10-301/601: Introduction to Machine Learning Lecture 7 – Perceptron

Henry Chai

5/15/25

Front Matter

- Announcements:
  - HW1 released on 5/13, due 5/16 (tomorrow!) at 11:59 PM
  - HW2 to be on released 5/16 (tomorrow!), due 5/20 at 11:59 PM
  - Quiz 1 on 5/16 (tomorrow!) at 11:00 AM in BH A36 (here)
    - Recitation (immediately after today's lecture) will be a review of this week's material to help you prepare for tomorrow's quiz

Recall: Fisher Iris Dataset

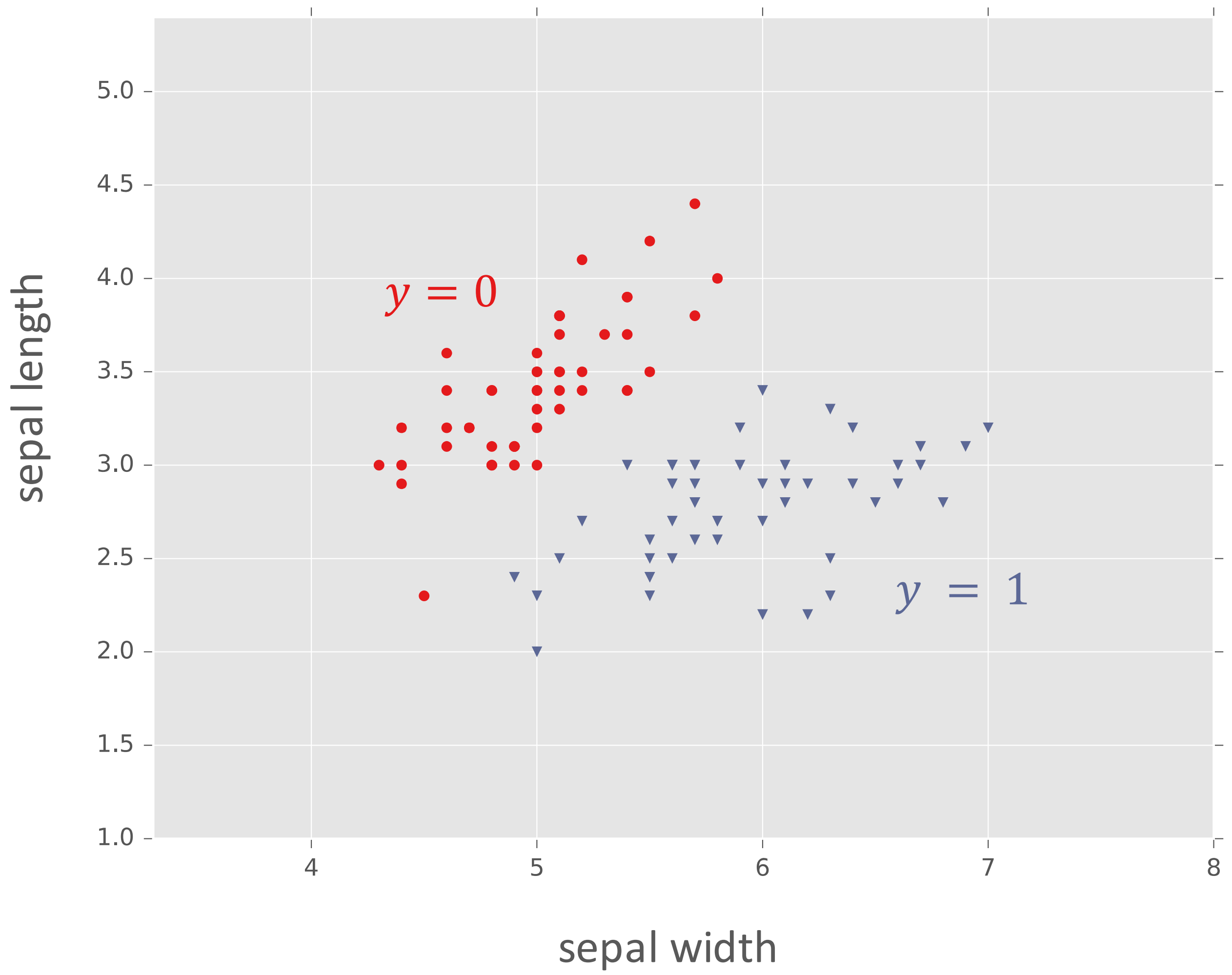


Figure courtesy of Matt Gormley

Linear Algebra Review

- Notation: in this class vectors will be assumed to be column vectors by default, i.e.,

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_D \end{bmatrix} \text{ and } \boldsymbol{a}^T = [a_1 \quad a_2 \quad \cdots \quad a_D]$$

- The dot product between two D-dimensional vectors is

$$\boldsymbol{a}^T\boldsymbol{b} = [a_1 \quad a_2 \quad \cdots \quad a_D] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_D \end{bmatrix} = \sum_{d=1}^{D} a_d b_d$$

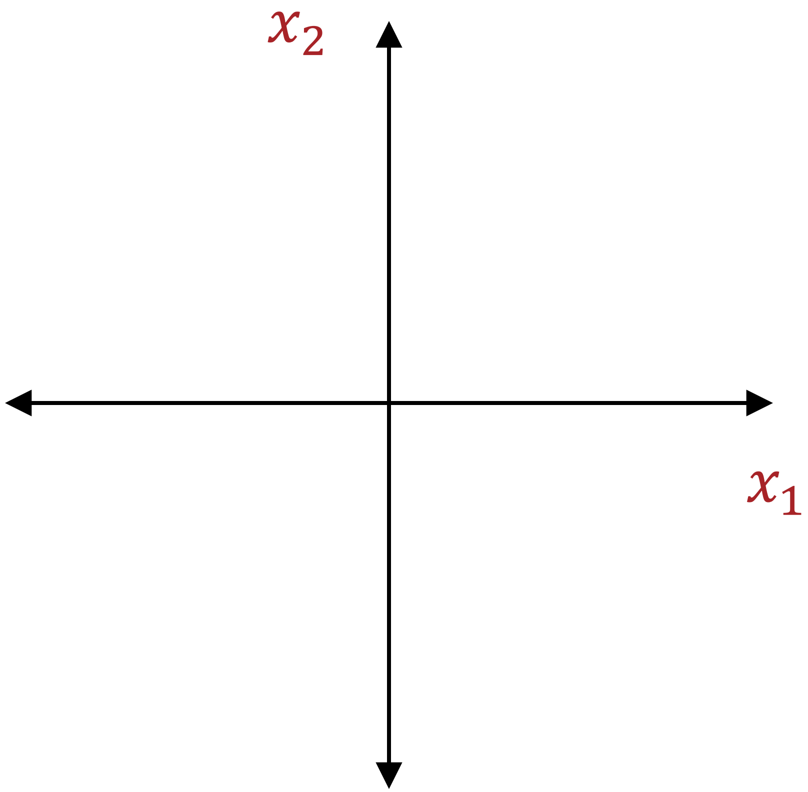
- The $L2$-norm of $\boldsymbol{a} = \|\boldsymbol{a}\|_2 = \sqrt{\boldsymbol{a}^T\boldsymbol{a}}$
- Two vectors are *orthogonal* iff

$$\boldsymbol{a}^T\boldsymbol{b} = 0$$

Geometry Warm-up

1. On the axes below, draw the region corresponding to
$$w_1 x_1 + w_2 x_2 + b > 0$$
where $w_1 = 1$, $w_2 = 2$ and $b = -4$.

2. Then draw the vector $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$



Geometry Warm-up

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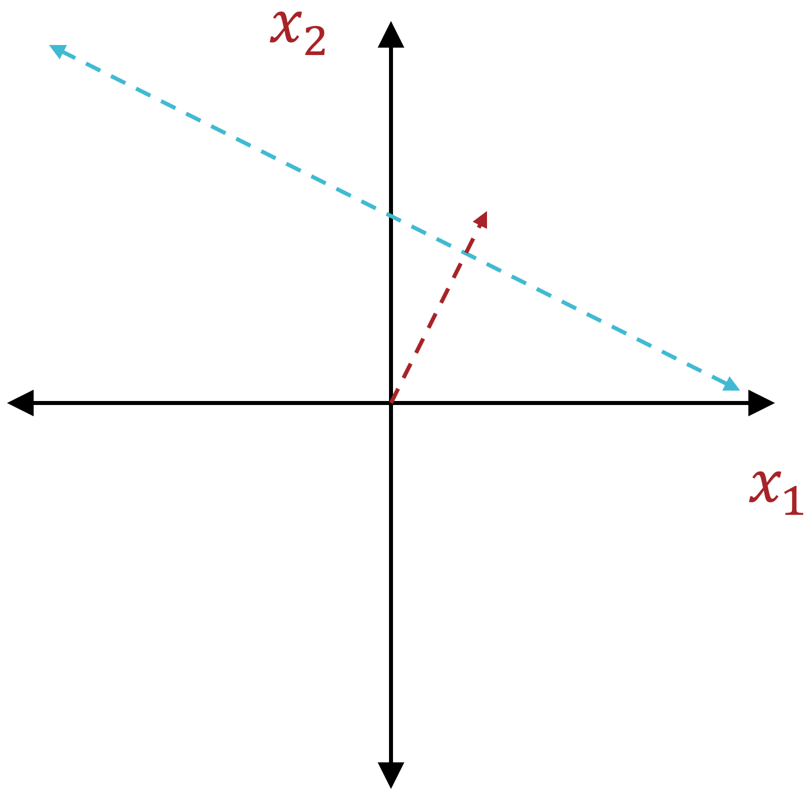

Linear Decision Boundaries

- In 2 dimensions, $w_1 x_1 + w_2 x_2 + b = 0$ defines a *line*
- In 3 dimensions, $w_1 x_1 + w_2 x_2 + w_3 x_3 + b = 0$ defines a *plane*
- In 4+ dimensions, $\boldsymbol{w}^T \boldsymbol{x} + b = 0$ defines a *hyperplane*
  - The vector $\boldsymbol{w}$ is always orthogonal to this hyperplane and always points in the direction where $\boldsymbol{w}^T \boldsymbol{x} + b > 0$!
- A hyperplane creates two *halfspaces*:
  - $\mathcal{S}_+ = \{\boldsymbol{x}\colon \boldsymbol{w}^T \boldsymbol{x} + b > 0\}$ or all $\boldsymbol{x}$ s.t. $\boldsymbol{w}^T \boldsymbol{x} + b$ is positive
  - $\mathcal{S}_- = \{\boldsymbol{x}\colon \boldsymbol{w}^T \boldsymbol{x} + b < 0\}$ or all $\boldsymbol{x}$ s.t. $\boldsymbol{w}^T \boldsymbol{x} + b$ is negative

Linear Decision Boundaries: Example

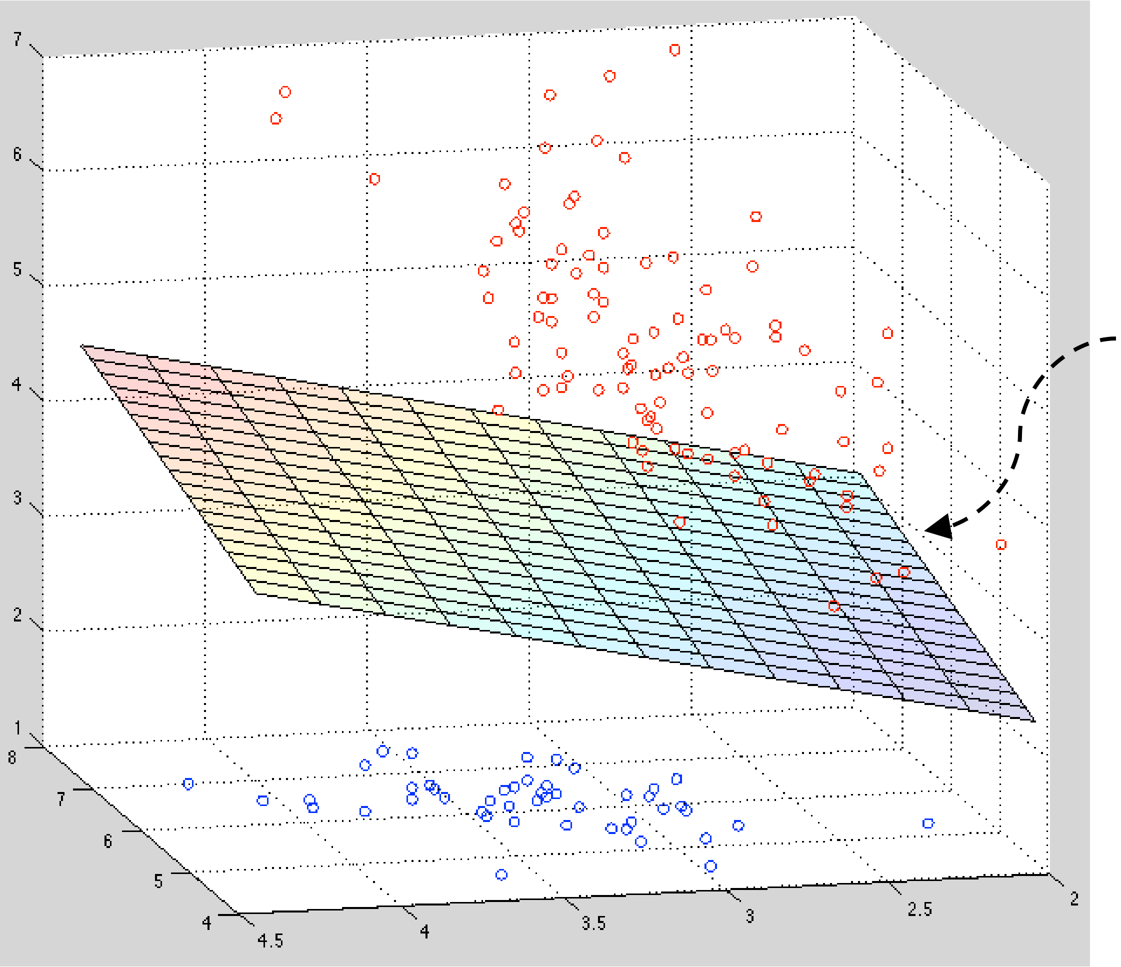

Figure courtesy of Matt Gormley

Goal: learn classifiers of the form $h(\boldsymbol{x}) = \text{sign}(\boldsymbol{w}^T\boldsymbol{x} + b)$ (assuming $y \in \{-1, +1\}$)

Key question: how do we learn the *parameters*, $\boldsymbol{w}$ and b?

Online Learning

- So far, we’ve been learning in the *batch* setting, where we have access to the entire training dataset at once
- A common alternative is the *online* setting, where examples arrive gradually and we learn continuously
- Examples of online learning:
  - Predicting stock prices
  - Recommender systems
  - Medical diagnosis
  - Robotics

Online Learning: Setup

- For $t = 1, 2, 3, \ldots$
  - Receive an unlabeled example, $\boldsymbol{x}^{(t)}$
  - Predict its label, $\hat{y} = h_{\boldsymbol{w},b}\left(\boldsymbol{x}^{(t)}\right)$
  - Observe its true label, $y^{(t)}$
  - Pay a penalty if we made a mistake, $\hat{y} \neq y^{(t)}$
  - Update the parameters, $\boldsymbol{w}$ and b

- Goal: minimize the number of mistakes made

(Online) Perceptron Learning Algorithm

- Initialize the weight vector and intercept to all zeros:

$$\boldsymbol{w} = [0 \quad 0 \quad \cdots \quad 0] \text{ and } b = 0$$

- For $t = 1, 2, 3, \ldots$
  - Receive an unlabeled example, $\boldsymbol{x}^{(t)}$
  - Predict its label, $\hat{y} = \text{sign}(\boldsymbol{w}^T \boldsymbol{x} + b) = \begin{cases} +1 \text{ if } \boldsymbol{w}^T \boldsymbol{x} + b \geq 0 \\ -1 \text{ otherwise} \end{cases}$
  - Observe its true label, $y^{(t)}$
  - If we misclassified a positive example ($y^{(t)} = +1, \hat{y} = -1$):
    - $\boldsymbol{w} \leftarrow \boldsymbol{w} + \boldsymbol{x}^{(t)}$
    - $b \leftarrow b + 1$
  - If we misclassified a negative example ($y^{(t)} = -1, \hat{y} = +1$):
    - $\boldsymbol{w} \leftarrow \boldsymbol{w} - \boldsymbol{x}^{(t)}$
    - $b \leftarrow b - 1$

(Online) Perceptron Learning Algorithm

- Initialize the weight vector and intercept to all zeros:

$$\boldsymbol{w} = [0 \quad 0 \quad \cdots \quad 0] \text{ and } b = 0$$

- For $t = 1, 2, 3, \ldots$
  - Receive an unlabeled example, $\boldsymbol{x}^{(t)}$
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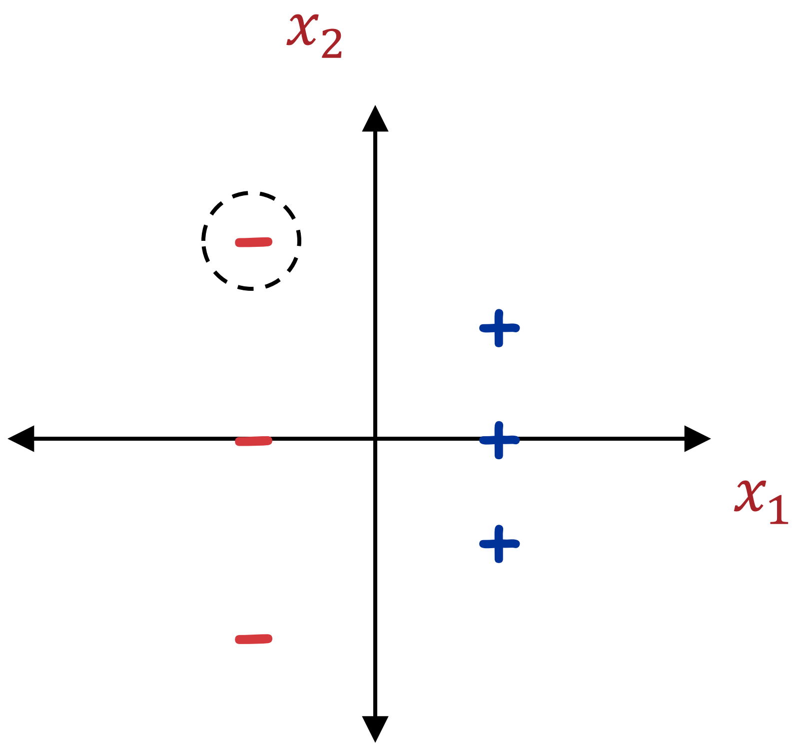
  - Observe its true label, $y^{(t)}$
  - If we misclassified an example ($y^{(t)} \neq \hat{y}$):
    - $\boldsymbol{w} \leftarrow \boldsymbol{w} + y^{(t)}\boldsymbol{x}^{(t)}$
    - $b \leftarrow b + y^{(t)}$

(Online) Perceptron Learning Algorithm: Example (no Intercept)

x_1	x_2	$\hat{y}$	y	Mistake?
-1	2	$+$	$-$	Yes

$$\boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



(Online) Perceptron Learning Algorithm: Example (no Intercept)

x_1	x_2	$\hat{y}$	y	Mistake?
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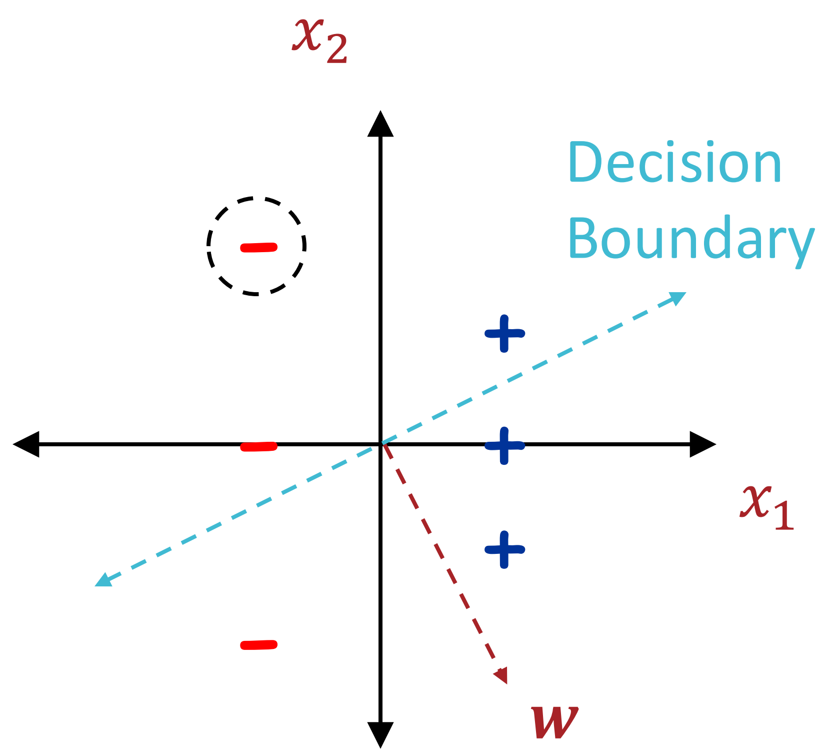


$$\boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

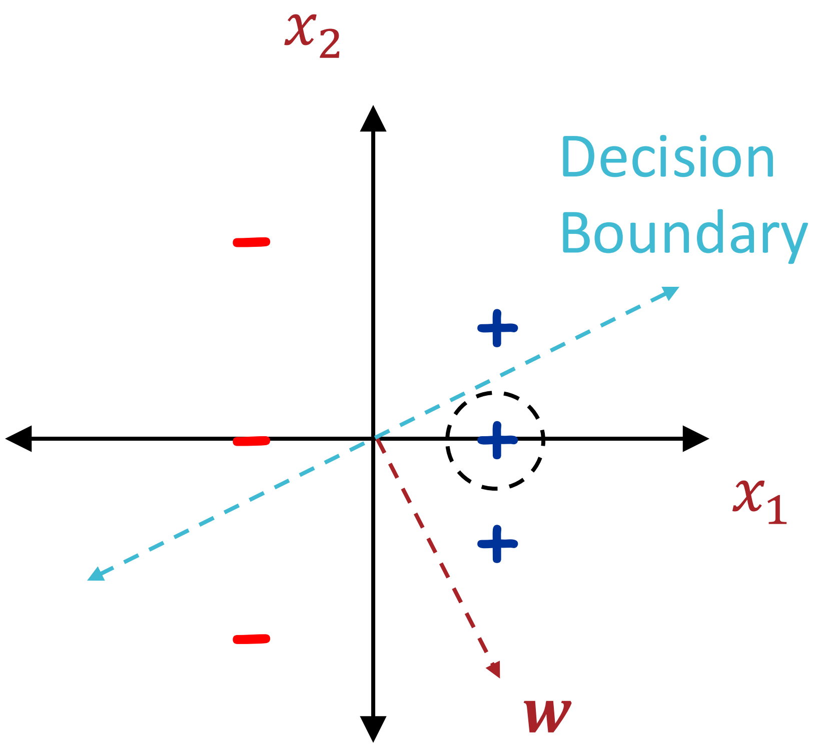
$$\boldsymbol{w} \leftarrow \boldsymbol{w} + y^{(1)}\boldsymbol{x}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Example courtesy of Nina Balcan

(Online) Perceptron Learning Algorithm: Example (no Intercept)

x_1	x_2	$\hat{y}$	y	Mistake?
-1	2	$+$	$-$	Yes
1	0	$+$	$+$	No

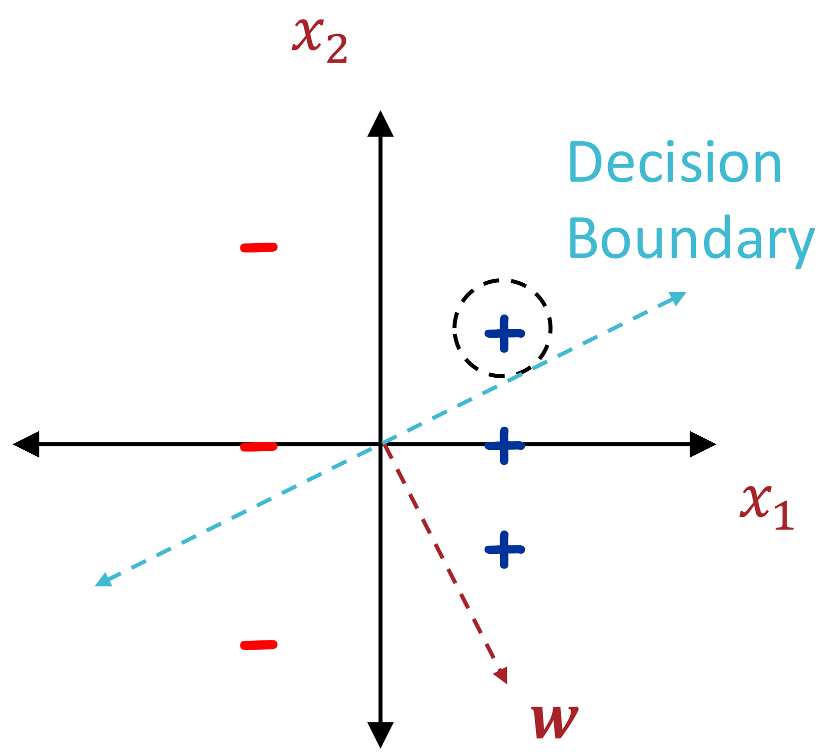
$$\boldsymbol{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$



Example courtesy of Nina Balcan

(Online) Perceptron Learning Algorithm: Example (no Intercept)

x_1	x_2	$\hat{y}$	y	Mistake?
-1	2	$+$	$-$	Yes
1	0	$+$	$+$	No
1	1	$-$	$+$	Yes

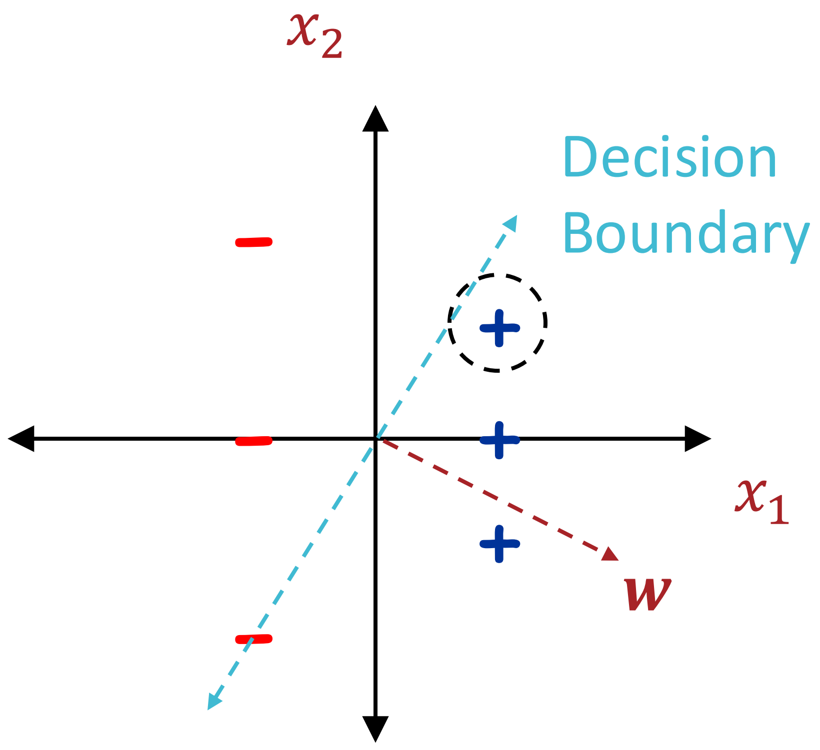


$$\boldsymbol{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\boldsymbol{w} \leftarrow \boldsymbol{w} + y^{(3)}\boldsymbol{x}^{(3)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(Online) Perceptron Learning Algorithm: Example (no Intercept)

x_1	x_2	$\hat{y}$	y	Mistake?
-1	2	$+$	$-$	Yes
1	0	$+$	$+$	No
1	1	$-$	$+$	Yes



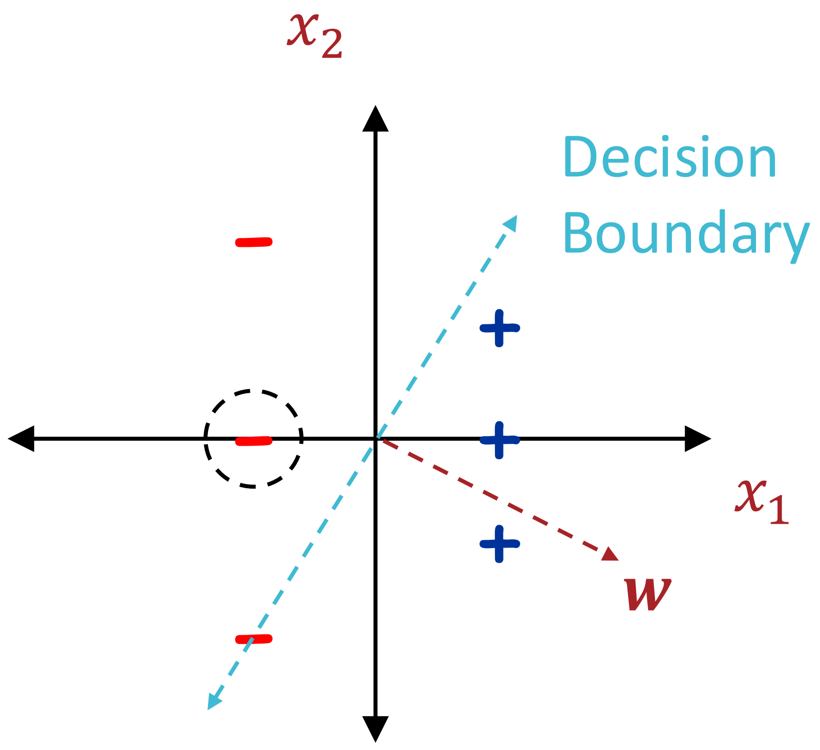
$$\boldsymbol{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\boldsymbol{w} \leftarrow \boldsymbol{w} + y^{(3)}\boldsymbol{x}^{(3)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(Online) Perceptron Learning Algorithm: Example (no Intercept)

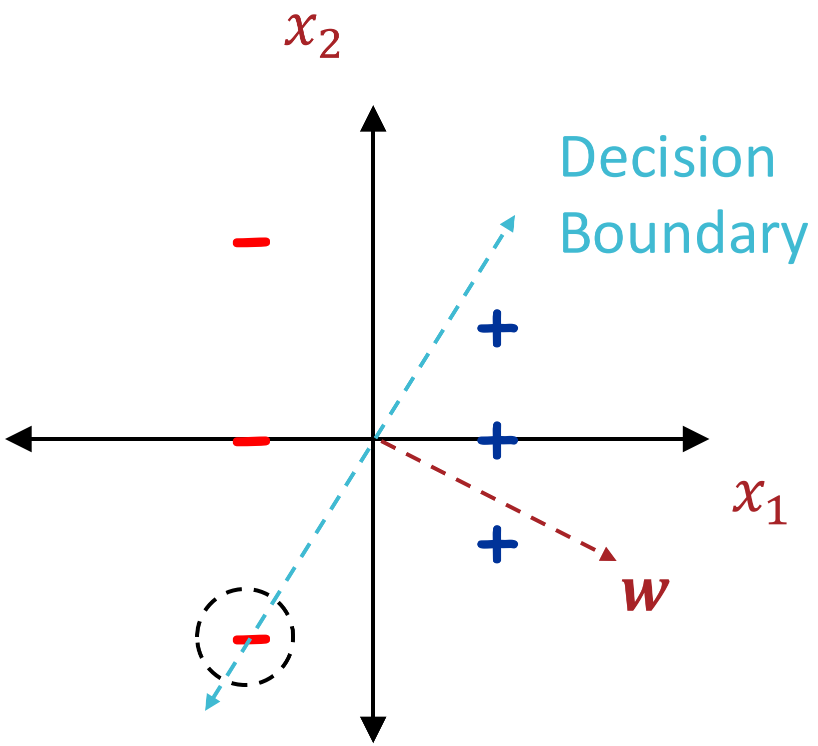
x_1	x_2	$\hat{y}$	y	Mistake?
-1	2	$+$	$-$	Yes
1	0	$+$	$+$	No
1	1	$-$	$+$	Yes
-1	0	$-$	$-$	No

$$\boldsymbol{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$



(Online) Perceptron Learning Algorithm: Example (no Intercept)

x_1	x_2	$\hat{y}$	y	Mistake?
-1	2	$+$	$-$	Yes
1	0	$+$	$+$	No
1	1	$-$	$+$	Yes
-1	0	$-$	$-$	No
-1	-2	$+$	$-$	Yes

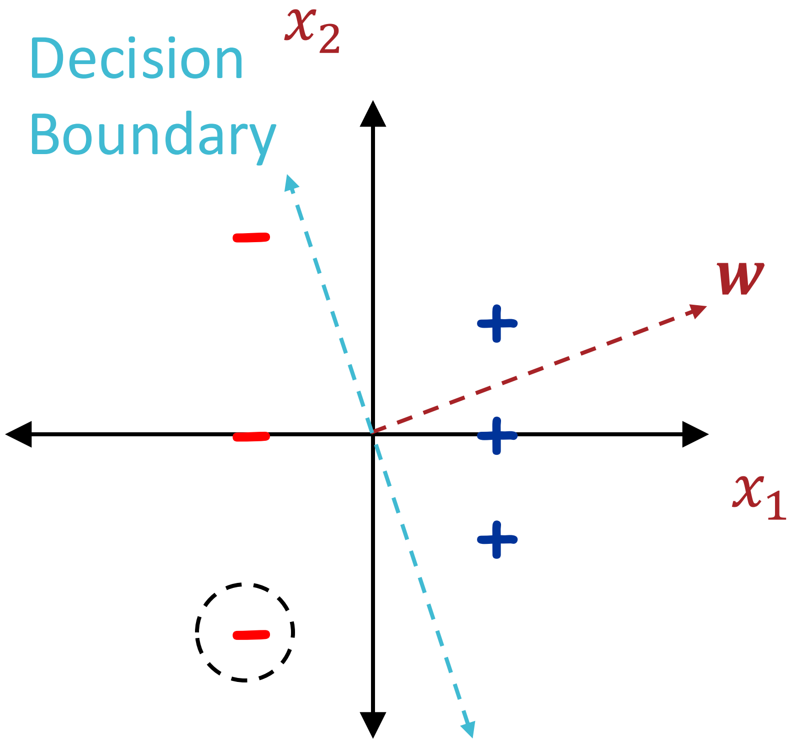


$$\boldsymbol{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\boldsymbol{w} \leftarrow \boldsymbol{w} + y^{(5)}\boldsymbol{x}^{(5)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(Online) Perceptron Learning Algorithm: Example (no Intercept)

x_1	x_2	$\hat{y}$	y	Mistake?
-1	2	$+$	$-$	Yes
1	0	$+$	$+$	No
1	1	$-$	$+$	Yes
-1	0	$-$	$-$	No
-1	-2	$+$	$-$	Yes



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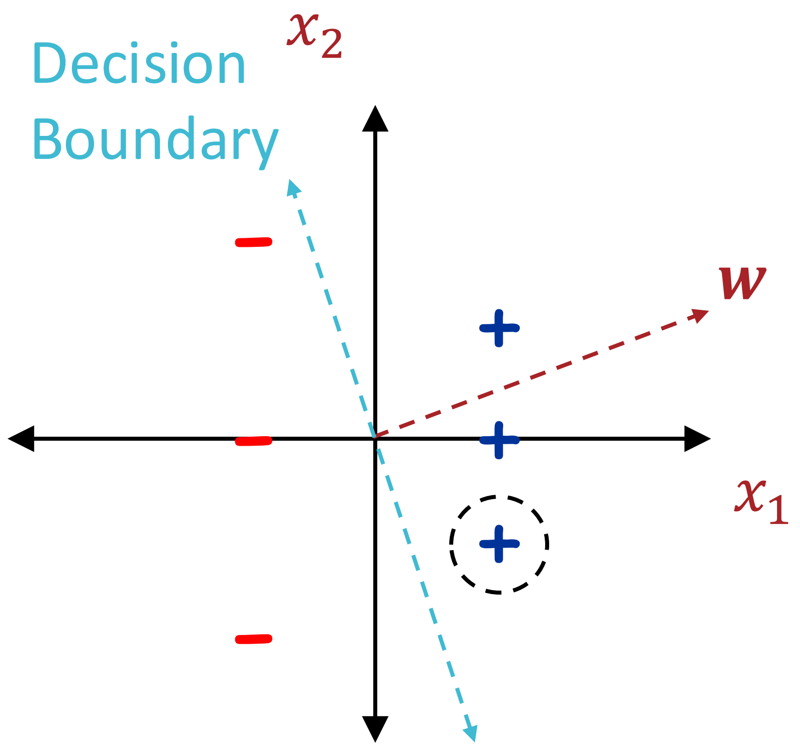
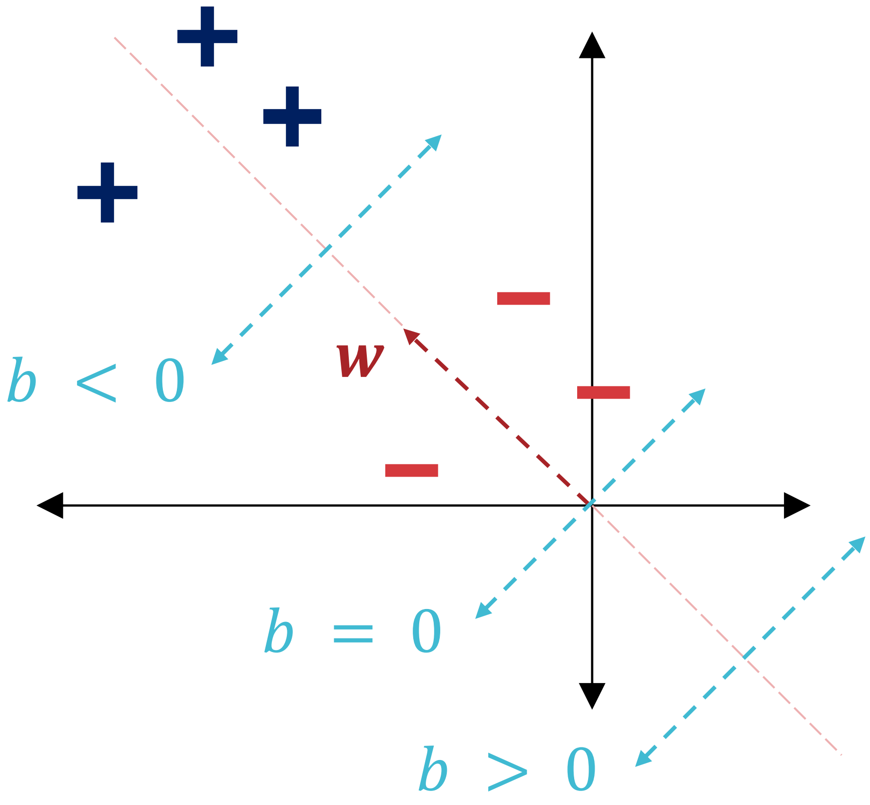
x_1	x_2	$\hat{y}$	y	Mistake?
-1	2	$+$	$-$	Yes
1	0	$+$	$+$	No
1	1	$-$	$+$	Yes
-1	0	$-$	$-$	No
-1	-2	$+$	$-$	Yes
1	-1	$+$	$+$	No

$$\boldsymbol{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



Updating the Intercept

- The intercept shifts the decision boundary off the origin
  - Increasing b shifts the decision boundary towards the negative side
  - Decreasing b shifts the decision boundary towards the positive side



Notational Hack

- If we add a 1 to the beginning of every example e.g.,

$$\boldsymbol{x}' = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ \vdots \\ x_D \end{bmatrix} \ldots$$

- … we can just fold the intercept into the weight vector!

$$\boldsymbol{\theta} = \begin{bmatrix} b \\ w_1 \\ w_2 \\ \vdots \\ w_D \end{bmatrix} \rightarrow \boldsymbol{\theta}^T \boldsymbol{x}' = \boldsymbol{w}^T \boldsymbol{x} + b$$

(Online) Perceptron Learning Algorithm

- Initialize the weight vector and intercept to all zeros:

$$\boldsymbol{w} = [0 \quad 0 \quad \cdots \quad 0] \text{ and } b = 0$$

- For $t = 1, 2, 3, \ldots$
  - Receive an unlabeled example, $\boldsymbol{x}^{(t)}$
  - Predict its label, $\hat{y} = \text{sign}(\boldsymbol{w}^T \boldsymbol{x} + b) = \begin{cases} +1 \text{ if } \boldsymbol{w}^T \boldsymbol{x} + b \geq 0 \\ -1 \text{ otherwise} \end{cases}$
  - Observe its true label, $y^{(t)}$
  - If we misclassified an example ($y^{(t)} \neq \hat{y}$):
    - $\boldsymbol{w} \leftarrow \boldsymbol{w} + y^{(t)} \boldsymbol{x}^{(t)}$
    - $b \leftarrow b + y^{(t)}$

(Online) Perceptron Learning Algorithm

- Initialize the parameters to all zeros:

$$\boldsymbol{\theta} = [0 \quad 0 \quad \cdots \quad 0]$$

1 prepended to $\boldsymbol{x}^{(t)}$

- For $t = 1, 2, 3, \ldots$
  - Receive an unlabeled example, $\boldsymbol{x}^{(t)}$
  - Predict its label, $\hat{y} = \text{sign}\left(\boldsymbol{\theta}^T \boldsymbol{x}'^{(t)}\right) = \begin{cases} +1 \text{ if } \boldsymbol{\theta}^T \boldsymbol{x}'^{(t)} \geq 0 \\ -1 \text{ otherwise} \end{cases}$
  - Observe its true label, $y^{(t)}$
  - If we misclassified an example ($y^{(t)} \neq \hat{y}$):
    - $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + y^{(t)} \boldsymbol{x}'^{(t)}$

Automatically handles updating the intercept!

Perceptron Learning Algorithm (Intuition)

- Suppose $(\boldsymbol{x}, y) \in \mathcal{D}$ is a misclassified training example and $y = +1$
  - $\boldsymbol{\theta}^T \boldsymbol{x}$ is negative
  - After updating $\boldsymbol{\theta}_{new} = \boldsymbol{\theta} + y\boldsymbol{x}$:
  $\boldsymbol{\theta}_{new}^T \boldsymbol{x} = (\boldsymbol{\theta} + y\boldsymbol{x})^T \boldsymbol{x} = \boldsymbol{\theta}^T \boldsymbol{x} + y\boldsymbol{x}^T \boldsymbol{x}$
  which is less negative than $\boldsymbol{\theta}^T \boldsymbol{x}$
    - Because $y > 0$ and $\boldsymbol{x}^T \boldsymbol{x} > 0$
  - Our prediction for $\boldsymbol{x}$ "improved"!
- A similar argument holds if $y = -1$

(Online) Perceptron Learning Algorithm: Inductive Bias

- The decision boundary is linear and *recent mistakes are more important than older ones* (and should be corrected immediately)

(Online) Perceptron Learning Algorithm

- Initialize the parameters to all zeros:

$$\boldsymbol{\theta} = [0 \quad 0 \quad \cdots \quad 0]$$

- For $t = 1, 2, 3, \ldots$
  - Receive an unlabeled example, $\boldsymbol{x}^{(t)}$
  - Predict its label, $\hat{y} = \text{sign}\left(\boldsymbol{\theta}^T \boldsymbol{x'}^{(t)}\right)$
  - Observe its true label, $y^{(t)}$
  - If we misclassified an example ($y^{(t)} \neq \hat{y}$):
    - $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + y^{(t)} \boldsymbol{x'}^{(t)}$

(Batch) Perceptron Learning Algorithm

- Input: $\mathcal{D} = \{(\boldsymbol{x}^{(1)}, y^{(1)}), (\boldsymbol{x}^{(2)}, y^{(2)}), \dots, (\boldsymbol{x}^{(N)}, y^{(N)})\}$
- Initialize the parameters to all zeros:

$$\boldsymbol{\theta} = [0 \quad 0 \quad \cdots \quad 0]$$

- While NOT CONVERGED
  - For $t \in \{1, \dots, N\}$
    - Predict the label of $\boldsymbol{x}'^{(t)}$, $\hat{y} = \text{sign}\left(\boldsymbol{\theta}^T \boldsymbol{x}'^{(t)}\right)$
    - Observe its true label, $y^{(t)}$
    - If we misclassified $\boldsymbol{x}'^{(t)}$ $(y^{(t)} \neq \hat{y})$:
      - $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + y^{(t)} \boldsymbol{x}'^{(t)}$

Perceptron Mistake Bound

- Definitions:
  - A dataset $\mathcal{D}$ is *linearly separable* if ∃ a linear decision boundary that perfectly classifies the examples in $\mathcal{D}$
  - The margin, γ, of a dataset $\mathcal{D}$ is the greatest possible distance between a linear separator and the closest example in $\mathcal{D}$ to that linear separator

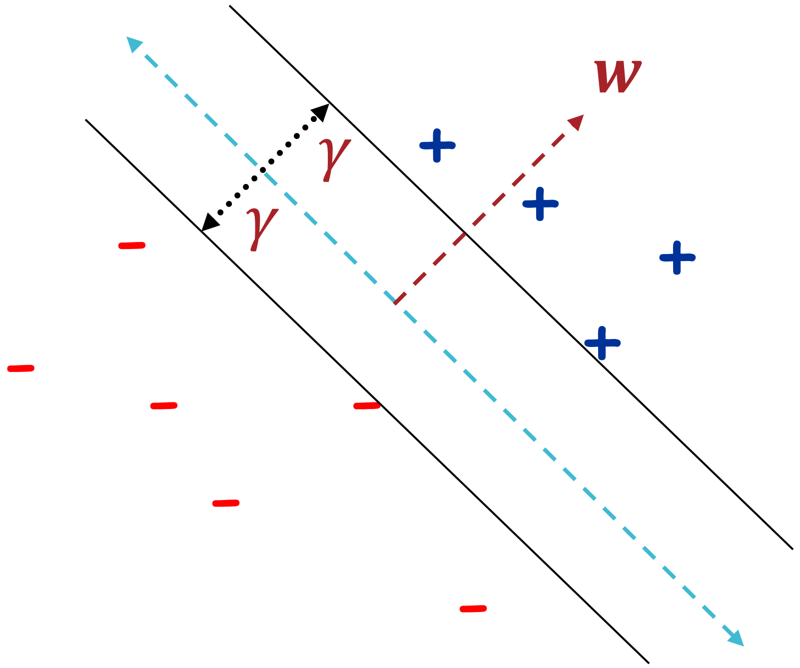


Figure courtesy of Nina Balcan

Perceptron Mistake Bound

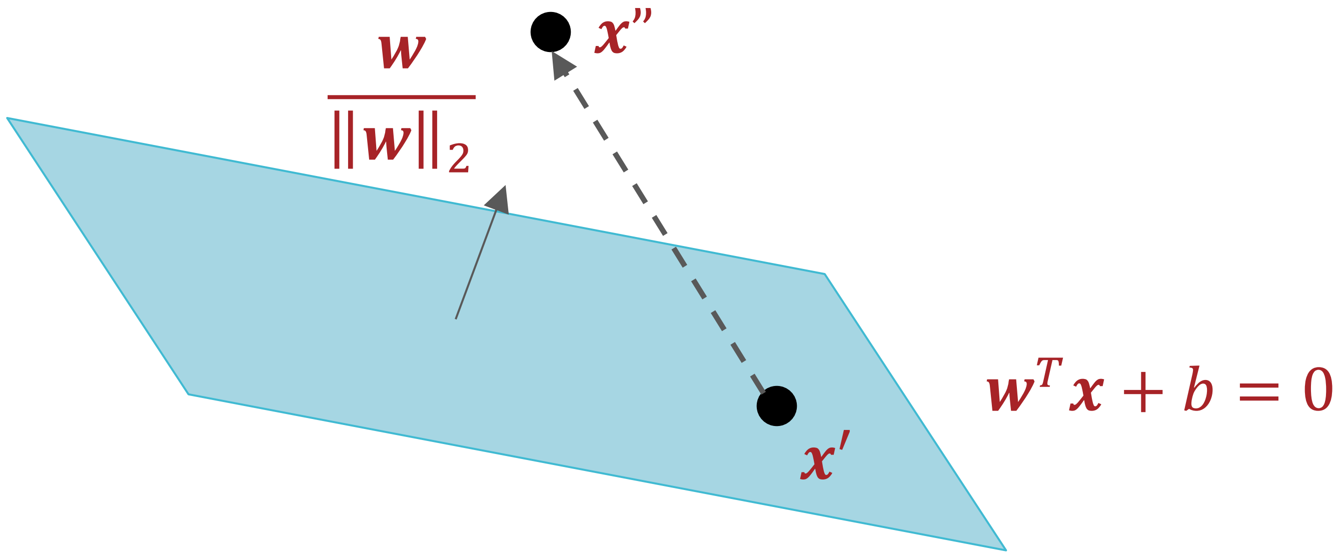
- Theorem: if the examples seen by the Perceptron Learning Algorithm (online and batch)
    1. lie in a ball of radius R (centered around the origin)
    2. have a margin of γ

then the algorithm makes at most $(R/\gamma)^2$ mistakes!

- Key Takeaway: if the training dataset is linearly separable, the batch Perceptron Learning Algorithm will converge (i.e., stop making mistakes on the training dataset or achieve 0 training error) in a finite number of steps!

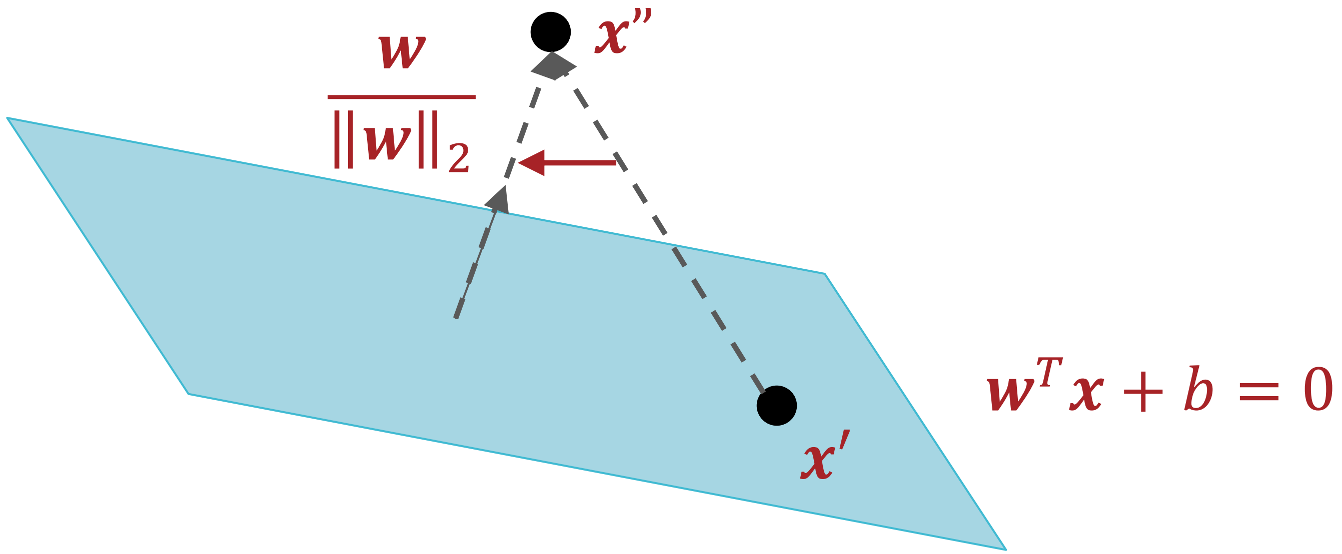
Computing the Margin (Bonus Content)

- Let $\boldsymbol{x}'$ be an arbitrary point on the hyperplane $\boldsymbol{w}^T\boldsymbol{x} + b = 0$ and let $\boldsymbol{x}''$ be an arbitrary point
- The distance between $\boldsymbol{x}''$ and $\boldsymbol{w}^T\boldsymbol{x} + b = 0$ is equal to the magnitude of the projection of $\boldsymbol{x}'' - \boldsymbol{x}'$ onto $\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|_2}$, the unit vector orthogonal to the hyperplane



Computing the Margin (Bonus Content)

- Let $\boldsymbol{x}'$ be an arbitrary point on the hyperplane $\boldsymbol{w}^T\boldsymbol{x} + b = 0$ and let $\boldsymbol{x}''$ be an arbitrary point
- The distance between $\boldsymbol{x}''$ and $\boldsymbol{w}^T\boldsymbol{x} + b = 0$ is equal to the magnitude of the projection of $\boldsymbol{x}'' - \boldsymbol{x}'$ onto $\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|_2}$, the unit vector orthogonal to the hyperplane



Computing the Margin (Bonus Content)

- Let $\boldsymbol{x}'$ be an arbitrary point on the hyperplane $\boldsymbol{w}^T\boldsymbol{x} + b = 0$ and let $\boldsymbol{x}''$ be an arbitrary point
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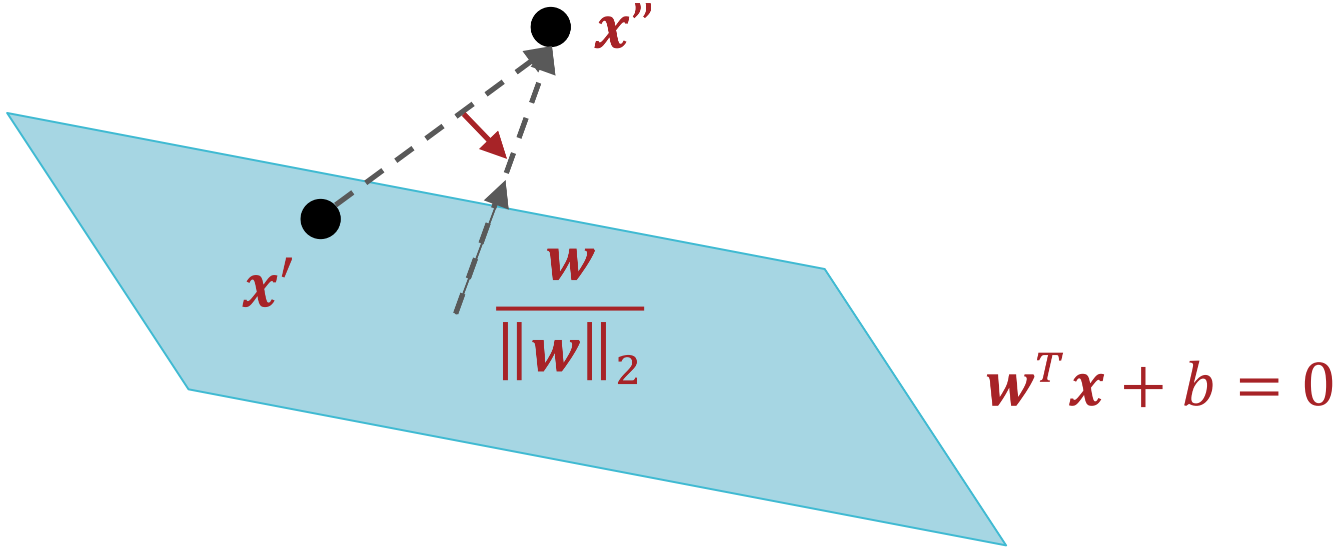

Computing the Margin (Bonus Content)

- Let $\boldsymbol{x}'$ be an arbitrary point on the hyperplane and let $\boldsymbol{x}''$ be an arbitrary point
- The distance between $\boldsymbol{x}''$ and $\boldsymbol{w}^T\boldsymbol{x} + b = 0$ is equal to the magnitude of the projection of $\boldsymbol{x}'' - \boldsymbol{x}'$ onto $\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|_2}$, the unit vector orthogonal to the hyperplane

$$\left|\frac{\boldsymbol{w}^T(\boldsymbol{x}'' - \boldsymbol{x}')}{\|\boldsymbol{w}\|_2}\right| = \frac{|\boldsymbol{w}^T\boldsymbol{x}'' - \boldsymbol{w}^T\boldsymbol{x}'|}{\|\boldsymbol{w}\|_2} = \frac{|\boldsymbol{w}^T\boldsymbol{x}'' + b|}{\|\boldsymbol{w}\|_2}$$

Key Takeaways

- Batch vs. online learning
- Perceptron learning algorithm for binary classification
- Impact of the bias term in perceptron
- Inductive bias of perceptron
- Convergence properties, guarantees and limitations for the batch Perceptron learning algorithm