10-301/601: Introduction to Machine Learning Lecture 27: Value and Policy Iteration

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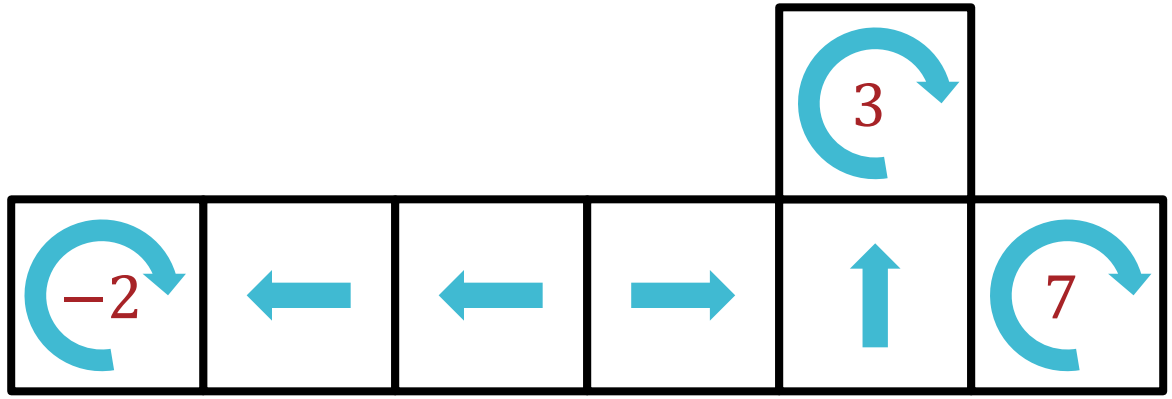
6/10/25

Front Matter

- Announcements:
  - HW6 released on 6/6, due 6/10 (today!) at 11:59 PM
  - HW7 to be released on 6/10 (today!), due 6/13 at 11:59 PM
  - Thursday's lecture will be a guest lecture by Alex Xie on Reinforcement Learning for LLMs
    - **Everyone who attends will have their lowest quiz grade down-weighted by 50%**
  - Final on 6/20 at **8:30 AM** in TBD
    - Lectures 17 – 30 are in-scope; **the guest lecture and next week's lectures will not be tested on the final**

Recall: Value Function

$\pi =$



$$R(s,a) = \begin{cases} -2 \text{ if entering state 0 (safety)} \\ 3 \text{ if entering state 5 (field goal)} \\ 7 \text{ if entering state 6 (touch down)} \\ 0 \text{ otherwise} \end{cases}$$

$\gamma = 0.9$

$V^{\pi} =$

0	−2	−1.8	2.7	3	0

$E[f(x)] = \sum_x p(x) f(x)$

deterministic rewards + stochastic transitions

Value Function

- $V^{\pi}(s) = \mathbb{E}[$discounted total reward of starting in state s and executing policy π forever$]$

$p(s'|s,a)$

$$= \mathbb{E}[R(s_0, \pi(s_0)) + \gamma R(s_1, \pi(s_1)) + \gamma^2 R(s_2, \pi(s_2)) + \cdots \mid s_0 = s]$$

$$= R(s, \pi(s)) + \gamma \mathbb{E}[R(s_1, \pi(s_1)) + \gamma R(s_2, \pi(s_2)) + \ldots \mid s_0 = s]$$

$$= R(s, \pi(s)) + \gamma \sum_{s_1 \in S} p(s_1 \mid s, \pi(s)) \big(R(s_1, \pi(s_1)) + \gamma \mathbb{E}\big[R(s_2, \pi(s_2)) + \cdots \mid s_1\big]\big)$$

Value Function

- $V^{\pi}(s) = \mathbb{E}[$discounted total reward of starting in state s and executing policy π forever$]$

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$$= R(s, \pi(s)) + \gamma \sum_{s_1 \in S} p(s_1 \mid s, \pi(s)) \Big(R(s_1, \pi(s_1)) + \gamma \mathbb{E}\big[R(s_2, \pi(s_2)) + \cdots \mid s_1 \big] \Big)$$

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- $V^{\pi}(s) = \mathbb{E}[$discounted total reward of starting in state s and executing policy π forever$]$

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Value Function

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Value Function

- $V^{\pi}(s) = \mathbb{E}[$discounted total reward of starting in state s and executing policy π forever$]$

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$$= R(s, \pi(s)) + \gamma \mathbb{E}[R(s_1, \pi(s_1)) + \gamma R(s_2, \pi(s_2)) + \ldots \mid s_0 = s]$$

$$= R(s, \pi(s)) + \gamma \sum_{s_1 \in S} p(s_1 \mid s, \pi(s)) \Big[\big(R(s_1, \pi(s_1)) + \gamma \mathbb{E}\big[R(s_2, \pi(s_2)) + \cdots \mid s_1\big]\big) \Big]$$

$$V^{\pi}(s) = R(s, \pi(s)) + \gamma \sum_{s_1 \in S} p(s_1 \mid s, \pi(s)) V^{\pi}(s_1)$$

Bellman equations

Optimality

- Optimal value function:

$$V^*(s) = \max_{a \in \mathcal{A}} \left(R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s' \mid s,a) V^*(s') \right)$$

  - System of $|\mathcal{S}|$ equations and $|\mathcal{S}|$ variables

- Optimal policy:

$$\pi^*(s) = \operatorname*{argmax}_{a \in \mathcal{A}} \left(R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s' \mid s,a) V^*(s') \right)$$

Immediate reward

(Discounted) Future reward

transition function

Fixed Point Iteration

- Iterative method for solving a system of equations
- Given some equations and initial values

$$x_1 = f_1(x_1, \dots, x_n)$$
$$\vdots$$
$$x_n = f_n(x_1, \dots, x_n)$$
$$x_1^{(0)}, \dots, x_n^{(0)}$$

- While not converged, do

$$x_1^{(t+1)} \leftarrow f_1\left(x_1^{(t)}, \dots, x_n^{(t)}\right)$$
$$\vdots$$
$$x_n^{(t+1)} \leftarrow f_n\left(x_1^{(t)}, \dots, x_n^{(t)}\right)$$

Fixed Point Iteration: Example

$$x_1 = x_1 x_2 + \frac{1}{2} = 0 \cdot 0 + \frac{1}{2} = \frac{1}{2} = x_1^{(1)}$$

$$x_2 = -\frac{3x_1}{2} = -\frac{3}{2}(0) = x_2^{(1)} = 0$$

$$x_1^{(0)} = x_2^{(0)} = 0$$

$$\hat{x}_1 = \frac{1}{3}, \hat{x}_2 = -\frac{1}{2}$$

t	$x_1^{(t)}$	$x_2^{(t)}$
0	0	0

Value Iteration

- Inputs: $R(s,a)$, $p(s' \mid s,a)$
- Initialize $V^{(0)}(s) = 0 \; \forall \; s \in \mathcal{S}$ (or randomly) and set $t = 0$
- While not converged, do:
  - For $s \in \mathcal{S}$

$$V^{(t+1)}(s) \leftarrow \max_{a \in \mathcal{A}} \underbrace{R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s' \mid s,a) V^{(t)}(s')}_{Q(s,a)}$$

  - $t = t + 1$
- For $s \in \mathcal{S}$

$$\pi^*(s) \leftarrow \operatorname*{argmax}_{a \in \mathcal{A}} R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s' \mid s,a) V^{(t)}(s')$$

- Return π^*

Value Iteration

- Inputs: $R(s,a)$, $p(s' \mid s,a)$
- Initialize $V^{(0)}(s) = 0 \; \forall \; s \in \mathcal{S}$ (or randomly) and set $t = 0$
- While not converged, do:
  - For $s \in \mathcal{S}$
    - For $a \in \mathcal{A}$

$$Q(s,a) = R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s' \mid s,a) V^{(t)}(s')$$

    - $V^{(t+1)}(s) \leftarrow \max_{a \in \mathcal{A}} Q(s,a)$
  - $t = t + 1$
- For $s \in \mathcal{S}$

$$\pi^*(s) \leftarrow \operatorname*{argmax}_{a \in \mathcal{A}} R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s' \mid s,a) V^{(t)}(s')$$

- Return π^*

Lecture 27 + 28 Polls

0 surveys completed

0 surveys underway

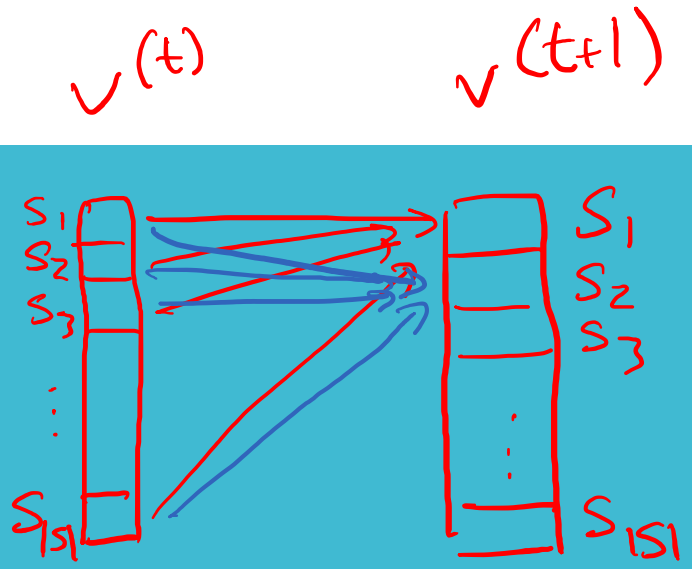
What is the runtime of one iteration of value iteration?

$O(|\mathcal{S}||\mathcal{A}|)$

$O(|\mathcal{S}|^2|\mathcal{A}|)$

$O(|\mathcal{S}||\mathcal{A}|^2)$

$O(|\mathcal{S}|^2|\mathcal{A}|^2)$



Synchronous Value Iteration

- Inputs: $R(s,a)$, $p(s' \mid s,a)$
- Initialize $V^{(0)}(s) = 0 \; \forall \; s \in \mathcal{S}$ (or randomly) and set $t = 0$
- While not converged, do:
  - For $s \in \mathcal{S}$
    - For $a \in \mathcal{A}$

$$Q(s,a) = R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s' \mid s,a) V^{(t)}(s')$$

    - $V^{(t+1)}(s) \leftarrow \max_{a \in \mathcal{A}} Q(s,a)$
  - $t = t + 1$
- For $s \in \mathcal{S}$

$$\pi^*(s) \leftarrow \operatorname{argmax}_{a \in \mathcal{A}} R(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s' \mid s,a) V^{(t)}(s')$$

- Return π^*

Asynchronous Value Iteration

- Inputs: $R(s, a)$, $p(s' \mid s, a)$
- Initialize $V^{(0)}(s) = 0 \; \forall \; s \in \mathcal{S}$ (or randomly)
- While not converged, do:
  - For $s \in \mathcal{S}$
    - For $a \in \mathcal{A}$

$$Q(s, a) = R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s' \mid s, a) V(s')$$

    - $V(s) \leftarrow \max_{a \in \mathcal{A}} Q(s, a)$

- For $s \in \mathcal{S}$

$$\pi^*(s) \leftarrow \operatorname{argmax}_{a \in \mathcal{A}} R(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s' \mid s, a) V(s')$$

- Return π^*

Value Iteration Theory

- **Theorem 1**: Value function convergence

V will converge to V^* if each state is "visited"

infinitely often (Bertsekas, 1989)

- **Theorem 2**: Convergence criterion

$$\text{if } \max_{s \in \mathcal{S}} \left| V^{(t+1)}(s) - V^{(t)}(s) \right| < \epsilon,$$

$$\text{then } \max_{s \in \mathcal{S}} \left| V^{(t+1)}(s) - V^{*}(s) \right| < \frac{2\epsilon\gamma}{1-\gamma} \text{ (Williams \& Baird, 1993)}$$

- **Theorem 3**: Policy convergence

The "greedy" policy, $\pi(s) = \underset{a \in \mathcal{A}}{\text{argmax}}\ Q(s,a)$, converges to the optimal π^* in a finite number of iterations, often before the value function has converged! (Bertsekas, 1987)

Policy Iteration

- Inputs: $R(s,a)$, $p(s' \mid s,a)$
- Initialize π randomly
- While not converged, do:
  - Solve the Bellman equations defined by policy π

$|S|$

$$V^{\pi}(s) = R\big(s,\pi(s)\big) + \gamma \sum_{s' \in S} p\big(s' \mid s,\pi(s)\big) V^{\pi}(s')$$

  - Update π

$$\pi(s) \leftarrow \underset{a \in \mathcal{A}}{\operatorname{argmax}}\ R(s,a) + \gamma \sum_{s' \in S} p(s' \mid s,a) V^{\pi}(s')$$

- Return π

Policy Iteration Theory

- In policy iteration, the policy improves in each iteration.
- Given finite state and action spaces, there are finitely many possible policies
- Thus, the number of iterations needed to converge is bounded!

- Policy iteration takes $O(|\mathcal{S}|^2|\mathcal{A}| + |\mathcal{S}|^3)$ time / iteration
  - However, empirically policy iteration requires fewer iterations to converge

Two big Q's

1. What can we do if the reward and/or transition functions/distributions are unknown?

2. How can we handle infinite (or just very large) state/action spaces?

Key Takeaways

- If the reward and transition functions are known, we can solve for the optimal policy (and value function) using value or policy iteration
  - Both algorithms are instances of fixed point iteration and are guaranteed to converge (under some assumptions)