11 Laplace Transforms

In Chapter 6, we covered a type of generating function known as the z-transform, which is particularly well suited to discrete, integer-valued, random variables. In this chapter, we will introduce a new type of generating function, called the Laplace transform, which is particularly well suited to common continuous random variables.

11.1 Motivating Example

We start with a motivating example.

Question: Let $X \sim \text{Exp}(\lambda)$. How can we derive $\mathbf{E}\left[X^3\right]$?

Answer: By definition,

$$\mathbf{E}\left[X^3\right] = \int_0^\infty t^3 \lambda e^{-\lambda t} dt.$$

While this is doable, it requires applying integration by parts many times – enough to guarantee that our answer will be wrong. In this chapter, we will see how Laplace transforms can be used to quickly yield the kth moment of $X \sim \text{Exp}(\lambda)$, for any k.

11.2 The Transform as an Onion

As in the case of the z-transform, we can think of the Laplace transform of a random variable (r.v.) as an onion, where the onion is an expression that contains all the moments of the r.v. The Laplace onion (Figure 11.1) looks different than the z-transform onion (Figure 6.1), but the basic point is the same: higher moments are stored deeper inside the onion and thus more peeling (tears) are required to get to them.

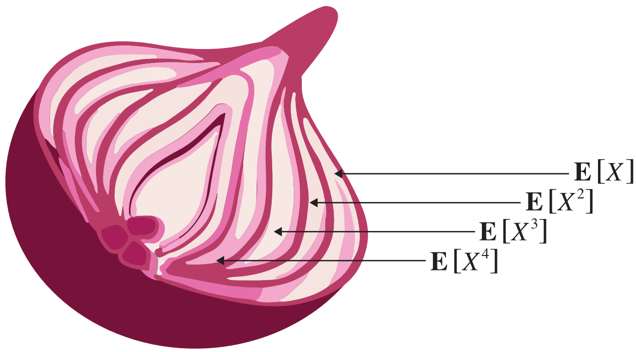

Figure 11.1 *The Laplace transform onion.*

Definition 11.1 *The* **Laplace transform**, $L_f(s)$, *of a continuous function,* $f(t)$, $t \geq 0$, *is defined as*

$$L_f(s) = \int_0^\infty e^{-st} f(t) dt.$$

Observe that the Laplace transform is a function of s. Here s should be thought of as a placeholder that keeps the layers of the onion separate, similar to the function of z in the z-transform.

When we speak of the Laplace transform of a continuous r.v. X, we are referring to the Laplace transform of the probability density function (p.d.f.), $f_X(t)$, associated with X.

Definition 11.2 *Let* X *be a non-negative continuous r.v. with p.d.f.* $f_X(t)$*. Then the* **Laplace transform** *of* X *is denoted by* $\widetilde{X}(s)$*, where*

$$\widetilde{X}(s) = L_{f_X}(s) = \int_0^\infty e^{-st} f_X(t) dt = \mathbf{E}\left[e^{-sX}\right].$$

Throughout, we will imagine that s is a constant where $s \geq 0$.

Question: What is $\widetilde{X}(0)$?

Theorem 11.3 *For all continuous random variables,* X,

$$\widetilde{X}(0) = 1.$$

Proof:

$$\widetilde{X}(0) = \mathbf{E}\left[e^{-0 \cdot X}\right] = 1. \qquad \blacksquare$$

11.3 Creating the Transform: Onion Building

The Laplace transform is defined so as to be really easy to compute for all the commonly used continuous random variables. Below are some examples.

Example 11.4 *Derive the Laplace transform of $X \sim Exp(\lambda)$:*

$$\widetilde{X}(s) = \int_0^\infty e^{-st} \lambda e^{-\lambda t} dt = \lambda \int_0^\infty e^{-(\lambda+s)t} dt = \frac{\lambda}{\lambda + s}.$$

Example 11.5 *Derive the Laplace transform of $X = a$, where a is some constant:*

$$\widetilde{X}(s) = e^{-sa}.$$

Example 11.6 *Derive the Laplace transform of $X \sim Uniform(a, b)$, $a, b \geq 0$:*

$$\begin{aligned}\widetilde{X}(s) &= \int_0^\infty e^{-st} f_X(t) dt \\ &= \int_a^b e^{-st} \frac{1}{b-a} dt \\ &= \left(\frac{-e^{-sb}}{s} + \frac{e^{-sa}}{s} \right) \frac{1}{b-a} \\ &= \frac{e^{-sa} - e^{-sb}}{s(b-a)}.\end{aligned}$$

Question: How do we know that the Laplace transform converges?

Theorem 11.7 (Convergence of Laplace transform) *$\widetilde{X}(s)$ is bounded for any non-negative continuous r.v. X, assuming $s \geq 0$.*

Proof: Observe that

$$e^{-t} \leq 1,$$

for all non-negative values of t. Since $s \geq 0$, it follows that

$$e^{-st} = \left(e^{-t}\right)^s \leq 1.$$

Thus:

$$\widetilde{X}(s) = \int_0^\infty e^{-st} f_X(t) dt \leq \int_0^\infty 1 \cdot f_X(t) dt = 1. \qquad \blacksquare$$

Question: Why don't we use the z-transform for continuous random variables?

Answer: We could, in theory. It just looks uglier. Consider, for example, the z-transform of $X \sim \text{Exp}(\lambda)$:

$$\widehat{X}(z) = \mathbf{E}\left[z^X\right] = \int_{t=0}^{\infty} z^t \cdot \lambda e^{-\lambda t} dt.$$

This doesn't look fun to integrate! However, it can be done, if we first express z^t as $e^{t \ln z}$. Try it!

11.4 Getting Moments: Onion Peeling

Once we have created the onion corresponding to r.v., X, we can "peel its layers" to extract the moments of X.

Theorem 11.8 (Onion peeling) *Let X be a non-negative, continuous r.v. with p.d.f. $f_X(t)$, $t \geq 0$. Then:*

$$\widetilde{X}'(s)\Big|_{s=0} = -\mathbf{E}[X]$$

$$\widetilde{X}''(s)\Big|_{s=0} = \mathbf{E}\left[X^2\right]$$

$$\widetilde{X}'''(s)\Big|_{s=0} = -\mathbf{E}\left[X^3\right]$$

$$\widetilde{X}''''(s)\Big|_{s=0} = \mathbf{E}\left[X^4\right]$$

$$\vdots$$

Note: If the above moments are not defined at $s = 0$, one can instead consider the limit as $s \to 0$.

Example 11.9 (Higher moments of Exponential) *Derive the kth moment of $X \sim Exp(\lambda)$:*

$$\widetilde{X}(s) = \frac{\lambda}{\lambda + s} = \lambda(\lambda + s)^{-1}$$

$$\widetilde{X}'(s) = -\lambda(\lambda + s)^{-2} \implies \mathbf{E}[X] = \frac{1}{\lambda}$$

$$\widetilde{X}''(s) = 2\lambda(\lambda + s)^{-3} \implies \mathbf{E}\left[X^2\right] = \frac{2}{\lambda^2}$$

$$\widetilde{X}'''(s) = -3!\lambda(\lambda + s)^{-4} \implies \mathbf{E}\left[X^3\right] = \frac{3!}{\lambda^3}$$

We can show via induction that:

$$\mathbf{E}\left[X^k\right] = \frac{k!}{\lambda^k}.$$

Proof: [Theorem 11.8] Below we provide a sketch of the proof argument. A more compact version of this proof is given in Exercise 11.3. However, for now we choose to write it out this way so that you can visualize exactly how the moments "pop" out of the transform when it's differentiated.

We start with the Taylor series expansion of e^{-st}:

$$e^{-st} = 1 - (st) + \frac{(st)^2}{2!} - \frac{(st)^3}{3!} + \frac{(st)^4}{4!} - \cdots$$

$$e^{-st} f(t) = f(t) - (st)f(t) + \frac{(st)^2}{2!} f(t) - \frac{(st)^3}{3!} f(t) + \frac{(st)^4}{4!} f(t) - \cdots$$

$$\int_0^\infty e^{-st} f(t)dt = \int_0^\infty f(t)dt - \int_0^\infty (st) f(t)dt + \int_0^\infty \frac{(st)^2}{2!} f(t)dt - \cdots$$

$$\widetilde{X}(s) = 1 - s\mathbf{E}[X] + \frac{s^2}{2!}\mathbf{E}\left[X^2\right] - \frac{s^3}{3!}\mathbf{E}\left[X^3\right] + \frac{s^4}{4!}\mathbf{E}\left[X^4\right] - \frac{s^5}{5!}\mathbf{E}\left[X^5\right] + \cdots$$

$$\widetilde{X}'(s) = -\mathbf{E}[X] + s\mathbf{E}\left[X^2\right] - \frac{1}{2!}s^2\mathbf{E}\left[X^3\right] + \frac{1}{3!}s^3\mathbf{E}\left[X^4\right] - \frac{1}{4!}s^4\mathbf{E}\left[X^5\right] + \cdots$$

$$\widetilde{X}'(0) = -\mathbf{E}[X] \checkmark$$

$$\widetilde{X}''(s) = \mathbf{E}\left[X^2\right] - s\mathbf{E}\left[X^3\right] + \frac{1}{2!}s^2\mathbf{E}\left[X^4\right] - \frac{1}{3!}s^3\mathbf{E}\left[X^5\right] + \cdots$$

$$\widetilde{X}''(0) = \mathbf{E}\left[X^2\right] \checkmark$$

$$\widetilde{X}'''(s) = -\mathbf{E}\left[X^3\right] + s\mathbf{E}\left[X^4\right] - \frac{1}{2!}s^2\mathbf{E}\left[X^5\right] + \cdots$$

$$\widetilde{X}'''(0) = -\mathbf{E}\left[X^3\right] \checkmark$$

And so on ... ∎

Question: At this point, you might be wondering why we don't define the Laplace transform of X to be $\mathbf{E}\left[e^{sX}\right]$, rather than $\mathbf{E}\left[e^{-sX}\right]$. What would be the pros and cons of using $\mathbf{E}\left[e^{sX}\right]$?

Answer: On the plus side, using $\mathbf{E}\left[e^{sX}\right]$ would obviate the need for the alternating negative signs. On the minus side, we would not have the convergence guarantee from Theorem 11.7.

As in the case of z-transforms, we will assume that the Laplace transform (when it converges) uniquely determines the distribution.

11.5 Linearity of Transforms

Just as we had a linearity theory for z-transforms, we have a similar result for Laplace transforms. Again, the random variables need to be independent!

Theorem 11.10 (Linearity) *Let X and Y be continuous, non-negative, independent random variables. Let $Z = X + Y$. Then,*

$$\widetilde{Z}(s) = \widetilde{X}(s) \cdot \widetilde{Y}(s).$$

Proof:

$$\begin{aligned} \widetilde{Z}(s) = \mathbf{E}\left[e^{-sZ}\right] &= \mathbf{E}\left[e^{-s(X+Y)}\right] \\ &= \mathbf{E}\left[e^{-sX} \cdot e^{-sY}\right] \\ &= \mathbf{E}\left[e^{-sX}\right] \cdot \mathbf{E}\left[e^{-sY}\right] \qquad \text{(because } X \perp Y\text{)} \\ &= \widetilde{X}(s) \cdot \widetilde{Y}(s). \end{aligned}$$

■

11.6 Conditioning

Conditioning also holds for Laplace transforms, just as it held for z-transforms:

Theorem 11.11 *Let X, A, and B be continuous random variables where*

$$X = \begin{cases} A & \text{w/prob } p \\ B & \text{w/prob } 1-p \end{cases}.$$

Then,

$$\widetilde{X}(s) = p \cdot \widetilde{A}(s) + (1-p) \cdot \widetilde{B}(s).$$

Proof:

$$\begin{aligned}\widetilde{X}(s) &= \mathbf{E}\left[e^{-sX}\right] \\ &= \mathbf{E}\left[e^{-sX}\middle| X = A\right]\cdot p + \mathbf{E}\left[e^{-sX}\middle| X = B\right]\cdot(1-p) \\ &= p\mathbf{E}\left[e^{-sA}\right] + (1-p)\mathbf{E}\left[e^{-sB}\right] \\ &= p\widetilde{A}(s) + (1-p)\widetilde{B}(s).\end{aligned}$$ ■

Theorem 11.12 is a generalization of Theorem 11.11, where we condition not just on two options, but a continuum of options. Theorem 11.12 is useful when you have a r.v. that depends on the value of another r.v.

Theorem 11.12 *Let Y be a non-negative continuous r.v., and let X_Y be a continuous r.v. that depends on Y. Then, if $f_Y(y)$ denotes the p.d.f. of Y, we have that*

$$\widetilde{X_Y}(s) = \int_{y=0}^{\infty} \widetilde{X_y}(s) f_Y(y)dy.$$

Proof: Observe that it is the fact that a transform is just an expectation that allows us to do the conditioning below:

$$\begin{aligned}\widetilde{X_Y}(s) = \mathbf{E}\left[e^{-sX_Y}\right] &= \int_{y=0}^{\infty} \mathbf{E}\left[e^{-sX_Y}\middle| Y = y\right]\cdot f_Y(y)dy \\ &= \int_{y=0}^{\infty} \mathbf{E}\left[e^{-sX_y}\right]\cdot f_Y(y)dy \\ &= \int_{y=0}^{\infty} \widetilde{X_y}(s)\cdot f_Y(y)dy.\end{aligned}$$ ■

An example of where Theorem 11.12 is used is given in Exercise 11.13. We will see many more examples when we get to later chapters on stochastic processes.

11.7 Combining Laplace and z-Transforms

Consider again the sum of a random number of random variables, similarly to what we did in Chapter 6, but this time where the random variables being summed are continuous.

Theorem 11.13 (Summing a random number of i.i.d. random variables) *Let $X_1, X_2, X_3, \ldots$ be i.i.d. continuous random variables, where $X_i \sim X$. Let N be a positive discrete r.v., where $N \perp X_i$ for all i. Let*

$$S = \sum_{i=1}^{N} X_i.$$

Then,

$$\widetilde{S}(s) = \widehat{N}\left(\widetilde{X}(s)\right),$$

that is, the z parameter of $\widehat{N}(z)$ has been replaced by $\widetilde{X}(s)$.

Example 11.14 (Transform of a Poisson number of i.i.d. Exponentials) *Derive the Laplace transform of a Poisson(λ) number of i.i.d. Exp(μ) random variables.*

Recall that for $N \sim \text{Poisson}(\lambda)$ we have that $\widehat{N}(z) = e^{-\lambda(1-z)}$. Recall likewise that for $X \sim \text{Exp}(\mu)$ we have that

$$\widetilde{X}(s) = \frac{\mu}{s+\mu}.$$

From this it follows that

$$\widetilde{S}(s) = \widehat{N}(\widetilde{X}(s)) = e^{-\lambda(1-z)}\Big|_{z=\frac{\mu}{s+\mu}} = e^{-\lambda\left(1-\frac{\mu}{s+\mu}\right)} = e^{-\frac{\lambda s}{s+\mu}}.$$

Proof: (Theorem 11.13) Let $\widetilde{S}(s \mid N = n)$ denote the Laplace transform of S given $N = n$. By Theorem 11.10, $\widetilde{S}(s \mid N = n) = \left(\widetilde{X}(s)\right)^n$. By conditioning,

$$\begin{aligned}\widetilde{S}(s) = \sum_{n=0}^{\infty} \mathbf{P}\{N = n\}\, \widetilde{S}(s \mid N = n) &= \sum_{n=0}^{\infty} \mathbf{P}\{N = n\} \left(\widetilde{X}(s)\right)^n \\ &= \widehat{N}\left(\widetilde{X}(s)\right).\end{aligned}$$

■

11.8 One Final Result on Transforms

Normally we look at the Laplace transform of the p.d.f., but we could also ask about the Laplace transform of an arbitrary function. Theorem 11.15 considers the Laplace transform of the cumulative distribution function (c.d.f.) and relates that to the Laplace transform of the p.d.f.

Theorem 11.15 *Let $B(x)$ be the c.d.f. corresponding to p.d.f. $b(t)$, where $t \geq 0$. That is,*

$$B(x) = \int_0^x b(t)dt.$$

Let

$$\widetilde{b}(s) = L_{b(t)}(s) = \int_0^\infty e^{-st} b(t)dt.$$

Let

$$\widetilde{B}(s) = L_{B(x)}(s) = \int_0^\infty e^{-sx} B(x)dx = \int_0^\infty e^{-sx} \int_0^x b(t)dtdx.$$

Then,

$$\widetilde{B}(s) = \frac{\widetilde{b}(s)}{s}.$$

Proof: The proof is just a few lines. See Exercise 11.4. ∎

11.9 Exercises

11.1 **Conditioning practice**
Let $X_1 \sim \text{Exp}(\mu_1)$. Let $X_2 \sim \text{Exp}(\mu_2)$. Assume $X_1 \perp X_2$. Let

$$X = \begin{cases} X_1 & \text{w/prob } \frac{1}{2} \\ X_1 + X_2 & \text{w/prob } \frac{1}{3} \\ 1 & \text{w/prob } \frac{1}{6} \end{cases}.$$

What is $\widetilde{X}(s)$?

11.2 **Effect of doubling**
Let $X \sim \text{Exp}(\lambda)$. Let $Y = 2X$. What is $\widetilde{Y}(s)$?

11.3 **Compact proof of onion peeling**
In this problem we provide a more compact proof of Theorem 11.8. Let X be a non-negative, continuous r.v. with p.d.f. $f_X(t), t \geq 0$. Prove that:

$$\left.\frac{d^k}{ds^k}\widetilde{X}(s)\right|_{s=0} = (-1)^k \mathbf{E}\left[X^k\right].$$

[Hint: Bring the derivative into the integral of $\widetilde{X}(s)$ and simplify.]

11.4 **Relating the transform of the c.d.f. to the transform of the p.d.f.**
Prove Theorem 11.15.

11.5 **Inverting the transform**
You are given that the Laplace transform of r.v. X is:

$$\widetilde{X}(s) = \frac{3e^{-3s}}{3 + 4s + s^2}.$$

How is X distributed? You can express X in terms of other random variables.

11.6 **Two species of onions**
We have defined two types of onions: the z-transform and the Laplace transform. Show that these are actually the same. Let X be a r.v.
(a) Show that $\widetilde{X}(s)$ becomes $\widehat{X}(z)$ when s is a particular function of z.
(b) Show that $\widehat{X}(z)$ becomes $\widetilde{X}(s)$ when z is a particular function of s.

11.7 **Sum of Geometric number of Exponentials**
Let $N \sim \text{Geometric}(p)$. Let $X_i \sim \text{Exp}(\mu)$, where the X_i's are independent. Let $S_N = \sum_{i=1}^{N} X_i$. Use transforms to prove that S_N is Exponentially distributed and derive the rate of S_N.

11.8 **Downloading files**
You need to download two files: file 1 and file 2. File 1 is available via source A or source B. File 2 is available only via source C. The time to download file 1 from source A is Exponentially distributed with rate 1. The time to download file 1 from source B is Exponentially distributed with rate 2. The time to download file 2 from source C is Exponentially distributed with rate 3. All of these download times are independent. You decide to download from *all three* sources simultaneously, in the hope that you get both file 1 and file 2 as soon as possible. Let T denote the time until you get *both* files. What is $\widetilde{T}(s)$?

11.9 **Two-sided Laplace transform: Normal distribution**
In the case where a distribution can take on negative values, we define the Laplace transform as follows: Let X be a r.v. with p.d.f. $f(t)$, $-\infty < t < \infty$:

$$\widetilde{X}(s) = L_f(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt.$$

Let $X \sim \text{Normal}(0, 1)$ be the standard Normal. Prove that

$$\widetilde{X}(s) = e^{\frac{s^2}{2}}. \tag{11.1}$$

Note: More generally, if $X \sim \text{Normal}(\mu, \sigma^2)$, then

$$\widetilde{X}(s) = e^{-s\mu + \frac{1}{2}s^2\sigma^2}. \tag{11.2}$$

You only need to prove (11.1).

11.10 **Sum of two Normals**
Let $X \sim \text{Normal}(\mu_x, \sigma_x^2)$. Let $Y \sim \text{Normal}(\mu_y, \sigma_y^2)$. Assume $X \perp Y$. Derive the distribution of $X + Y$. First try doing this without Laplace transforms. After you give up, use Laplace transforms, specifically (11.2).

11.11 **Those tricky interview questions**
Let $X, Y \sim \text{Normal}(0, 1)$ be i.i.d. random variables. Derive $\mathbf{P}\{X < 3Y\}$.

11.12 **Heuristic proof of Central Limit Theorem (CLT) via transforms**
You will derive a heuristic proof of the CLT. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. non-negative random variables, each with distribution X and mean μ and variance σ^2. CLT says that the distribution of

$$\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \tag{11.3}$$

tends to the standard Normal as $n \to \infty$. Specifically,

$$\mathbf{P}\left\{\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le a\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx, \text{ as } n \to \infty.$$

We'll show that the Laplace transform of (11.3) (roughly) converges to that of the standard Normal (11.1), hence the underlying distributions are the same. Let

$$S = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}.$$

(a) Start with the case where $\mu = 0$ and $\sigma^2 = 1$.
 (i) Show that

$$\widetilde{S}(s) \approx \left(1 - \frac{s\mathbf{E}[X]}{\sqrt{n}} + \frac{s^2\mathbf{E}[X^2]}{2n}\right)^n.$$

 (ii) Using what you know about μ and σ^2, show that

$$\widetilde{S}(s) \to \widetilde{N_{(0,1)}}(s), \text{ as } n \to \infty.$$

(b) Now go back to the case where $\mu \neq 0$ and $\sigma^2 \neq 1$.
 (i) Define $Y_i = \frac{X_i - \mu}{\sigma}$. What are the mean and variance of Y_i?
 (ii) Based on (a), what can you say about $\mathbf{P}\left\{\frac{Y_1 + \cdots + Y_n}{\sqrt{n}} \le a\right\}$?
 (iii) What does (ii) tell us about $\mathbf{P}\left\{\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le a\right\}$?

11.13 **Random variable with random parameters**
The time until a light bulb burns out is Exponentially distributed with mean somewhere between $\frac{1}{2}$ year and 1 year. We model the lifetime using r.v. X_Y where $X_Y \sim \text{Exp}(Y)$ and $Y \sim \text{Uniform}(1, 2)$. Derive $\widetilde{X_Y}(s)$.