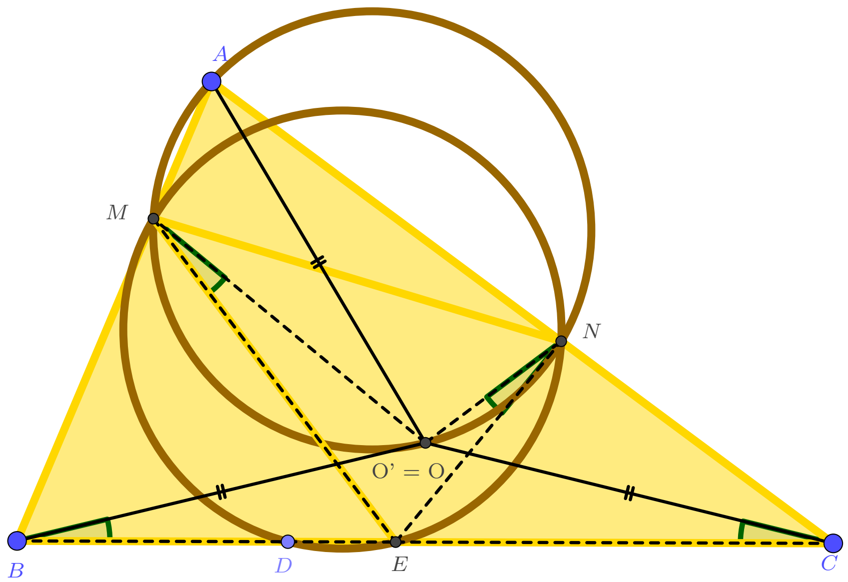
Observe that OM and ON are the perpendicular bisectors of EB and EC respectively. Let $X = OM \cap BE$, $Y = ON \cap CE$, then $EXOY$ is a cyclic quadrilateral.

Hence

$$\angle MON = 180^0 - \angle XEY = 180^0 - A,$$

which means $AMON$ is cyclic. (QED)

Proof 6



Let $E \neq D$ be the other intersection point of (DMN) and BC.

We have $\angle EMN = \angle EDN = C,\ A = \angle MDN = \angle MEN$ so $\triangle EMN \sim \triangle ACB$.

Let O' be the orthocenter of $\triangle EMN$. We observe that $BMO'E, AMO'N, CNO'E$ are cyclic quadrilaterals, hence

$$\angle O'BE = \angle O'ME = \angle O'NE = \angle O'CE,$$

so $O'C = O'B$. Similarly we have $O'A = O'B$ $(= O'C)$ and thus o' is the orthocenter of $\triangle ABC$. In other words, $O' \equiv O$.

We also have $\angle MON = 180^0 - \angle MDN = 180^0 - A$, which means $AMON$ is cyclic.

Remark 1. We can observe the relation among the six solutions given above. While all solutions involve the additional construction of some geometric objects (new point, new line, or new circle), the more sophisticated this construction is, the simpler the subsequent proof becomes.

- Solution 1 is the most simple approach: by considering the problem in the context of Cartesian coordinates. It is then straightforward to calculate the coordinates of every point. The proof does apply a small twist, however, by invoking Simson's theorem to turn the problem of cyclicity into that of collinearity, thereby avoiding equations of circles which are usually fairly complicated.

- Solution 2 and 3 "reverse" the problem by constructing the point $N' \neq A$ as the intersection of AC and (AMO). The motivation here is that, instead of having to deduce cyclicity, we assume cyclicity is already given. What remains is to show that both N' and N share a common unique property, which in turn implies that N' is indeed N. This is a recommended approach when one has not figured out how to effectively use all of the information provided by the problem.

- Solution 4 builds upon the realization that all properties of M and N depend only on the location of D on BC, i.e., the length of the segment DB. Hence we try to convert all computations to those involving only the segments on BC. Thanks to this approach, there are much less algebraic work to do.

- Solution 5 and 6 are two purely geometric solutions that one needs rich experience in geometry to come up with. In particular, the idea of solution 5 is that since $\angle MDN = A$, we create a reflection of D in order to obtain a cyclic quadrilateral. In solution 6, we use the following lemma:

  Lemma 1. *Given $\triangle ABC$ with three points X, Y, Z on BC, CA, AB respectively such that $\triangle XYZ \sim \triangle ABC$. It follows that the orthocenter of $\triangle XYZ$ is also the circumcenter of $\triangle ABC$.*

While solutions such as 5 and 6 are always desirable, it can be difficult to identify the underlying geometrical insights that in turn lead to such concise solutions, especially for students not strong in geometry[1]. The goal of this article, then, is to provide a systematic approach to tackling geometry problems using the tools from algebra and calculus. We will show how algebraic techniques along the line of Solution 1 and 2 can be utilized to solve a variety of Olympiad-level problems.

Remark 2. The given problem is in fact one in a series of similar geometric properties.

Problem 1

Let O, I and H be the circumcenter, incenter and orthocenter of $\triangle ABC$ respectively. Consider three points $D \in BC,\ M \in CA,\ N \in AB$.

- Prove that if $BD = BM$ and $CD = CN$ then $I \in (AMN)$.
- Prove that if $MD = MB$ and $ND = NC$ then $O \in (AMN)$.
- Prove that if $DM = DB$ and $DN = DC$ then $H \in (AMN)$.

[1] including us, the authors.

2 Background

We first highlight a number of theorems and solutions that are highly applicable. Note that there are a number of textbook formulas - Pythagorean's Theorem, Thales' theorem, Law of sines and Law of cosines - that are not introduced but also relevant here.

Theme 1. *Useful equatilies.*

- (Ptolemy) If $ABCD$ is a cyclic quadrilateral then

$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

- (Stewart) For any collinear points A, B, C and arbitrary point M,

$$MA^2 \cdot \overline{BC} + MB^2 \cdot \overline{CA} + MC^2 \cdot \overline{AB} = -\overline{AB} \cdot \overline{BC} \cdot \overline{CA}.$$

- (Carnot) For any four arbitrary points A, B, C, D,

$$AD \perp BC \Leftrightarrow AB^2 - AC^2 = DB^2 - DC^2.$$

- (Heron) The area of $\triangle ABC$ with side lengths a, b and c and semiperimeter $p = \frac{a+b+c}{2}$ is

$$S_{\triangle ABC} = \sqrt{p(p-a)(p-b)(p-c)}.$$

Theme 2. *Triangle lengths.*

Given $\triangle ABC$ $(AB < AC)$ with altitude AD and bisector AF. Let E be the tangent point of $\triangle ABC$'s incircle and BC. We then see that

$$BD = \frac{a^2 + c^2 - b^2}{2a}, \quad BE = \frac{a+c-b}{2}, \quad BF = \frac{ac}{b+c},$$

which in turn yields

$$DE = BE - BD = \frac{(b-c)(b+c-a)}{2a}.$$

Furthermore, from Stewart's theorem, we have

$$AD = \sqrt{AB \cdot AC - DB \cdot DC} = \frac{2\sqrt{bcp(p-a)}}{b+c} = \frac{2bc}{b+c} \cdot \cos\frac{A}{2}.$$

Theme 3. *Trigonometric expressions.*

In any $\triangle ABC$, we have

$$\cos\frac{A}{2} = \sqrt{\frac{p(p-a)}{bc}}, \quad \tan\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}},$$
$$\cot A = \frac{b^2 + c^2 - a^2}{4S}, \quad \cot\frac{A}{2} = \frac{b+c-a}{2r}.$$

Theme 4. *Concurrency and collinearity.*

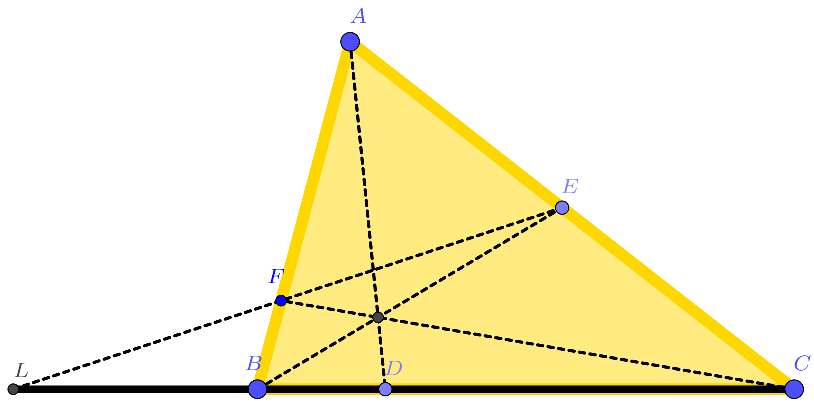
In $\triangle ABC$ consider points $D \in BC, E \in CA, F \in AB$. Let L be another point on BC.

- (Ceva) AD, BE, CF are concurrent if and only if

$$\frac{\overline{DB}}{\overline{DC}} \cdot \frac{\overline{EA}}{\overline{EB}} \cdot \frac{\overline{FC}}{\overline{FA}} = -1.$$

- (Menelaus) L, E, F are collinear if and only if

$$\frac{\overline{LB}}{\overline{LC}} \cdot \frac{\overline{EA}}{\overline{EB}} \cdot \frac{\overline{FC}}{\overline{FA}} = 1.$$



Combining the two theorems also shows that

$$\frac{\overline{DB}}{\overline{DC}} : \frac{\overline{LB}}{\overline{LC}} = -1.$$

In other words, if AD, BE, CF are concurrent and L, E, F are collinear then $(DLBC) = -1 = (LDBC)$.

Theme 5. *Oriented angles.*

Oriented angles are considered in mod π and have the following properties:

- $(AB, CD) \equiv (AB, DC) \equiv (BA, CD)$.
- $a \parallel b$ if and only if $(a, b) \equiv 0$.
- $a \perp b$ if and only if $(a, b) \equiv \frac{\pi}{2}$.
- $(a, b) \equiv -(b, a)$.
- $(a, b) \equiv (a, c) + (c, b)$.

- A, B, C, D are cyclic if and only if $(AB, AD) \equiv (CB, CD)$.
- AT is tangent to (ABC) if and only if $(AT, AB) \equiv (CA, CB)$.

Theme 6. *Cartesian coordinates.*

Besides textbook formulas on vectors and line equations, we introduce the expressions of several useful geometric constructs in the context of Cartesian coordinates.
Given $\triangle ABC$ with centroid G, incenter I, orthocenter H, circumcenter O and excenter J opposite to A, we have:

- $a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC} = \overrightarrow{0}$, so

$$I\left(\frac{ax_A + bx_B + cx_C}{a+b+c}, \frac{ay_A + by_B + cy_C}{a+b+c}\right).$$

- $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{0}$, so

$$G\left(\frac{x_A + x_B + x_C}{3}, \frac{y_A + y_B + y_C}{3}\right).$$

- $\overrightarrow{AH} \cdot \overrightarrow{BC} = 0$ and $\overrightarrow{BH} \cdot \overrightarrow{AC} = 0$, so

$$\begin{aligned} x_H(x_C - x_B) + y_H(y_C - y_B) &= x_A(x_C - x_B) + y_A(y_C - y_B), \\ x_H(x_C - x_A) + y_H(y_C - y_A) &= x_B(x_C - x_A) + y_B(y_C - y_A). \end{aligned}$$

- $OA^2 = OB^2 = OC^2$, so

$$\begin{aligned} &- 2x_Ox_A + x_A^2 - 2y_Oy_A + y_A^2 \\ =&- 2x_Ox_B + x_B^2 - 2y_Oy_B + y_B^2 \\ =&- 2x_Ox_C + x_C^2 - 2y_Oy_C + y_C^2. \end{aligned}$$

- $(\overrightarrow{AJ}, \overrightarrow{AB}) = (\overrightarrow{AJ}, \overrightarrow{AC})$ and $(\overrightarrow{BJ}, \overrightarrow{BC}) = (\overrightarrow{BJ}, \overrightarrow{AB})$, so

$$\frac{\overrightarrow{AJ} \cdot \overrightarrow{AB}}{c} = \frac{\overrightarrow{AJ} \cdot \overrightarrow{AC}}{b}, \quad \frac{\overrightarrow{BJ} \cdot \overrightarrow{BC}}{a} = \frac{\overrightarrow{BJ} \cdot \overrightarrow{AB}}{c},$$

  which yields

$$J\left(\frac{-ax_A + bx_B + cx_C}{-a+b+c}, \frac{-ay_A + by_B + cy_C}{-a+b+c}\right).$$

3 Example problems

Problem 2

Consider $\triangle ABC$ with incircle (I) touching BC at D. Let DE be a diameter of (I). The excircle to vertex A touches BC at L. Prove that A, E, L are collinear.

Proof. We know from the definition of L that $CL = BD = \frac{a+c-b}{2}$. To prove that A, E, L are collinear, let L' be the intersection of AE and BC. It remains to show that $CL' = CL$, so $L' \equiv L$. To make use of $DE \perp BC$ let's create an altitude AH. In this case, $\triangle AHL$ has $DE \parallel AH$ - since we already know DH, DE and AH, Thales would give us CL', as desired.

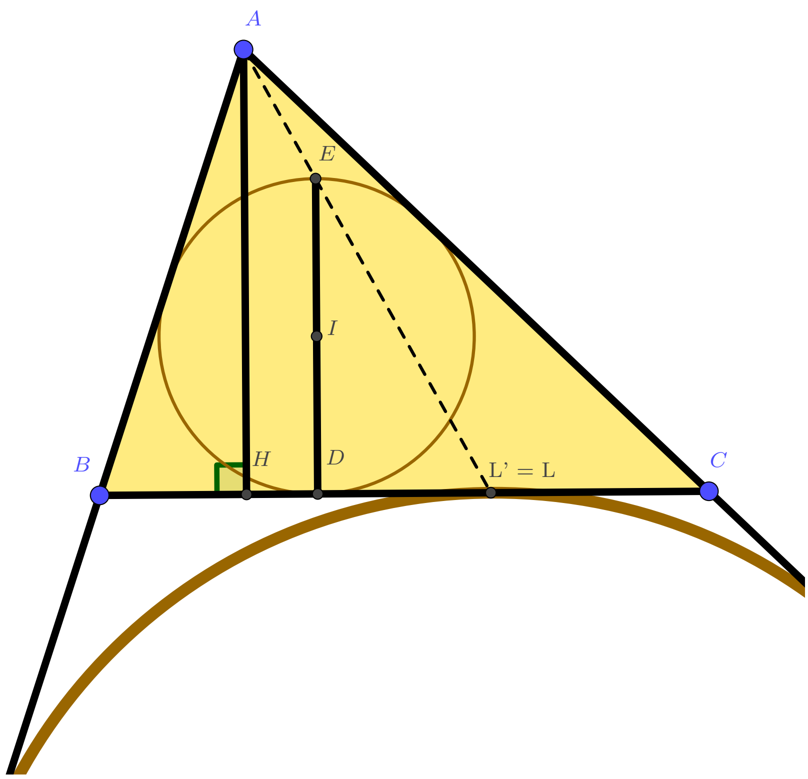


Figure 2: Prove that A, E, L are collinear.

Let L' be the intersection of AE and BC. Draw an altitude AH of $\triangle ABC$. WLOG assume $AB \leq AC$, so H lies between B and D.

Let $BL' = x$. As $DE \parallel AH$ we have

$$\frac{DL'}{HL'} = \frac{ED}{AH} = \frac{2r}{AH} = \frac{a}{p} = \frac{2a}{a+b+c},$$

which leads to

$$\frac{x - BD}{x - BH} = \frac{DL'}{HL'} = \frac{2a}{a+b+c}.$$

Having

$$BH = \frac{a^2 + c^2 - b^2}{2a}, \; BD = \frac{a+c-b}{2}$$

then gives us

$$x = \frac{-2a \cdot BH + (a+b+c) \cdot BD}{b+c-a} = \frac{b+a-c}{2},$$

so

$$CL' = a - x = \frac{a+c-b}{2} = CL.$$

In other words, $L' \equiv L$ and therefore A, E, L are collinear. □

Remark 3. This example serves as a strong lemma for many Olympiad problems. Let's take a look at one of them next.

Problem 3: USAMO 2001

The incircle (I) of $\triangle ABC$ touches CA and CB at E_1 and D_1 respectively. Consider $E_2 \in CA$ and $D_2 \in CB$ such that $CE_2 = AE_1$ and $CD_2 = BD_1$. AD_2 intersects BE_2 at P. Let Q be the intersection of (I) and AD_2 that is closer to A. Prove that $AQ = D_2P$.

Proof. Construct an altitude AH of $\triangle ABC$. As $BD_2 = CD_1 = \frac{a+b-c}{2}$, D_1Q is in fact a diameter of (I), so $D_1Q \parallel AH$, from which we can calculate AQ.

To make use of the intersecion P, we can apply Menelaus to the transversal line BPE_2 of $\triangle CAD_2$, which would give us PD_2.

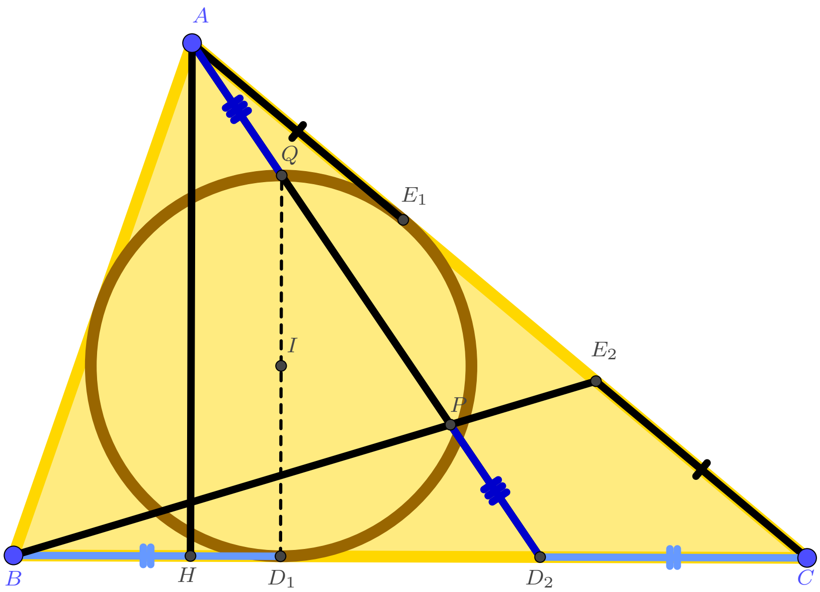


Figure 3: Prove that $AQ = DP_2$.

Construct an altitude AH of $\triangle ABC$. According to Problem 2, D_1Q is the diameter of (I), so $AH \parallel D_1Q$. This would imply that

$$\frac{QD_2}{AD_2} = \frac{QD_1}{AH} \Rightarrow \frac{AD_2 - QD_2}{AD_2} = \frac{AH - QD_1}{AH}.$$

In other words,

$$\frac{AQ}{AD_2} = \frac{AH - QD_1}{AH} = \frac{\frac{2S}{a} - \frac{2S}{p}}{\frac{2S}{a}} = \frac{p-a}{p} = \frac{b+c-a}{b+c+a}. \tag{7}$$

According to Menelaus, since BPE_2 is a transversal line of $\triangle CAD_2$,

$$\frac{PD_2}{PA} \cdot \frac{BC}{BD_2} \cdot \frac{E_2A}{E_2C} = 1$$
$$\frac{PD_2}{PA} \cdot \frac{2a}{a+b-c} \cdot \frac{a+b-c}{b+c-a} = 1$$
$$\frac{PD_2}{PA} = \frac{b+c-a}{2a}$$
$$\frac{PD_2}{AD_2} = \frac{b+c-a}{b+c-a+2a} = \frac{b+c-a}{b+c+a}. \tag{8}$$

From (7) and (8) we have $AQ = D_2P$. □

Remark 4. Many more applications of the lemma introduced in Problem 2 can be found in [1]. For now we will move on to a different application of Menelaus' theorem.

Problem 4: USAMO 2008

In an acute, scalene triangle $\triangle ABC$, let M, N, P be the midpoints of BC, CA, AB respectively. The perpendicular bisectors of AB and AC intersect ray AM at D and E respectively. BD intersects CE at F. Prove that $APFN$ is a cyclic quadrilateral.

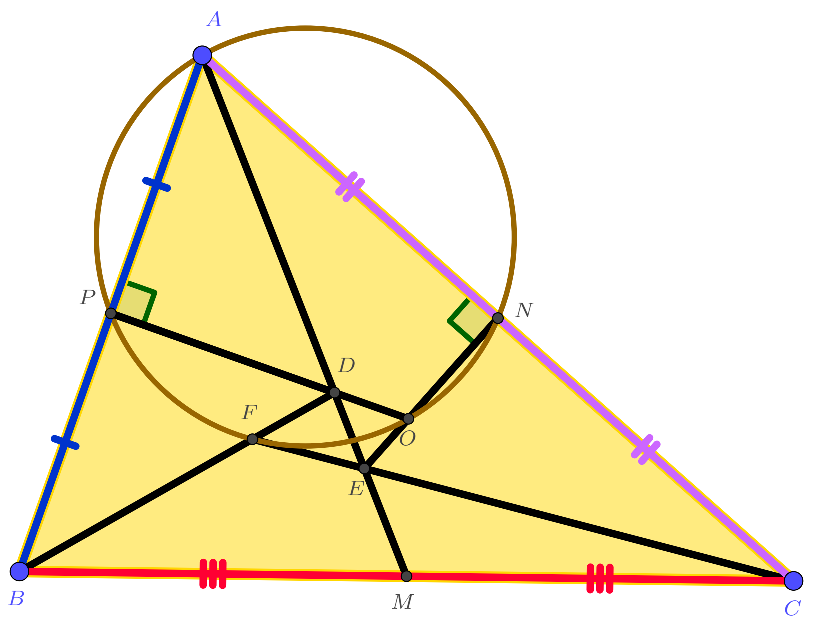


Figure 4: Prove that $APFN$ is a cylic quadrilateral.

Proof. To make use of the intersection F, we can apply Menelaus to the transversal line DEM or FEC. Note that using DEM would involve the ratio $\frac{MB}{MC} = 1$, since M is the midpoint of BC; this would make the calculations simpler.

It's easy to see that $DA = DB$ and $FA = FC$. Let O be the circumcenter of $\triangle ABC$, then O lies on both EN and DP.

$\triangle BFC$ has transversal line DEM so by Menelaus:

$$1 = \frac{DF}{DB} \cdot \frac{EC}{EF} \cdot \frac{MB}{MC} = \frac{DF}{DB} \cdot \frac{EC}{EF},$$

so

$$\frac{DF}{EF} = \frac{DB}{EC} = \frac{AD}{AE}.$$

Hence FA is the external bisector at F of $\triangle DEF$. Observe that $\angle APO = \angle ANO = 90^\circ$ so $APON$ is cyclic. To show that $APFN$ is also cyclic, we will prove that $\angle AFO = 90^\circ$ as well, or, in other words, FO is the internal bisector at F of $\triangle DEF$.

Indeed, $\triangle ADB$ has $DA = DB$ and median DP so DP is also the angular bisector of $\angle ADB$. In other words, DO is the external bisector at D of $\triangle DEF$. Similarly, EO is the external bisector at F of $\triangle DEF$. Hence O is the excenter relative to F of $\triangle DEF$. It then follows that FO is the internal bisector at F of $\triangle DEF$. □

Problem 5: VMO 2006, Board A

Let $ABCD$ be a convex quadrilateral. Consider a moving point M on AB such that $M \neq A$ and $M \neq B$. Let $N \neq M$ be the other intersection of (MAC) and (MBD). Prove that N is always on a fixed circle and MN always goes through a fixed point.

Proof. Let I be the intersection of AC and BD, then

$$\begin{aligned}(CI, CN) &\equiv (CA, CN) \equiv (MA, MN) \\ &\equiv (MB, MN) \equiv (DB, DN) \\ &\equiv (DI, DN).\end{aligned}$$

So C, I, D, N are cyclic or, in other words, N is always on the fixed circle (CDI).

Let t be the line that goes through I and is parallel to AB. t intersects MN at K.

Observe $(MA, MN) \equiv (KI, KN)$ because $AM \parallel KI$, while $(MA, MN) \equiv (CI, CN)$ as well because A, M, C, N are cyclic. Hence $(KI, KN) \equiv (CI, CN)$ so C, I, K, N are cyclic. It follows that K is the other intersection of t and (CDI) besides I.

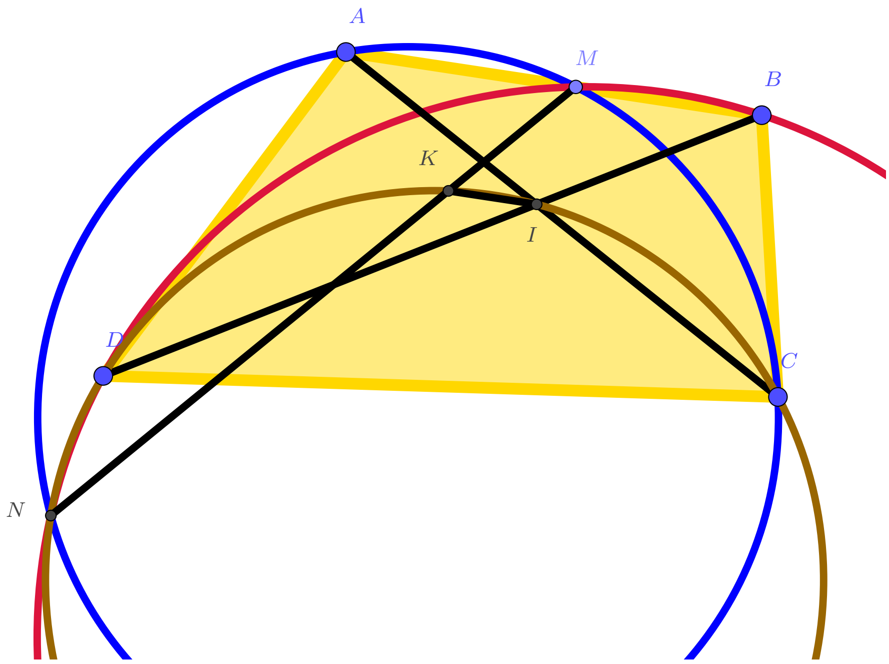


Figure 5: Prove that $APFN$ is a cylic quadrilateral.

As both t and (CDI) are fixed, K is a fixed point, and this is the point that MN always goes through. □

Remark 5. This problem is an example of when oriented angles work best. Due to the positioning of M, it's very hard to prove that C, I, D, N are cyclic with normal angles; both cases of $\angle ICN = \angle IDN$ or $\angle ICN = 180° - \angle IDN$ can happen under different circumstances. With oriented angle, we can simply express this condition as $(CI, CN) \equiv (DI, DN)$.

Problem 6: Vietnam TST 2014

Prove that:

(a) In $\triangle ABC$ with altitude AD, consider a point P on AD. PB intersects AC at E and PC intersects AB at F. If $AEDF$ is a cyclic quadrilateral, prove that

$$\frac{PA}{PD} = (\tan B + \tan C) \cdot \cot \frac{A}{2}.$$

(b) Consider $\triangle ABC$ with orthocenter H. Let P be a moving point on AH. The line perpendicular to AC at C intersects BP at M. The line perpendicular to AB at B intersects CP at N. Let K be the projection of A on MN. Prove that $\angle BKC + \angle MAN$ is constant.

Proof. We first state, without proof, a common application of the harmonic series:

Lemma 2. *Consider $\triangle ABC$ with altitude AD and a point P on AD. BP intersects AC at E and CP intersects AB at F. Then DA is the angular bisector of $\angle DHF$.*

Using this lemma, we see that $\angle AEF = \angle ADF = \angle ADE = \angle AFE$, so $\triangle AEF$ is isosceles at A. Now consider two cases:

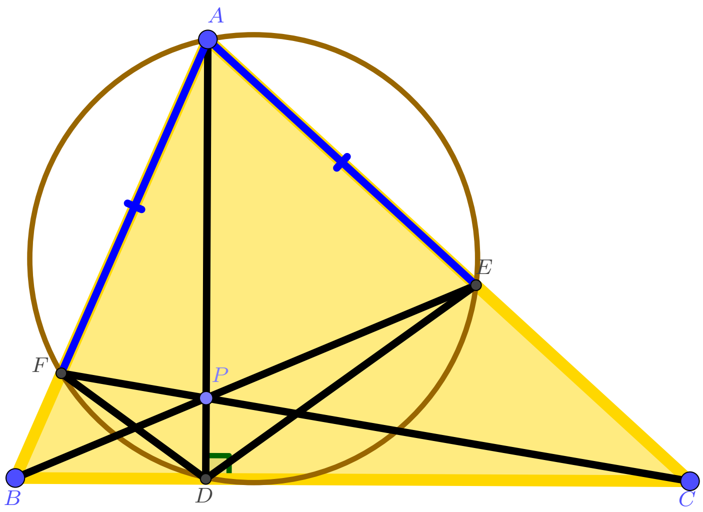


Figure 6: Prove that $\frac{PA}{PD} = (\tan B + \tan C) \cdot \cot \frac{A}{2}$.

(a)
- If $AB = AC$ then AD is the angular bisector of A. It then follows that $DF \perp AB, DE \perp AC$. Hence
$$EC = \frac{DC^2}{AC} = \frac{a^2}{4b}, \; EA = AC - EC = \frac{4b^2 - a^2}{4b}.$$
Applying Menelaus to the transversal line EPB of $\triangle ADC$ yields
$$\frac{PA}{PD} = \frac{EA}{EC} \cdot \frac{BC}{BD} = \frac{4b^2 - a^2}{2a^2}.$$
Note that $\angle B = \angle C = 90^\circ - \frac{\angle A}{2}$, so we also have
$$(\tan B + \tan C) \cdot \cot \frac{A}{2} = 2\cot^2 \frac{A}{2} = \frac{(b + c - a)^2}{2r^2} = \frac{(2b - a)^2}{r^2},$$
Using algebraic manipulations and noting that $\triangle ABC$ has $b = c, r = \frac{S}{p}$, we see that
$$\begin{aligned}\frac{(2b - a)^2}{r^2} &= (2b - a)^2 \cdot \frac{p^2}{S^2} \\ &= (2b - a)^2 \cdot \frac{p^2}{p(p - a)(p - b)(p - c)} \\ &= (2b - a)^2 \cdot \frac{4(2b + a)^2}{a^2(2b + a)(2b - a)} \\ &= \frac{4b^2 - a^2}{2a^2}.\end{aligned}$$
Hence $\frac{PA}{PD} = (\tan B + \tan C) \cot \frac{A}{2}$.

- If $AB \neq AC$ then $DB \neq DC$. Applying Lemma 2, we see that DA is the angular bisector of $\angle EDF$. Furthermore, $AFDE$ is cyclic, so $AE = AF = x$.

  We now see that

$$DB = \frac{a^2 + c^2 - b^2}{2a}, \ DC = \frac{a^2 + b^2 - c^2}{2a}, AD = \frac{2S}{a}, \ \cot\frac{A}{2} = \frac{b + c - a}{2r},$$

  which implies

$$\begin{aligned}(\tan B + \tan C) \cdot \cot\frac{A}{2} &= AD\left(\frac{1}{DB} + \frac{1}{DC}\right) \cdot \cot\frac{A}{2} \\ &= \frac{2S}{a}\left(\frac{2a}{a^2 + c^2 - b^2} + \frac{2a}{a^2 + b^2 - c^2}\right) \cdot \frac{b + c - a}{2r} \\ &= \frac{2a^2(b + c - a)(b + c + a)}{(a^2 + c^2 - b^2)(a^2 + b^2 - c^2)}. \qquad (9)\end{aligned}$$

  As AD, BE, CF are collinear, according to Ceva,

$$1 = \frac{DB}{DC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = \frac{DB}{DC} \cdot \frac{x}{c - x} \cdot \frac{b - x}{x}.$$

  which yields

$$x = \frac{b \cdot DB - c \cdot DC}{DB - DC}, \ b - x = \frac{DC(c - b)}{DB - DC}.$$

  Applying Menelaus to the transversal line EPB of $\triangle ADC$:

$$\frac{PA}{PD} = \frac{EA}{EC} \cdot \frac{BC}{BD} = \frac{x}{b - x} \cdot \frac{a}{BD} = \frac{b \cdot DB - c \cdot DC}{DC(c - b)} \cdot \frac{a}{DB}. \qquad (10)$$

  Substituting $DB = \frac{a^2+c^2-b^2}{2a}$ and $DC = \frac{a^2+b^2-c^2}{2a}$ into (10) and simplifying the expression, we get

$$\frac{PA}{PD} = \frac{2a^2(b + c - a)(b + c + a)}{(a^2 + c^2 - b^2)(a^2 + b^2 - c^2)} = (\tan B + \tan C) \cdot \cot\frac{A}{2}. \qquad (11)$$

(b) Let I be the intersection of NB and MC, then I lies on (O).

Note that $AKBN$ and $AKCM$ are cyclic, so

$$\angle NAM + \angle BKC = \angle KBI + \angle KCI + \angle BKC = 360^\circ - \angle BIC,$$

which is constant.

□

Problem 7: VMO 2008

Consider $\triangle ABC$ with median AD and a line d perpendicular to AD. Let M be a mobile point on d and let E, F be the midpoints of MB, MC respectively. The line passing through E and perpendicular to d intersects AB at P. The line passing through F and perpendicular to d intersects AC at Q. Let d' be the line passing through M and perpendicular to PQ. Prove that d' always goes through a fixed point.

Proof. Since this problem only concerns perpendicular and parallel lines, a natural solution would be to consider it the context of a two-dimensional coordinate Oxy. To simplify subsequent calculations we will pick D as the origin.

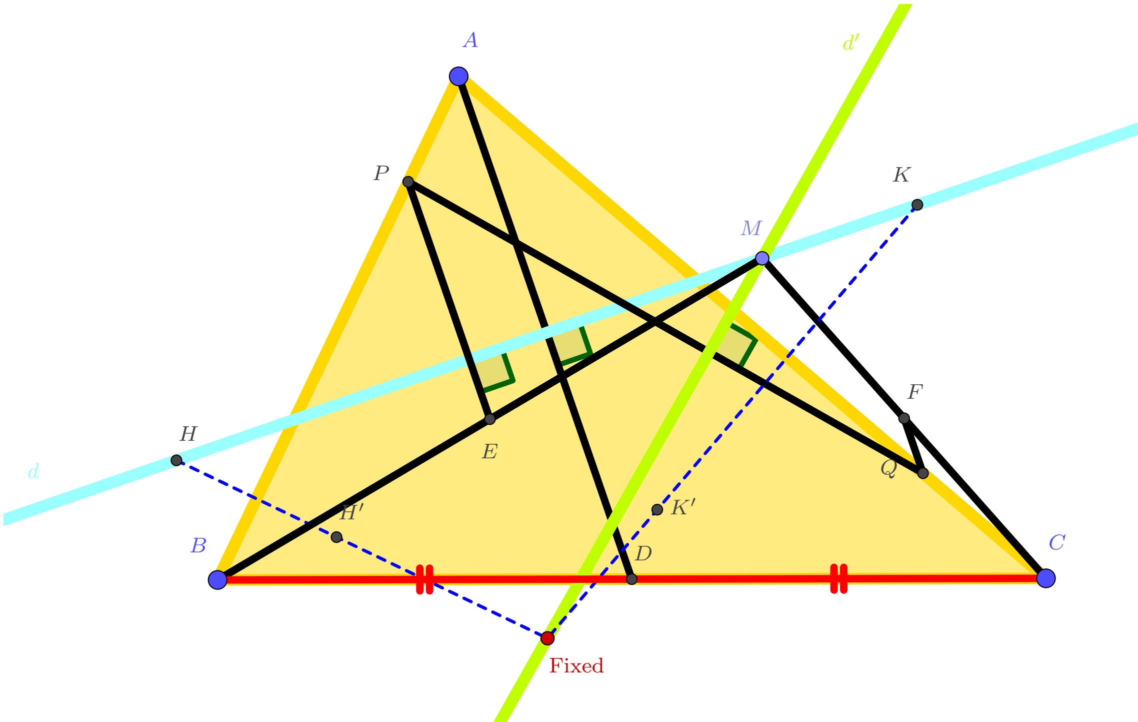


Figure 7: Prove that d' always goes through a fixed point.

Consider the Cartesian coordinates Dxy with origin D and $Dy \equiv DA$. As $Dx \perp Dy$ and $d \perp DA$, we have $Dx \parallel d$.

Now consider the following point coordinates: $A(0, a), B(b, c), C(-b, -c)$, where $a, b \neq 0$. We can then calculate

$$\overrightarrow{AB} = (b, c - a), \ \overrightarrow{AC} = (-b, -c - a),$$

which yields

$$(AB) : (a - c)x + by - ab = 0,$$

$$(AC) : (a + c)x - by + ab = 0.$$

Note that d is fixed and $d \parallel Dx$ so y_M is fixed. Let $y_M = h$, then

$$E\left(\frac{b+x_M}{2}, \frac{h+c}{2}\right),\ F\left(\frac{x_M-b}{2}, \frac{h-c}{2}\right).$$

Let d_1, d_2 be the lines passing through E, F respectively and perpendicular to d. It follows that

$$(d_1) : x = \frac{x_M+b}{2},\ (d_2) : x = \frac{x_M-b}{2},$$

so

$$P = d_1 \cap AB = \left(\frac{x_M+b}{2}, a - \frac{(a-c)(x_M+b)}{2b}\right),$$
$$Q = d_2 \cap AC = \left(\frac{x_M-b}{2}, a + \frac{(a+c)(x_M-b)}{2b}\right).$$

As a result, $\overrightarrow{PQ} = \left(-b, \frac{ax_M - bc}{b}\right)$. We can now calculate

$$(d') : -b(x - x_M) + \frac{ax_M - bc}{b} \cdot (y - h) = 0,$$

which can be rewritten as

$$b^2\left(x - \frac{bc}{a}\right) - (ax_M - bc)\left(y - h + \frac{b^2}{a}\right) = 0.$$

Hence d' always passes through a fixed point with coordinates $\left(\frac{bc}{a}, h - \frac{b^2}{a}\right)$. □

Remark 6. This was a very difficult problem in VMO 2008, with 81.2% participants getting a 0/7 score on it. A number of purely geometrical solutions to this problem can be viewed here. The fixed point in question is the intersection of HH' and KK', where H, K are the projections of B, C on d respectively and H', K' are the reflections of H about AB and K about AC respectively. An analytical solution, in our case, turns out to be much simpler.

4 Practice Exercises

Exercise 1 (China Girls MO 2002)**.** *An acute $\triangle ABC$ has three heights AD, BE and CF respectively. Prove that the perimeter of triangle DEF is not over half of the perimeter of triangle ABC.*

Exercise 2 (Italy TST 2000)**.** *Let ABC be an isosceles right triangle and M be the midpoint of its hypotenuse AB. Points D and E are taken on the legs AC and BC respectively such that $AD = 2DC$ and $BE = 2EC$. Lines AE and DM intersect at F. Show that FC bisects $\angle DFE$.*

Exercise 3 (Mixtinilear Incircle)**.** *Consider $\triangle ABC$ with circumcircle (O). A circle ω touches (O) and also touches AB, AC at D, E respectively. Let J be the incenter of $\triangle ABC$. Prove that D, J, E are collinear.*

Exercise 4 (USA TST 2004). *Let ABC be a triangle. Choose a point D in its interior. Let ω_1 be a circle passing through B and D and ω_2 be a circle passing through C and D so that the other point of intersection of the two circles lies on AD. Let ω_1 and ω_2 intersect side BC at E and F, respectively. Denote by X the intersection of DF, AB and Y the intersection of DE, AC. Show that $XY \parallel BC$.*

Exercise 5 (Ho Chi Minh City Regional Team Selection 2012). *$\triangle ABC$ has circumcircle (O) and heights AM, BN. Let D be a point on the arc $\hat{BC}$ that does not contain A ($D \neq B, D \neq C$). DA intersects BN at Q; DB intersects AM at P. Let I be the midpoint of PQ. Prove that M, N, I are collinear.*

Exercise 6 (IMO 2013). *Let the excircle of $\triangle ABC$ opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of $(A_1B_1C_1)$ lies on the circumcircle (ABC). Prove that $\triangle ABC$ is right-angled.*

Exercise 7 (IMO Shortlist 2005). *Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of $\triangle ABC$, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.*

Exercise 8 (IMO Shortlist 2011). *Let ABC be a triangle with incentre I and circumcircle ω. Let D and E be the second intersection points of ω with AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and B meet at a point K. Prove that the three lines AE, BD and KP are either parallel or concurrent.*

Exercise 9 (All-Russian MO 2009). *Let be given a parallelogram $ABCD$ and two points A_1, C_1 on its sides AB, BC, respectively. Lines AC_1 and CA_1 meet at P. Assume that the circumcircles of triangles AA_1P and CC_1P intersect at the second point Q inside triangle ACD. Prove that $\angle PDA = \angle QBA$.*

Exercise 10 (Balkan MO 2007). *Let $ABCD$ a convex quadrilateral with $AB = BC = CD$, with AC not equal to BD and E be the intersection point of it's diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120$.*

Exercise 11 (China TST 2008)**.** *Let ABC be a triangle, let $AB > AC$. Its incircle touches side BC at point E. Point D is the second intersection of the incircle with segment AE (different from E). Point F (different from E) is taken on segment AE such that $CE = CF$. The ray CF meets BD at point G. Show that $CF = FG$.*

Exercise 12 (Balkan MO 2013)**.** *In $\triangle ABC$, the excircle ω_a opposite A touches AB at P and AC at Q, while the excircle ω_b opposite B touches BA at M and BC at N. Let K be the projection of C onto MN and let L be the projection of C onto PQ. Show that the quadrilateral $MKLP$ is cyclic.*

Exercise 13 (IMO Shortlist 2005 *)**.** *Given $\triangle ABC$ satisfying $AC + BC = 3 \cdot AB$. The incircle of $\triangle ABC$ has center I and touches the sides BC and CA at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I. Prove that the points A, B, K, L lie on one circle.*

5 Solution to Practice Exercises

Solution to Exercise 1: China Girls MO 2002

An acute triangle ABC has three heights AD, BE and CF respectively. Prove that the perimeter of triangle DEF is not over half of the perimeter of triangle ABC.

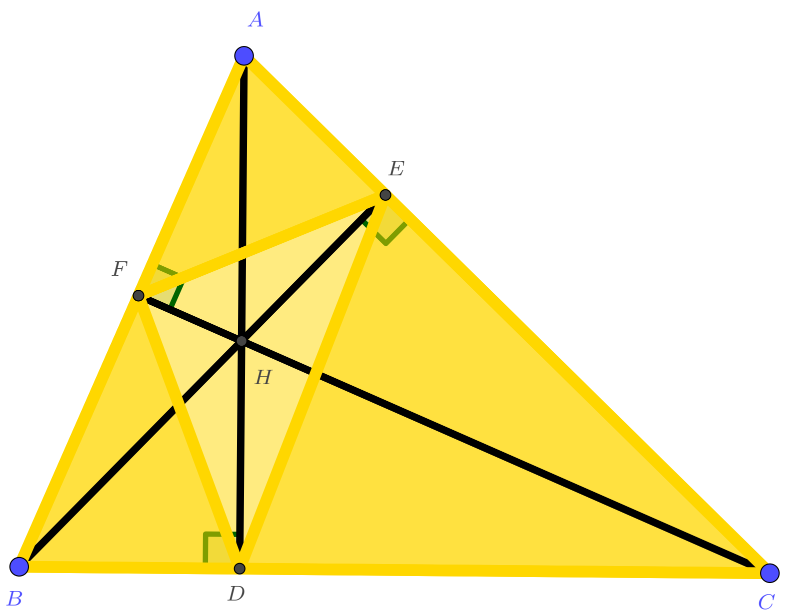


Figure 8: Prove that $DE + EF + FD \leq \frac{1}{2}(AB + BC + CA)$.

Proof. From $\triangle AEF \sim \triangle ABC \sim \triangle DBF \sim \triangle DEC$ we get

$$EF = BC \cdot \cos A = a \cdot \cos A$$
$$DE = AB \cdot \cos C = c \cdot \cos C$$
$$FD = AC \cdot \cos C = b \cdot \cos B$$

We need to prove that

$$EF + DE + FD \leq \frac{1}{2}(AB + BC + CA),$$

which is equivalent to

$$\begin{aligned} &a \cdot \cos A + b \cdot \cos B + c \cdot \cos C \leq \frac{1}{2}(a+b+c) \\ \Leftrightarrow &\sum a \cdot \frac{b^2 + c^2 - a^2}{2bc} \leq \frac{1}{2}(a+b+c) \\ \Leftrightarrow &\sum a^2(b^2 + c^2 - a^2) \leq abc(a+b+c) \\ \Leftrightarrow &2(a^2b^2 + b^2c^2 + c^2a^2) \leq a^4 + b^4 + c^4 + abc(a+b+c). \end{aligned}$$

According to Schur's Inequality, we have

$$a^4 + b^4 + c^4 + abc(a+b+c) \geq \sum (a^3b + ab^3).$$

while based on AM-GM we have

$$\sum (a^3b + ab^3) \geq \sum 2\sqrt{a^3b \cdot ab^3} = 2\sum a^2b^2.$$

Hence $2(a^2b^2 + b^2c^2 + c^2a^2) \leq a^4 + b^4 + c^4 + abc(a+b+c)$. □

Solution to Exercise 2: Italy TST 2000

Let ABC be an isosceles right triangle and M be the midpoint of its hypotenuse AB. Points D and E are taken on the legs AC and BC respectively such that $AD = 2DC$ and $BE = 2EC$. Lines AE and DM intersect at F. Show that FC bisects $\angle DFE$.

Proof. Let $AC = BC = x$ we have

$$AM = \frac{x\sqrt{2}}{2}, \ AD = BE = \frac{2x}{3}, \ CD = CE = \frac{x}{3}, DE^2 = \frac{2x^2}{9},$$

which yields

$$ME^2 = MB^2 + BE^2 - 2MB \cdot BE \cdot \cos\frac{\pi}{4} = \frac{5x^2}{18}.$$

Hence $AM^2 - AD^2 = ME^2 - DE^2$. According to Carnot's Theorem, $AE \perp DM$.

Hence $DCEF$ is cyclic, and we also have $CD = CE$, so $\angle DFC = \angle DEC = \angle CDE = \angle CEF$.

In other words, FE bisects $\angle DFE$. □

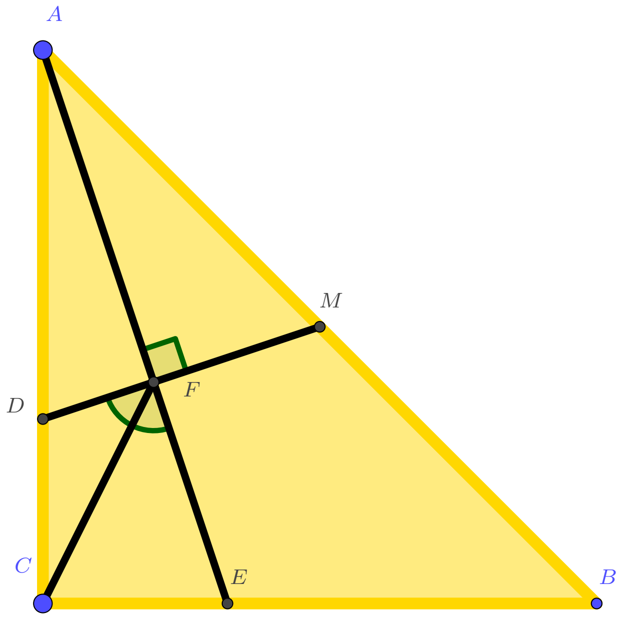


Figure 9: Prove that FC bisects $\angle DFE$.

Solution to Exercise 3: Mixtilinear Incircle

Consider $\triangle ABC$ with circumcircle (O). A circle ω touches (O) and also touches AB, AC at D, E respectively. Let J be the incenter of $\triangle ABC$. Prove that D, J, E are collinear.

Proof. Denote I as the center of ω and let $ID = IE = x$, we have

$$IA = \frac{x}{\sin\frac{A}{2}},\ JA = \frac{r}{\sin\frac{A}{2}},\ OI = R - x,\ OJ = \sqrt{R^2 - 2Rr}.$$

According to Stewart's Theorem,

$$OA^2 \cdot IJ + OI^2 \cdot JA = OJ^2 \cdot AI + IA \cdot AJ \cdot JI,$$

which is equivalent to

$$R^2(x-r) + (R-x)^2 r = (R^2 - 2Rr)x + \frac{rx(x-r)}{\sin^2\frac{A}{2}},$$

which implies that $\sin^2\frac{A}{2} = \frac{x-r}{x}$, so

$$\begin{aligned} IJ \cdot IA &= (AI - AJ) \cdot AI = AI^2 - AI \cdot AJ \\ &= \frac{1}{\sin^2\frac{A}{2}}(x^2 - rx) = \frac{x}{x-r} \cdot (x^2 - rx) \\ &= x^2 = IE^2. \end{aligned}$$

Hence $JE \perp AI$. By similar arguments, $JD \perp AI$. Thus D, J, E are collinear. □

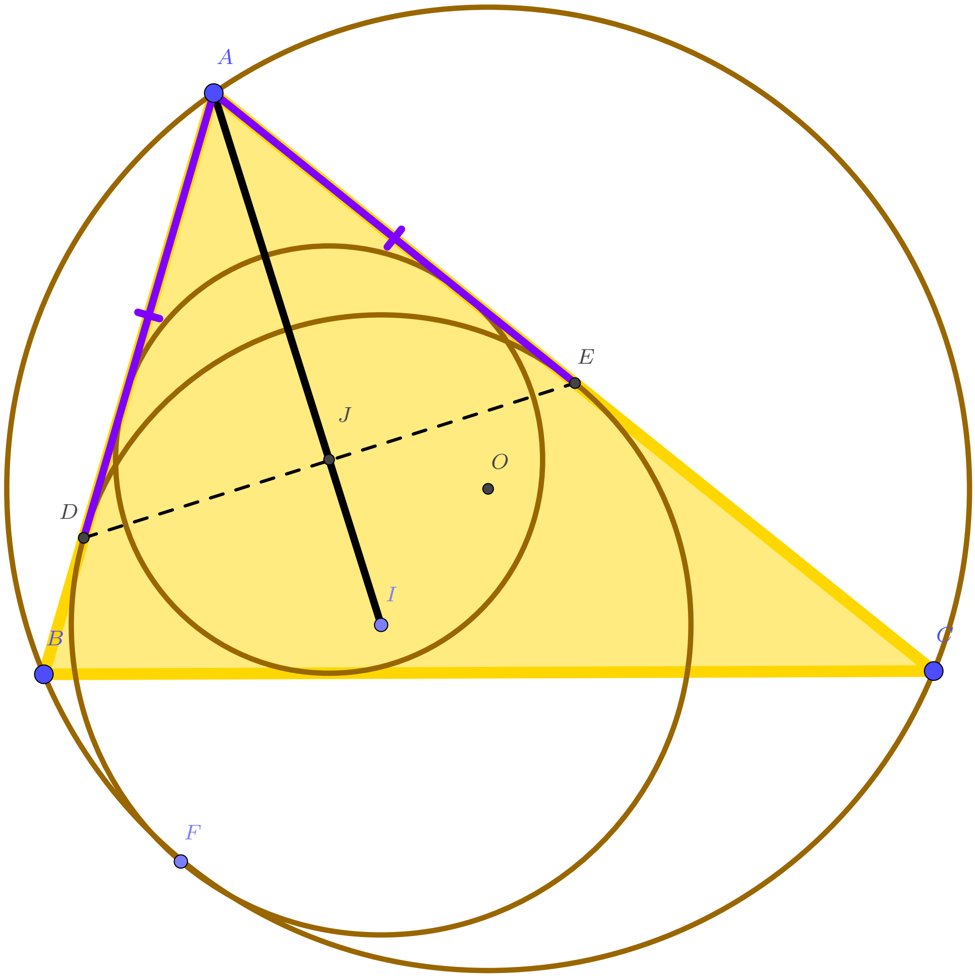


Figure 10: Prove that E, D, F are collinear.

Solution to Exercise 4: USA TST 2004

Let ABC be a triangle. Choose a point D in its interior. Let ω_1 be a circle passing through B and D and ω_2 be a circle passing through C and D so that the other point of intersection of the two circles lies on AD. Let ω_1 and ω_2 intersect side BC at E and F, respectively. Denote by X the intersection of DF, AB and Y the intersection of DE, AC. Show that $XY \parallel BC$.

Proof. Let S be the intersection of AD and BC and R be the other intersection of ω_1 and ω_2 besides D, then R lies on AD.

Applying Menelaus to a transversal line XDF $\triangle ABS$ and line ACS of $\triangle EDY$ we have

$$\frac{\overline{AX}}{\overline{XB}} = \frac{\overline{AD}}{\overline{SD}} \cdot \frac{\overline{SF}}{\overline{BF}} \text{ and } \frac{\overline{AY}}{\overline{CY}} = \frac{\overline{AD}}{\overline{SD}} \cdot \frac{\overline{SE}}{\overline{CE}}.$$

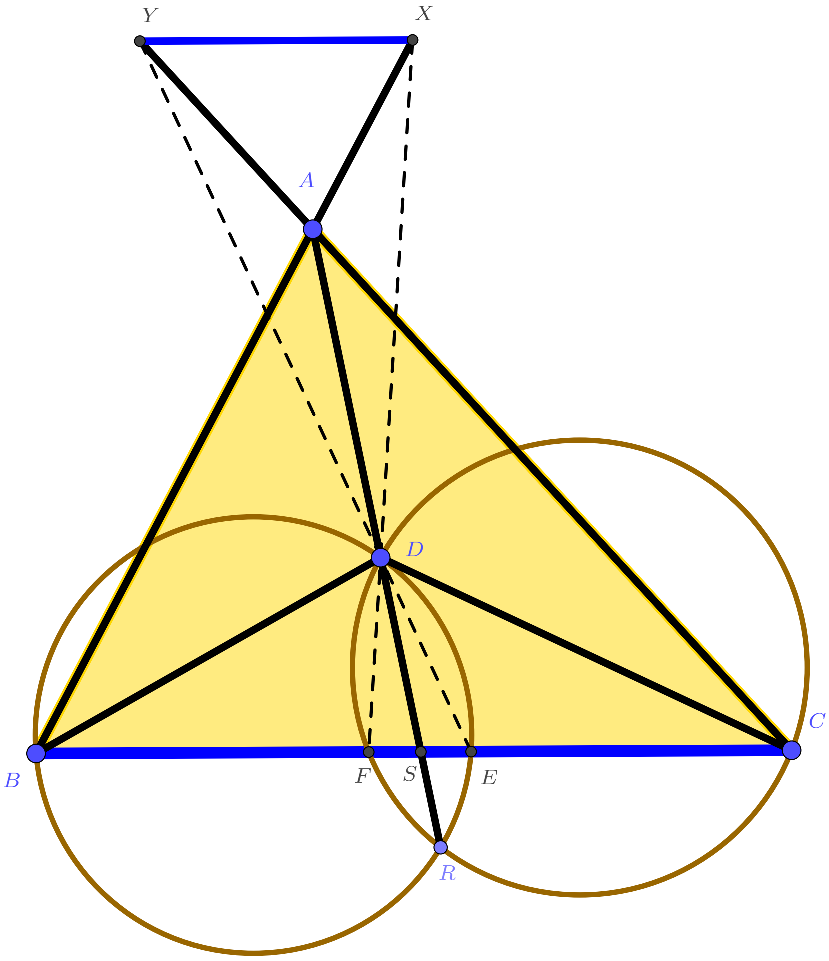


Figure 11: Prove that $XY \parallel BC$.

To prove that $XY \parallel BC$ we need $\frac{\overline{AX}}{\overline{BX}} = \frac{\overline{AY}}{\overline{CY}}$, or, in other words, $\frac{\overline{SF}}{\overline{BF}} = \frac{\overline{SE}}{\overline{CE}}$.
Observe that $\overline{SF} \cdot \overline{SC} = \overline{SR} \cdot \overline{SD} = \overline{SE} \cdot \overline{SB}$, so

$$\overline{SF} \cdot \overline{CE} = \overline{SF}(\overline{SE} - \overline{SC}) \text{ and } \overline{SE} \cdot \overline{BF} = \overline{SE}(\overline{SF} - \overline{SB}).$$

We note that $\triangle SDC \sim \triangle SFR$ and $\triangle SDB \sim \triangle SER$ so

$$\overline{SF} \cdot \overline{SC} = \overline{SD} \cdot \overline{SR} = \overline{SE} \cdot \overline{SB},$$

which in turn means $\overline{SF} \cdot \overline{CE} = \overline{SE} \cdot \overline{BF}$. Thus $XY \parallel BC$. □

Solution to Exercise 5: Ho Chi Minh City Regional Team Selection 2012

$\triangle ABC$ has circumcircle (O) and heights AM, BN. Let D be a point on the arc $\hat{BC}$ that does not contain A $(D \neq B, D \neq C)$. DA intersects BN at Q; DB intersects AM at P. Let I be the midpoint of PQ. Prove that M, N, I are collinear.

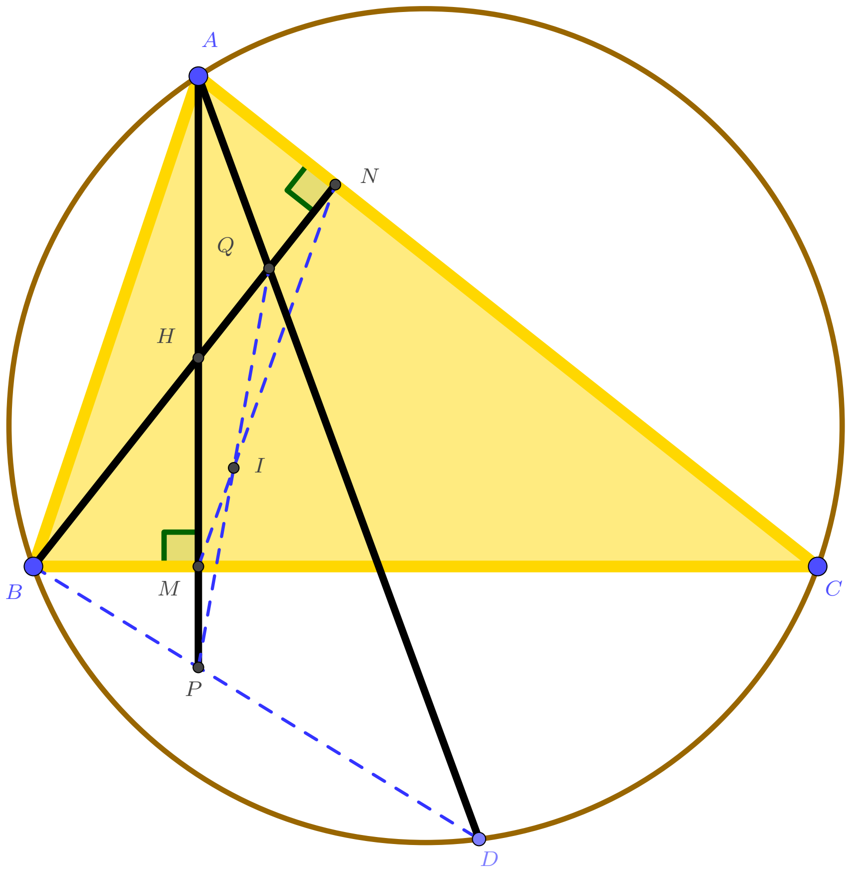


Figure 12: Prove that M, N, I are collinear.

Proof. Let $\angle CAD = \angle CDB = x$. Consider the following ratios:

$$\frac{NQ}{NH} = \frac{S_{ANQ}}{S_{ANH}} = \frac{AN \cdot AQ \cdot \sin x}{AN \cdot AH \cdot \sin \angle HAC} = \frac{AQ \sin x}{AH \cos C}, \tag{12}$$

$$\frac{MP}{MH} = \frac{S_{BMP}}{S_{BMH}} = \frac{BM \cdot BP \cdot \sin \widehat{HBC}}{BM.BH. \sin x} = \frac{BP \sin x}{BH \cos C}. \tag{13}$$

Observe that $\angle AQH + \angle BPH = 180°$, so $DPHQ$ is cyclic. Therefore

$$\frac{\sin \angle AHQ}{\sin \angle AQH} = \frac{\sin \angle BHP}{\sin \angle BPH}.$$

It then follows that $\frac{AQ}{AH} = \frac{BP}{BH}$, and, according to (12) and (13), $\frac{NQ}{NH} = \frac{MP}{MH}$.
Now consider three points M, I, N and $\triangle HPQ$, we see that

$$\frac{IP}{IQ} \cdot \frac{NQ}{NH} \cdot \frac{MH}{MP} = \frac{NQ}{NH} \cdot \frac{MH}{MP} = 1,$$

According to the converse Menelaus' theorem, M, N, I are collinear. □

Solution to Exercise 6: IMO 2013

Let the excircle of $\triangle ABC$ opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of $(A_1B_1C_1)$ lies on the circumcircle (ABC). Prove that $\triangle ABC$ is right-angled.

Proof. We will extend the problem and prove both directions: *$\triangle ABC$ is right-angled if and only if the circumcentre of $(A_1B_1C_1)$ lies on the circumcircle of $\triangle ABC$.*

- **If $\triangle ABC$ is right-angled**, WLOG assume $\angle BAC = 90°$. Let O be the midpoint of BC and M be the midpoint of the arc $\widehat{BC}$ that contains A in the circumcircle (ABC). As

$$MC = MB,\ CB_1 = BC_1 \text{ and } \angle MCB_1 = \angle MBC_1,$$

we get $\triangle MCB_1 = \triangle MBC_1$ and therefore $MB_1 = MC_1$. We now show that $MA_1 = MC_1$. Note that

$$BA_1 = \frac{a+b-c}{2},\ OM = \frac{a}{2},\ OA_1 = \frac{c-b}{2}, MA_1^2 = \frac{a^2+(b-c)^2}{4}.$$

Let H be the projection of M on AB. $\triangle MBC$ is a right isosceles triangle so $MB = MC = \frac{a}{\sqrt{2}}$. By Carnot's Theorem,

$$MB^2 - MA^2 = BH^2 - AH^2 = (BH + AH)(BH - AH),$$

which would give us MA:

$$\frac{a^2}{2} - MA^2 = c(c - \sqrt{2} \cdot MA)$$
$$\frac{b^2+c^2}{2} - MA^2 = c^2 - \sqrt{2} \cdot c \cdot MA$$
$$MA = \frac{\sqrt{2}}{2}(c-b).$$

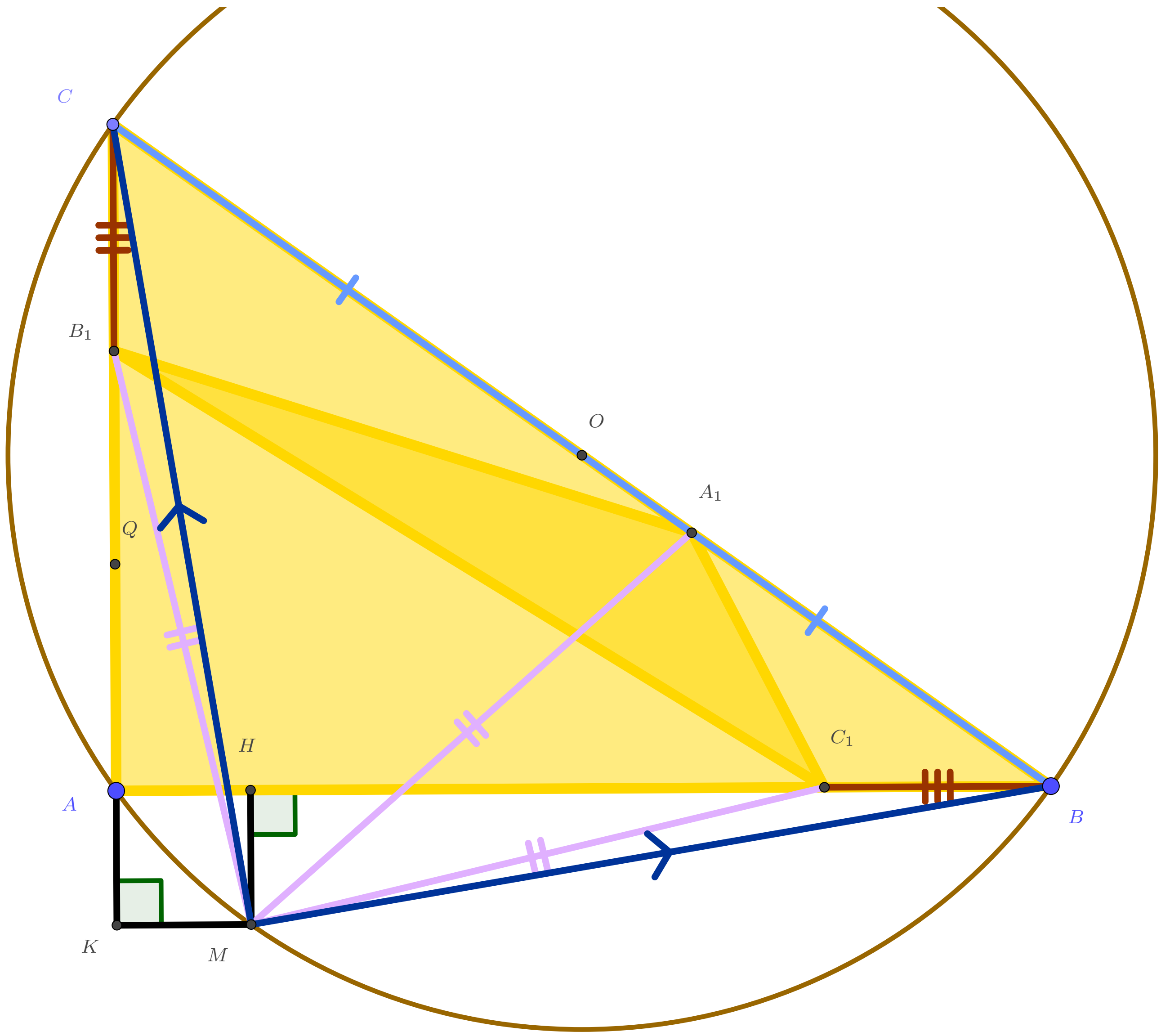


Figure 13: Prove that $\triangle ABC$ is right-angled.

Applying the law of cosine on $\triangle MAC_1$:

$$\begin{aligned} MC_1^2 &= MA^2 + AC_1^2 - 2 \cdot MA \cdot AC_1 \cdot \cos 45^\circ \\ &= \frac{(b-c)^2}{2} + \frac{(a+c-b)^2}{4} - 2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{a+c-b}{2} \cdot \frac{c-b}{\sqrt{2}} \\ &= \frac{a^2 + (b-c)^2}{4}. \end{aligned}$$

Hence $MA_1^2 = MC_1^2$, so $MA_1 = MC_1 = MB_1$, which means M is the circumcenter of $(A_1B_1C_1)$.

- **If the circumcenter of** $(A_1B_1C_1)$ **lies on the circumcircle** (ABC), WLOG assume M, the circumcenter of $(A_1B_1C_1)$, lies on the arc $\widehat{BC}$ that contains A of (ABC).

We then have $MB_1 = MC_1$, and also note that from the property of excircle, $BC_1 = CB_1$. It follows that $\triangle MB_1C = \triangle MC_1B$, so $MB = MC$, or, in other words, M is the midpoint of arc $\widehat{BC}$.

Let H and K be the projections of M on AB and AC respectively. As $AB > AC$, note that H lies inside the segment AB and K lies outside the segment AC. It's easy to see that $\triangle MHB = \triangle MKC$, so $MH = MK$ and therefore $AH = AK$. In other words,

$$AB - AM \cdot \sin\left(\frac{180^\circ - A}{2}\right) = AC + AM \cdot \sin\left(\frac{180^\circ - A}{2}\right),$$

$$AB - AC = 2AM\cos\frac{A}{2},$$

$$\sin C - \sin B = 2\sin\left(\frac{C-B}{2}\right)\cos\frac{A}{2},$$

which means $\tan\frac{A}{2} = 1$ and therefore $\angle BAC = 90^\circ$. Thus, $\triangle ABC$ is right-angled.

□

Solution to Exercise 7: IMO Shortlist 2005

Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of $\triangle ABC$, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.

Proof. Let O and O' be the center of the circumcircles (ABC) and (ADE) respectively. Let I and P be the midpoints of AH and the arc $\widehat{BC}$ not containing A of (ABC) respectively. AP intersects HM at N and (ADE) at N'. We will show that $N \equiv N'$.

Draw the diameter AA' of (ABC). As $BHCA'$ is a parallelogram, H, M, A' are collinear and OM is the midsegment of $\triangle AHA'$. Now observe that

$$AH = 2OM = 2R\cos A \text{ and } MP = OP - OM = R(1 - \cos A).$$

Furthermore, since $AH \parallel MP$,

$$\frac{AN}{AP} = \frac{AH}{AH + MP} = \frac{2\cos A}{1 + \cos A}. \tag{14}$$

Note that

$$\sin\angle AHD = \sin\left(B + \frac{A}{2}\right), \ \sin\angle ADE = \cos\frac{A}{2}.$$

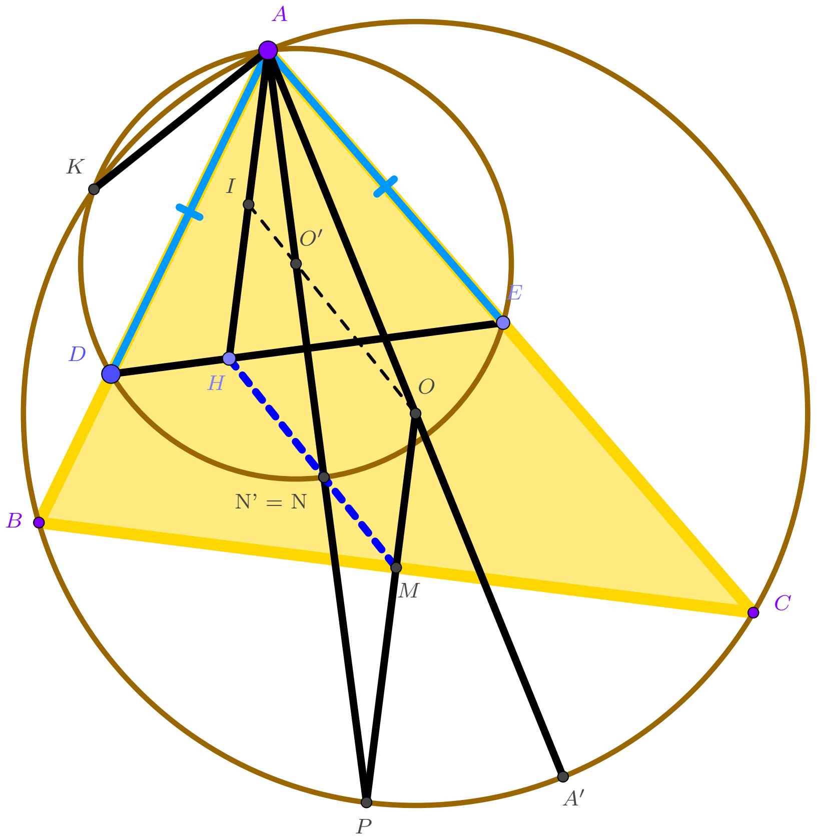


Figure 14: Prove that $HM \perp AK$.

Applying the sine rule in $\triangle ADH$ we get

$$AD = \frac{1}{\cos\frac{A}{2}} \cdot AH \sin\left(B + \frac{A}{2}\right). \tag{15}$$

As $\triangle ADE$ is isosceles and AP is the angular bisector of A, AN' is the diameter of (ADE). Hence $AD = AN' \cos\frac{A}{2}$. Combining this with (15) we get

$$AN' = \frac{1}{\cos^2\frac{A}{2}} \cdot AH \sin\left(B + \frac{A}{2}\right) = \frac{4R\cos A}{1+\cos A} \cdot \sin\left(B + \frac{A}{2}\right). \tag{16}$$

On the other hand,

$$AP = 2R \sin\angle ABP = 2R \sin\left(B + \frac{A}{2}\right),$$

so (14) yields

$$AN = \frac{2\cos A}{1+\cos A} \cdot AP = \frac{4R\cos A}{1+\cos A} \cdot \sin\left(B + \frac{A}{2}\right).$$

Combining this with (16) we get $AN = AN'$; in other words, $N \equiv N'$. Hence H, N, A' are collinear. Also, O, O', I' are collinear and lie on the midsegment of $\triangle AHA'$.

As a result, $HM \parallel OI$, while we also have $OI \perp AN$, so $HM \perp AN$. □

Remark 7. A similar problem has appeared in Vietnam TST 2006: *Consider $\triangle ABC$ with orthocenter H. The exterior angular bisector of $\angle BHC$ intersects AB and AC at D and E*

respectively. The angular bisector of $\angle BAC$ interesects (ADE) at $K \neq A$. Prove that HK goes through the midpoint of BC.

Solution to Exercise 8: IMO Shortlist 2011

Let ABC be a triangle with incentre I and circumcircle ω. Let D and E be the second intersection points of ω with AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and B meet at a point K. Prove that the three lines AE, BD and KP are either parallel or concurrent.

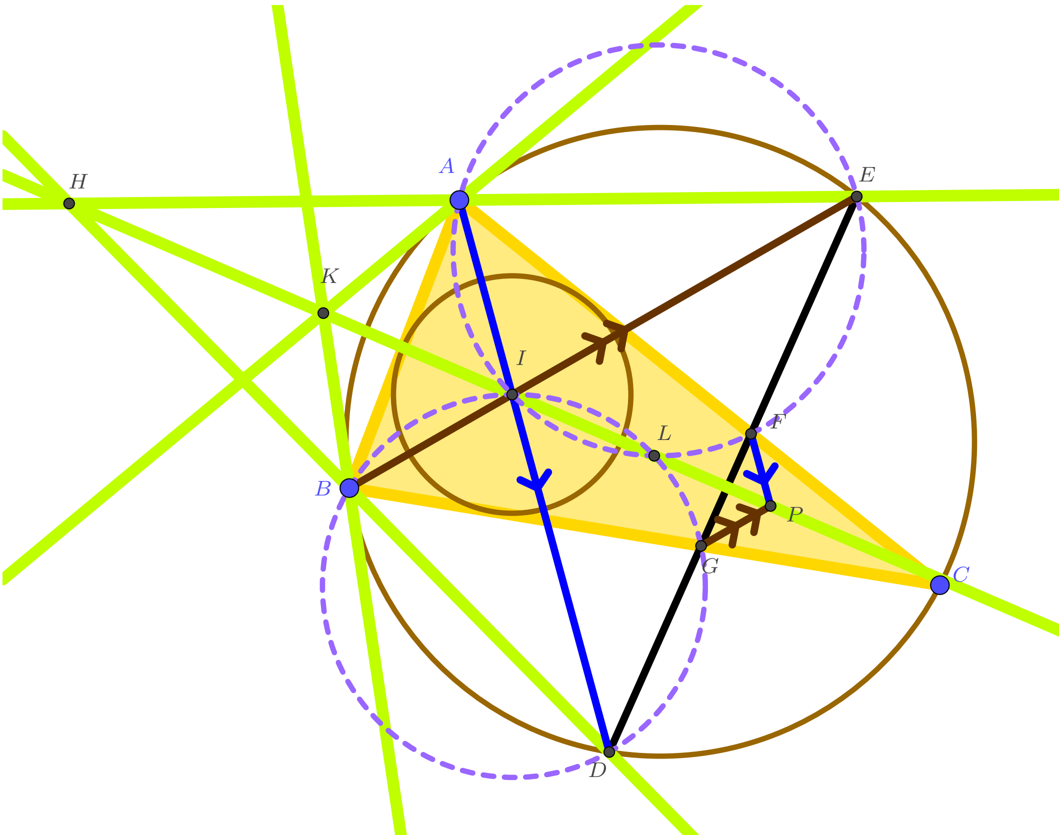


Figure 15: Prove that AE, BD and KP are either parallel or concurrent.

Proof. As $\angle IAF = \angle DAC = \angle BAD = \angle DEB = \angle FEI$, we see that $AIFE$ is cyclic. Similarly, $BDGI$ is cyclic. Denote ω_1 as the circle passing through A, I, F, E and ω_2 as the circle passing through B, D, G, I.

If ω_1 touches ω_2 then denote d as the two circles' common tangent line. Otherwise, let L be the other intersection point of ω_1 and ω_2 besides I and denote d as the line IL.

d intersects (ABL) and (FGL) at $K', P' \neq L$ respectively. Note that

$$(AB, BK') \equiv (AL, LK') \equiv (AL, LI) \equiv (AE, EI) \equiv (AE, EB) \equiv (AB, AB).$$

Therefore $BK' \equiv BK$. Similarly, $AK' \equiv AK$. It follows that $K \equiv K'$. On the other hand,

$$(P'F, FG) \equiv (P'L, LG) \equiv (IL, LG) \equiv (ID, DG) \equiv (AD, DE) \equiv (PF, FG).$$

Hence $P'F \equiv PF$ and similarly $P'G \equiv PG$. As a result, $P' \equiv P$. Thus d passes through K and P. Finally, as KP, AE, BD are [...] of ω_1 and ω_2, ω and ω_1, ω and ω_2 respectively, they are either parallel or concurrent. □

Solution to Exercise 9: All-Russian MO 2009

Consider a parallelogram $ABCD$ and two points A_1, C_1 on its sides AB, BC, respectively. Lines AC_1 and CA_1 meet at P. Assume that the circumcircles of triangles AA_1P and CC_1P intersect at the second point Q inside triangle ACD. Prove that $\angle PDA = \angle QBA$.

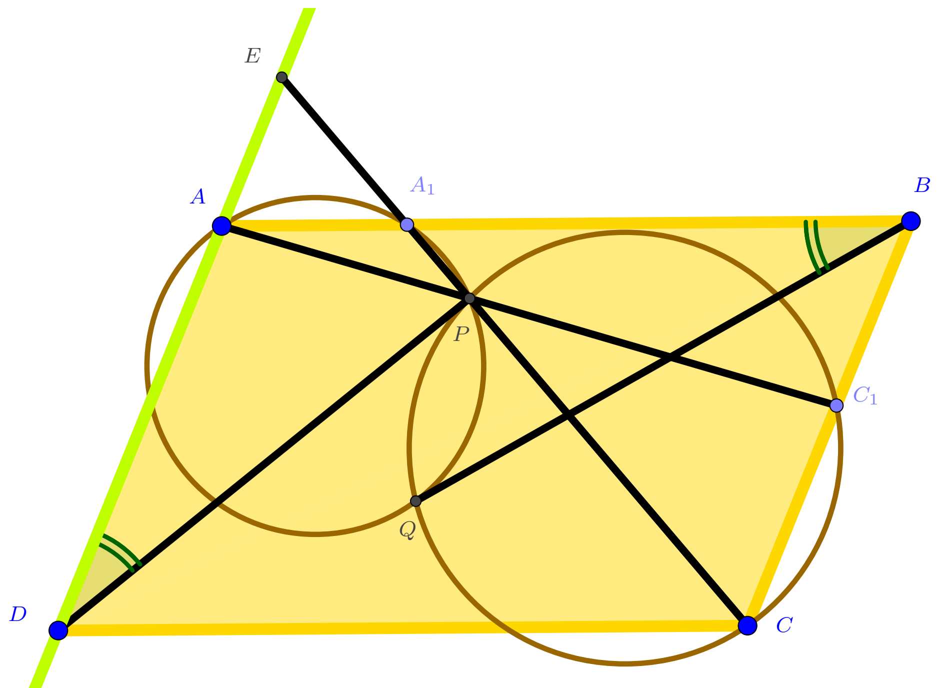


Figure 16: Prove that $\angle PDA = \angle QBA$.

Proof. Let E be the intersection of AD and CP, then

$$\frac{\sin \angle ABP}{\sin \angle CBQ} = \frac{AA_1}{CC_1}, \quad \frac{\sin \angle ADP}{\sin \angle CDP} \cdot \frac{EP}{PC} = \frac{CD}{ED}.$$

In addition,

$$\frac{EP}{PC} = \frac{EA}{CC_1}, \quad \frac{AD}{ED} = 1 - \frac{EA}{ED} = 1 - \frac{AA_1}{CD} = \frac{CD - AA_1}{CD},$$

so

$$\begin{aligned}
ED &= \frac{CD \cdot AD}{CD - AA_1} = \frac{CD \cdot AD}{AB - AA_1} = \frac{CD \cdot AD}{BA_1} \\
\Rightarrow \frac{\sin \angle ADP}{\sin \angle CDP} &= \frac{EA}{CC_1} \cdot \frac{CD}{\frac{CD \cdot AD}{BA_1}} = \frac{EA}{BC} \cdot \frac{BA_1}{CC_1} \\
\Rightarrow \frac{\sin \angle ADP}{\sin \angle CDP} &= \frac{AA_1}{BA_1} \cdot \frac{BA_1}{CC_1} = \frac{AA_1}{CC_1} = \frac{\sin \angle ABQ}{\sin \angle CBQ}. \qquad (17)
\end{aligned}$$

Let $\angle ABC = \angle ADC = \alpha$ and consider

$$f : (0, \pi) \to \mathbb{R}, \;\; f(x) = \frac{\sin(\alpha - x)}{\sin x}.$$

Note that f is a decreasing function on $(0, \pi)$, and from (17) we see that $f(\angle PDA) = f(\angle QBA)$, so $\angle PDA = \angle QBA$. □

Solution to Exercise 10: Balkan MO 2007

Let $ABCD$ a convex quadrilateral with $AB = BC = CD$, with AC not equal to BD and E be the intersection point of it's diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120$.

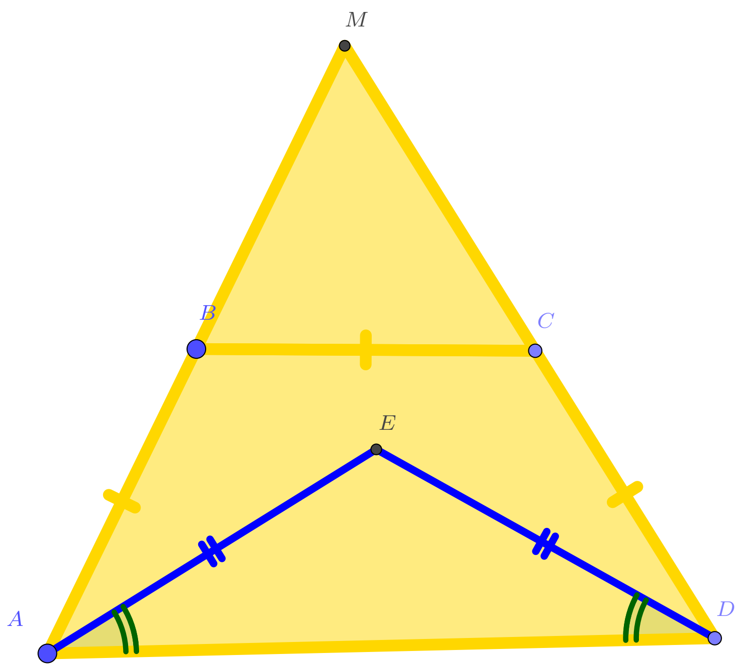


Figure 17: Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120$.

Proof. We will prove both directions:

- **If** $AE = ED$, then $\triangle EAD$ is isoceles at E so $\angle EAD = \angle EDA = \beta$. Applying the sine rule in $\triangle ABD$ and $\triangle ACD$ we get

$$\frac{AB}{\sin\beta} = \frac{AD}{\sin\angle ABD}, \ \frac{CD}{\sin\beta} = \frac{AD}{\sin\angle ACD}.$$

Also, $AB = CD$ so $\sin\angle ABD = \sin\angle ACD$, or $\angle ABD + \angle ACD = 180°$.

Let $\angle ACD = x$ and denote M as the intersection of AB and CD. We now have

$$\angle MBD = \angle BAD + \angle BDC = \angle BAC + 2\beta,$$

which implies $x - 2\beta = \angle BAC = \angle CBA$. By similar arguments,

$$\angle CDB = 180° - x - 2\beta.$$

On the other hand, $\angle BEA = \angle EBC + \angle BCE$, so $\angle DBC = 4\beta - x$. Combining this with $\angle CBD = \angle CDB$, we get the equation

$$180° - x - 2\beta = 4\beta - x,$$

which yields $x = 30°$, from which it's easy to deduce that $\angle BAD + \angle ADC = 120$.

- **If** $\angle BAD + \angle ADC = 120°$, let $\angle EAD = \alpha$ and $\angle EDA = \beta$. It follows that $\alpha + \beta = 60°$. Applying the sine rule in $\triangle ABD$ and $\triangle ACD$ we get

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin\angle ACD}{\sin\angle ABD}.$$

Let $\angle CDB = \angle CBD = x$, we can calculate

$$\angle ACD = 180° - \alpha - \beta - x = 120° - x.$$

Now note that $\angle AEB = \angle EBC + \angle ECB$, so

$$\alpha + \beta - x = \angle ACB = \angle CAB,$$

which implies

$$\angle ABD = 180° - 2\alpha - 2\beta + x = 60° + x.$$

Finally, we see that

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin\angle ACD}{\sin\angle ABD} = \frac{60° + x}{120° - x} = 1,$$

which means $\sin\alpha = \sin\beta$, so $\alpha = \beta$. Consequently, $AE = DE$.

□

Solution to Exercise 11: China TST 2008

Let ABC be a triangle, let $AB > AC$. Its incircle touches side BC at point E. Point D is the second intersection of the incircle with segment AE (different from E). Point F (different from E) is taken on segment AE such that $CE = CF$. The ray CF meets BD at point G. Show that $CF = FG$.

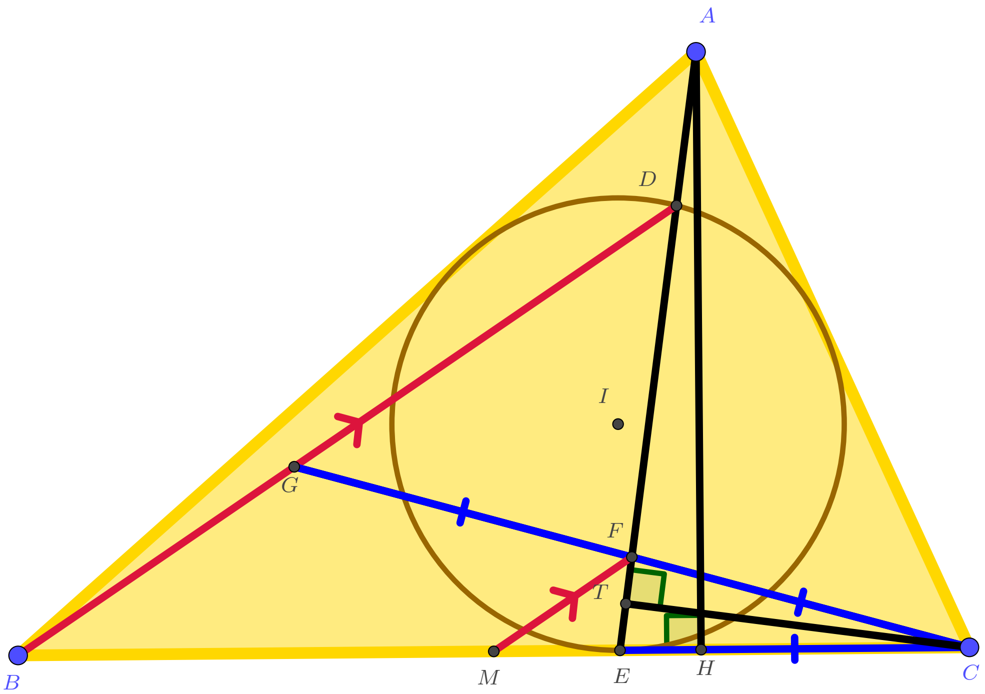


Figure 18: Prove that $CF = FG$.

Proof. Let I be the incenter of $\triangle ABC$ and M be the midpoint of BC. Construct the heights AH of $\triangle ABC$ and CT of $\triangle CEF$.

To prove that $CF = FG$, we will show that $MF \parallel BD$ or, equivalently, $\frac{EM}{EB} = \frac{EF}{ED}$. Note that

$$\frac{EM}{EB} = \frac{BE - BM}{EB} = \frac{\frac{a+c-b}{2} - \frac{a}{2}}{\frac{a+c-b}{2}} = \frac{c-b}{a+c-b}.$$

On the other hand,

$$\begin{aligned} AE^2 &= AH^2 + HE^2 = \frac{4S_{ABC}^2}{a^2} + \left(\frac{a+b-c}{2} - \frac{a^2+b^2-c^2}{2a}\right)^2 \\ &= \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4a^2} + \frac{(c-b)^2(b+c-a)^2}{4a^2}. \end{aligned} \tag{18}$$

(I) touches AB at K, then

$$AD \cdot AE = AK^2 = \left(\frac{b+c-a}{2}\right)^2. \tag{19}$$

As $ED \cdot EA = AE^2 - AD \cdot AE$, from (18) and (19) we get

$$ED \cdot EA = \frac{4S^2}{a^2} + \frac{(c-b)^2(b+c-a)^2}{4a^2} - \left(\frac{b+c-a}{2}\right)^2$$
$$= \frac{4S^2}{a^2} + \frac{(b+c-a)^2(c-b-a)(c-b+a)}{4a^2}$$
$$= \frac{(a+b-c)(b+c-a)(c+a-b)}{2a}. \tag{20}$$

As $ATHC$ is cyclic,

$$EF \cdot EA = 2EI \cdot EA = 2 \cdot EH \cdot EC$$
$$= 2 \cdot \left(\frac{a^2+c^2-b^2}{2a} - \frac{a+c-b}{2}\right) \cdot \frac{a+b-c}{2}$$
$$= \frac{(c-b)(b+c-a)(a+b-c)}{2a}. \tag{21}$$

From (20) and (21), we get

$$\frac{EF}{ED} = \frac{c-b}{a+c-b} = \frac{EM}{EB},$$

which yields $CF = FG$. □

Solution to Exercise 12: Balkan MO 2013

In $\triangle ABC$, the excircle ω_a opposite A touches AB at P and AC at Q, while the excircle ω_b opposite B touches BA at M and BC at N. Let K be the projection of C onto MN and let L be the projection of C onto PQ. Show that the quadrilateral $MKLP$ is cyclic.

Proof. Let I_b, I_c be the excenters opposite B and C in $\triangle ABC$ respectively. Let ω_c be the excircle opposite to C. ω_c touches BC at T. S is the intersection of MN and PQ.

Let X, Y be the projections of M on PQ and P on MN respectively. AX intersects PY at H. HC intersects MN at S_1 and PQ at S_2.

Observe that

$$CK = \frac{b+c-a}{2} \cdot \cos\frac{B}{2},\ CL = \frac{a+c-b}{2} \cdot \cos\frac{A}{2}.$$

On the other hand, $HA \cdot HX = HP \cdot HY$ so

$$\frac{YH}{XH} = \frac{MH}{PH} = \frac{\sin\frac{B}{2}}{\sin\frac{A}{2}}.$$

In addition,

$$\tan\frac{A}{2} = \frac{b+c-a}{2TI_c},\ \tan\frac{B}{2} = \frac{c+a-b}{2TI_c},$$

which yields

$$(a+c-b) \cdot \tan\frac{B}{2} = (b+c-a) \cdot \tan\frac{A}{2},$$

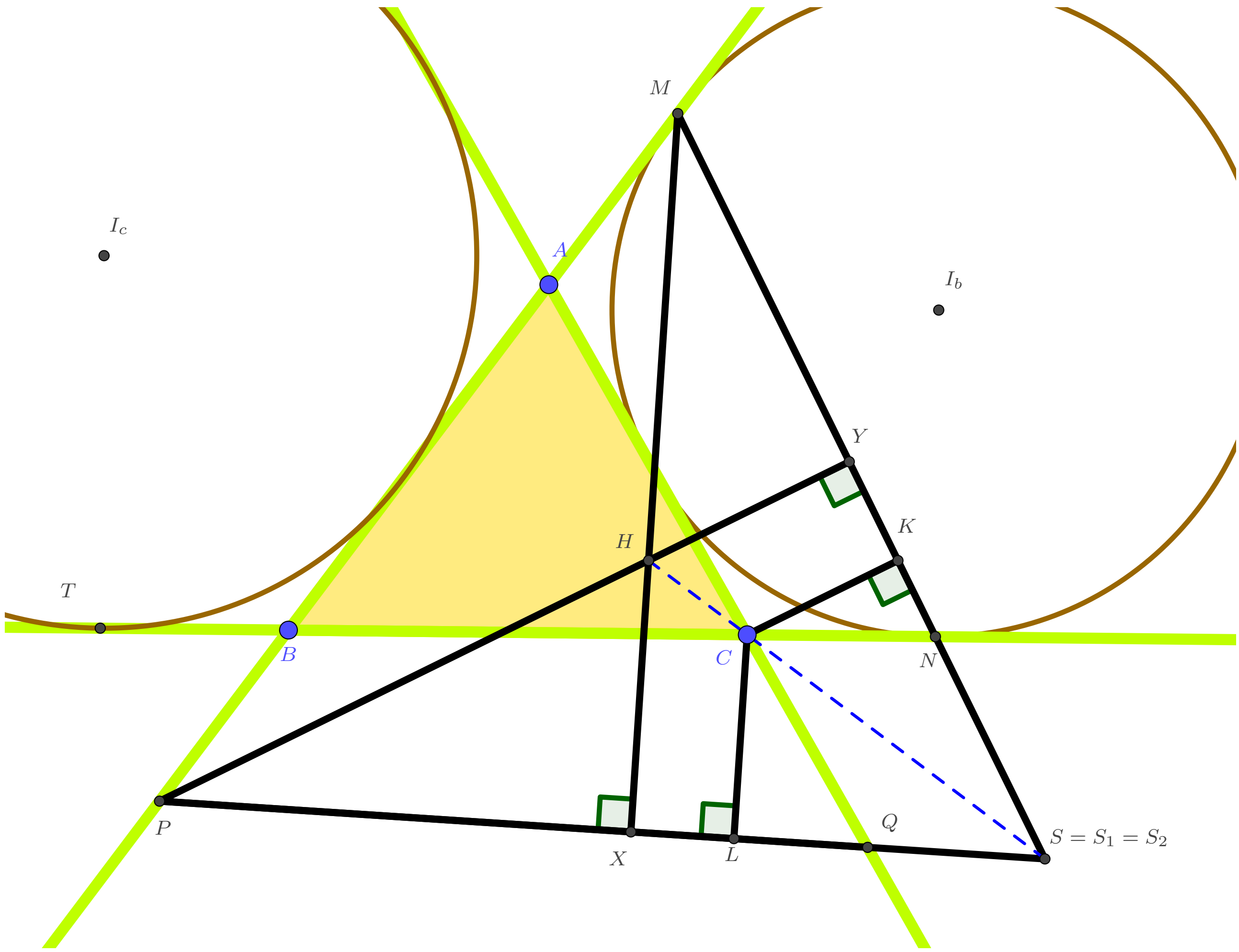


Figure 19: Prove that $MKLP$ is cyclic.

which implies that

$$\frac{CK}{CL} = \frac{(b+c-a)\cdot\cos\frac{B}{2}}{(a+c-b)\cos\frac{A}{2}} = \frac{\sin\frac{B}{2}}{\sin\frac{A}{2}} = \frac{YH}{XH}.$$

Now, as $CK \parallel YH$ and $CL \parallel XH$,

$$\frac{S_1C}{S_1H} = \frac{CK}{YH} = \frac{CL}{XH} = \frac{S_2C}{S_2H},$$

so $S_1 \equiv S_2$ and therefore H, C, S are collinear.

As $\angle SKL = 90^0 - \angle CKL$ and $\angle SYX = 90^0 - \angle HYX$, $\angle SKL = \angle HYX$, so $KL \parallel XY$. It follows that $\angle YXP = \angle KLP$, while we also have $\angle YXP = 180° - \angle PMY$. Thus $\angle KLP = 180° - \angle PMY$ and so $MPLK$ is cyclic. $\square$

Solution to Exercise 13: IMO Shortlist 2005

Given $\triangle ABC$ satisfying $AB+BC=3CA$. The incircle of $\triangle ABC$ has center I and touches the sides AB and BC at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I. Prove that the points A, C, K, L lie on one circle.

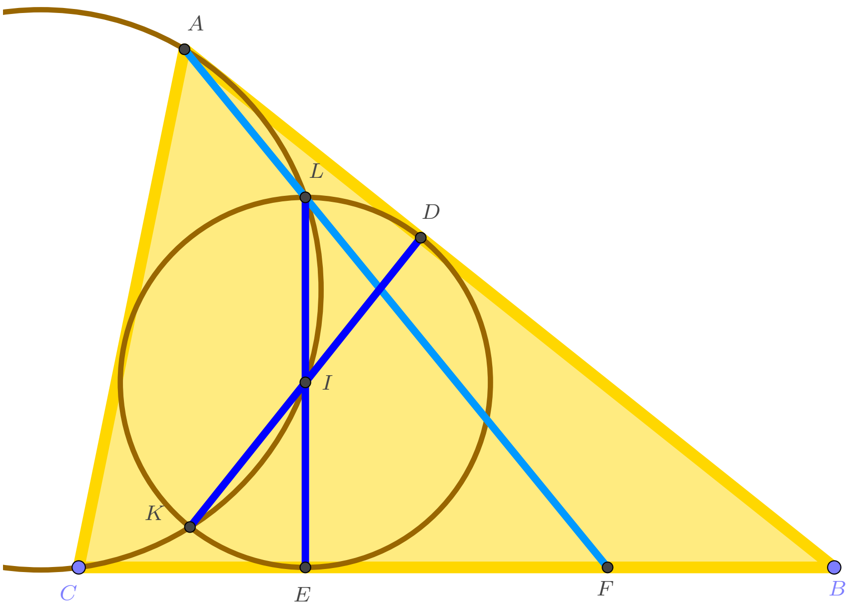


Figure 20: Prove that A, C, K, L lie on one circle.

Proof. We have $c+a=3b$. Note that if $b=c$ then $a=2b=b+c$, which contradicts the triangle inequality. Hence $b\neq c$, and WLOG assume $b<c$.

AL intersects BC at F. Assume the positive direction is that of vector $\overrightarrow{BC}$, then

$$\overline{EF}=\overline{CF}-\overline{CE}=-\overline{BE}-\overline{CE}=\frac{a+c-b}{2}-\frac{a+b-c}{2}=c-b,$$

so $EF=|\overline{EF}|=c-b$. Now $\triangle LFE$ and $\triangle CLE$ have

$$\tan\angle LFC=\frac{LE}{EF}=\frac{2r}{c-b},\ \tan\angle LCE=\frac{LE}{CE}=\frac{4r}{a+c-b},$$

so

$$\begin{aligned}\tan\angle ALC=\tan(\angle LFC+\angle LCE)&=\frac{\tan\angle LFC+\tan\angle LCE}{1-\tan\angle LFC\cdot\tan\angle LCE}\\&=\left(\frac{2r}{c-b}+\frac{4r}{a+b-c}\right):\left(1-\frac{8r^2}{(c-b)(a+b-c)}\right)\\&=\frac{2r(a+c-b)}{(c-b)(a+b-c)-8r^2}.\end{aligned}$$

As $a + c = 3b$, we can compute the inradius r of $\triangle ABC$ from

$$r^2 = \frac{S^2}{p^2} = \frac{(p-a)(p-b)(p-c)}{p} = \frac{(b+c-a)(a+b-c)}{8}.$$

It then follows that

$$\begin{aligned}\tan\angle ALC &= \frac{2r(a+c-b)}{(c-b)(a+b-c)-(b+c-a)(a+b-c)} \\ &= \frac{2r(a+c-b)}{(a+b-c)(a-2b)} = \frac{2br}{(2b-c)(b-c)}.\end{aligned}$$

By similar arguments,

$$\tan\angle AKC = \frac{2br}{(2b-a)(b-a)}.$$

From $a + c = 3b$ it is easy to see that $(2b - c)(b - c) = (2b - a)(b - a)$. Thus $\tan\angle ALC = \tan\angle AKC$. In other words, $\angle ALC = \angle AKC$, so $ALKC$ is cyclic. $\square$