15-853:Algorithms in the Real World

Linear and Integer Programming III

- Integer Programming
  - Applications
  - Algorithms

Integer (linear) Programming

$$\begin{array}{ll} \textbf{minimize:} & c^T x \\ \textbf{subject to:} & Ax \leq b \\ & x \geq 0 \\ & x \in Z^n \end{array}$$

Related Problems

- Mixed Integer Programming (MIP)
- Zero-one programming
- Integer quadratic programming
- Integer nonlinear programming

History

- Introduced in 1951 (Dantzig)
- TSP as special case in 1954 (Dantzig)
- First convergent algorithm in 1958 (Gomory)
- General branch-and-bound technique 1960 (Land and Doig)
- Frequently used to prove bounds on approximation algorithms (late 90s)

Current Status

- Has become "dominant" over linear programming in past decade
- Saves industry Billions of Dollars/year
- Can solve 10,000+ city TSP problems
- 1 million variable LP approximations
- Branch-and-bound, Cutting Plane, and Separation all used in practice
- General purpose packages do not tend to work as well as with linear programming --- knowledge of the domain is critical.

Subproblems/Applications

- **Facility location**
  Locating warehouses or franchises (e.g. a Burger King)
- **Set covering and partitioning**
  Scheduling airline crews
- **Multicommodity distribution**
  Distributing auto parts
- **Traveling salesman and extensions**
  Routing deliveries
- **Capital budgeting**
- **Other Applications**
  VLSI layout, clustering

Knapsack Problem

Integer (zero-one) Program:

maximize $c^T x$
subject to: $ax \le b$
x **binary**

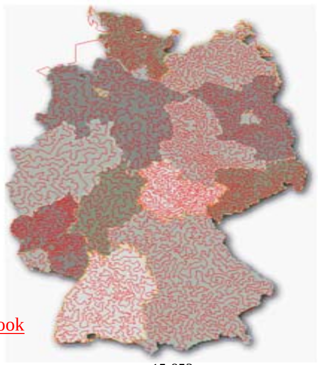
where:

b = maximum weight
c_i = utility of item i
a_i = weight of item i
x_i = 1 if item i is selected, or 0 otherwise

The problem is NP-hard.

Traveling Salesman Problem

Find shortest tours that visit all of n cities.



courtesy: Applegate, Bixby, Chvátal, and Cook

Traveling Salesman Problem

minimize: $$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to: $$\sum_{j=0}^{n} x_{ij} = 2 \quad 1 \le i \le n$$ (path enters and leaves)

$$x_{ji} = x_{ij} \text{ , binary}$$

$c_{ij} = c_{ji}$ = distance from city i to city j
(assuming **symmetric version**)
x_{ij} if tour goes from i to j or j to i, and 0 otherwise

Anything missing?

Traveling Salesman Problem

minimize: $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$

subject to: $\sum_{j=0}^{n} x_{ij} = 1 \qquad 1 \le i \le n$ (out degrees = 1)

$\sum_{i=0}^{n} x_{ij} = 1 \qquad 1 \le j \le n$ (in degrees = 1)

$t_i - t_j + nx_{ij} \le n-1 \qquad 2 \le i, j \le n$ (??)

c_{ij} = distance from city i to city j
x_{ij} = 1 if tour visits i then j, and 0 otherwise (binary)
t_i = arbitrary real numbers we need to solve for

Traveling Salesman Problem

minimize: $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$

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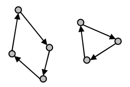
c_{ij} = distance from city i to city j
x_{ij} = 1 if tour visits i then j, and 0 otherwise (binary)
t_i = arbitrary real numbers we need to solve for

Traveling Salesman Problem

The last set of constraints: $t_i - t_j + nx_{ij} \le n-1 \quad \boxed{2} \le i, j \le n$
prevents "subtours":



Consider a cycle that goes from some node 4 to 5,
$t_4 - t_5 + nx_{4,5} \le n-1 \Rightarrow t_5 \ge t_4 + 1$
Similarly t has to increase by 1 along each edge of the cycle that does not include vertex 1.
Therefore, for a tour of length m that does not go through vertex 1, $t_4 \ge t_4 + m$, a contradiction.
Every cycle must go through vertex 1.
Together with other constraints, it forces one cycle.

Traveling Salesman Problem

Many "Real World" applications based on the TSP.
- They typically involve more involved constraints
- Not just routing type problems.
  Consider a drug company with k drugs they can make at a lab. They can only make the drugs one at a time. The cost of converting the equipment from making drug i to drug j is c_{ij}

Current best solutions are based on IP
- Applegate, Bixby, et. al., have solutions for more than 15K cities in Germany
  > 150,000 CPU hours (more info)
- Involves "branch-and-bound" and "cutting planes"

Set Covering Problem

Find cheapest sets that cover all elements

Courtesy: Darmstadt University of Technology

Set Covering and Partitioning

Given m sets and n items:

$$A_{ij} = \begin{cases} 1, & \text{if set j includes item i} \\ 0, & \text{otherwise} \end{cases}$$

$$c_i = \text{cost of set j}$$

$$x_j = \begin{cases} 1, & \text{if set j is included} \\ 0, & \text{otherwise} \end{cases}$$

Columns = sets
Rows = items

Set covering: **minimize:** $c^T x$
subject to: $Ax \geq 1$, x binary

Set partitioning: **minimize:** $c^T x$
subject to: $Ax = 1$, x binary

Set Covering and Partitioning

set	members	cost
s_1	{a,c,d}	.5
s_2	{b,c}	.2
s_3	{b,e}	.3
s_4	{a,d}	.1
s_5	{c,e}	.2
s_6	{b,c,e}	.6
s_7	{c,d}	.2

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Best cover: s_2, s_4, s_5 = .5
Best partition: s_4, s_6 = .7

Set Covering and Partitioning

Applications:

- **Facility location**.
  Each set is a facility (e.g. warehouse, fire station, emergency response center).
  Each item is an area that needs to be covered
- **Crew scheduling**.
  Each set is a route for a particular crew member (e.g. NYC->Pit->Atlanta->NYC).
  Each item is a flight that needs to be covered.

Constraints Expressible with IP

Many constraints are expressible with integer programming:

- logical constraints (e.g. x implies not y)
- k out of n
- piecewise linear functions
- … and many more

Constraints Expressible with IP

Logical constraints (x_1, x_2 binary):

Either x_1 or x_2	$\Rightarrow$	$x_1 + x_2 \geq 1$
If x_1 then x_2	$\Rightarrow$	$x_1 - x_2 \leq 0$
k out of n	$\Rightarrow$	$\sum_{1 \leq i \leq n} x_i = k$

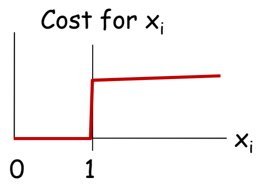
Combining constraints:

$$\text{Either } a_1x \leq b_1 \text{ or } a_2x \leq b_2 \;\Rightarrow\; \begin{array}{ll} a_1x - My & \leq b_1 \\ a_2x - M(1-y) & \leq b_2 \end{array}$$

y is a binary variable, M needs to be "large", a_1, a_2, and x can be vectors

Constraints Expressible with IP

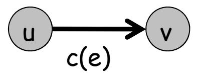
- **Discrete variables**: x_i in $\{k_1, k_2, …, k_n\}$
  - Create new binary vars z_j and add constraints $x_i = \sum_{1 \leq i \leq n} z_i k_i$ and $\sum_{1 \leq i \leq n} z_i = 1$
- **Piecewise linear functions:**
  - If $x_i \geq 1$ then $c_i \geq a_i x_i$
  - Convert to $(x_i < 1)$ or $(c_i \geq a_i x_i)$ and use prev. method.



Tricks for Expressing Constraints

- ***Covering Constraints:*** $Ax \geq b$ (non-negative a_{ij}'s)
- ***Packing constraints:*** $Ax \leq b$ (non-negative a_{ij}'s)
- ***Connectivity constraints*** (e.g. for network design):
  - flow formulation: to connect s and t, buy enough edges to support a unit s-t flow.
- ***Cut constraints*** (e.g., for clustering):
  - Distance formulation: e.g., separate p and q

    Variables d, x. Edge costs/lengths c(e):
    $d(p, v) \leq d(p, u) + x(e)$ for each edge
    $d(p,v) \geq 0$ for each v
    $d(p,q) \geq 1$



Algorithms

1. Use a linear program
   - round to integer solution (what if not feasible?)
2. Search
   - Branch and bound (integer ranges)
   - Implicit (0-1 variables)
3. Cutting planes
   - Many variants

Important Properties

- LP solution is an upper bound on IP solution (assuming maximization)
- If LP is infeasible then IP is infeasible
- If LP solution is integral (all variables have integer values), then it is the IP solution.

Linear Programming Solution

1. Some LP problems will always have integer solutions
   - transportation problem
   - assignment problem
   - min-cost network flow

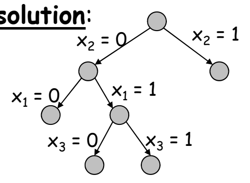
   These are problems with a unimodular matrix A.
   (unimodular matrices have det(A) = 1).
2. Solve as linear program and round. Can violate constraints, and be non-optimal. Works OK if
   - integer variables take on large values
   - accuracy of constraints is questionable

Branch and Bound

Lets first consider **0-1** programs.

Exponential solution: try all $\{0,1\}^n$

Branch-and-bound solution:



Traverse tree keeping current best solution.

If it can be shown that a subtree never improves on the current solution, **or** is infeasable, **prune** it.

Zero-One Branch and Bound

minimize: $z = c^T x$, subject to: $Ax \leq b, x \geq 0, x \in \{0,1\}^n$

Assume all elements of c are non-negative

```
function ZO_r(A, b, c, x_f, z*)
        // x_f: a fixed setting for a subset of the variables
        // z* is the cost of current best solution
    x = x_f + 0    // set unconstrained variables to zero
    if (cx ≥ z*) or (no feasible completion of x_f) return z*
    if (Ax ≤ b) then return cx
    pick an unconstrained variable x_i from x
    z_0* = ZO_r(A, b, x_f ∪ {x_i = 0}, c, z*)
    z_1* = ZO_r(A, b, x_f ∪ {x_i = 1}, c, z_0*)
    return z_1*
function ZO(A, b, c) = ZO_r(A, b, c, ∅, ∞)
```

Zero-One Branch and Bound

Checking for feasible completions: check each constraint and find if minimum of left is greater than right.

Example:

$x_f = \{x_1 = 1, x_3 = 0\}$

and one of the constraints is

$3x_1 + 2x_2 - x_3 + x_4 \leq 2$

then

$3 \quad + 2x_2 - 0 \ + x_4 \leq 2$

$2x_2 + x_4 \leq -1$

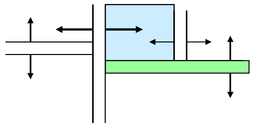
which is impossible.

Integer Branch and Bound

The zero-one version is sometimes called "implicit enumeration" since it might enumerate all possibilities.

An integer version **cannot** branch on all possible integer values for a variable. Even if the integer range is bounded, it is not practical.

Will "bound" by adding inequalities to split the two branches.



Since solutions are integral, each split can remove a strip 🟩 of width 1

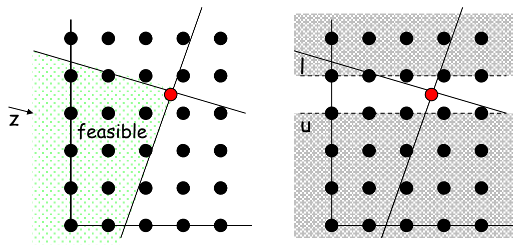
Integer Branch and Bound

maximize: $z = c^T x$, subject to: $Ax \leq b, x \geq 0, x \in Z^n$

```
function IP_r(A_e, b_e, c, z*)
        // A_e, b_e are A and b with additional constraints
        // z* is the cost of current best solution
    z, x, f = LP(A,b,c)    // f indicates whether feasible
    if not(f) or (z < z*) return z*
    if (integer(x)) return z
    pick a non-integer variable x_i' from x
    z_l* = IP(extend A_e,b_e with x_i ≤ ⌊x_i'⌋, c, z*)
    z_g* = IP(extend A_e,b_e with –x_i ≤ -⌈x_i'⌉, c, z_l*)
    return z_g*
function IP(A, b, c) = IP_r(A, b, c, -∞)
```

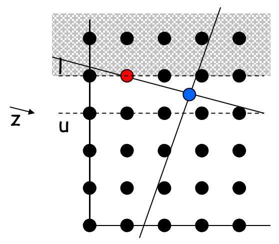
Note use of z_l*

Example



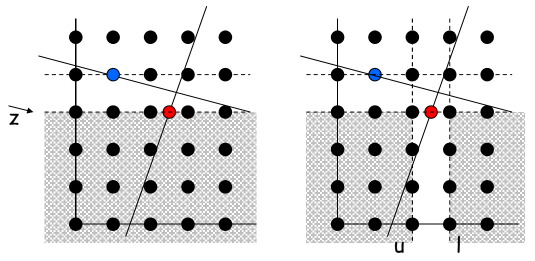
Find optimal solution.
Cut along y axis, and make two recursive calls

Example



Find optimal solution.
Solution is integral, so return it as current best z*

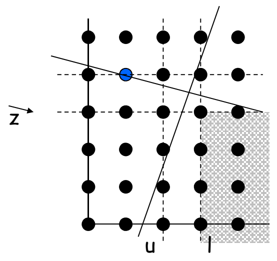
Example



● = z*

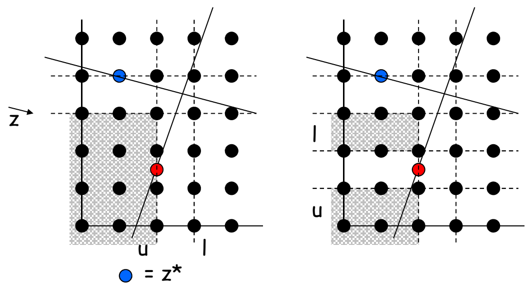
Find optimal solution. It is better than z*.
Cut along x axis, and make two recursive calls

Example



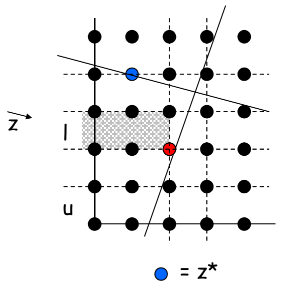
Infeasible, Return.

Example



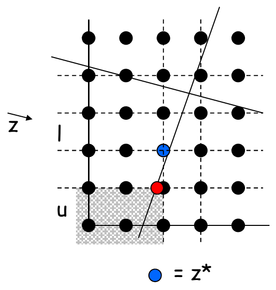
Find optimal solution. It is better than z*.
Cut along y axis, and make two recursive calls

Example



Find optimal solution. Solution is integral and better than z*. Return as new z*.

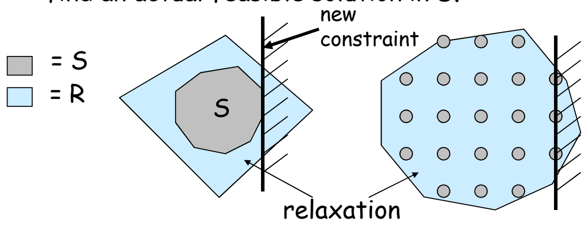
Example



Find optimal solution. Not as good as z*, return.

Cutting Plane

The idea is to start with a "relaxation" R of the problem and then add constraints on the fly to find an actual feasible solution in S.



Example 1

Example 2
A "linear" relaxation

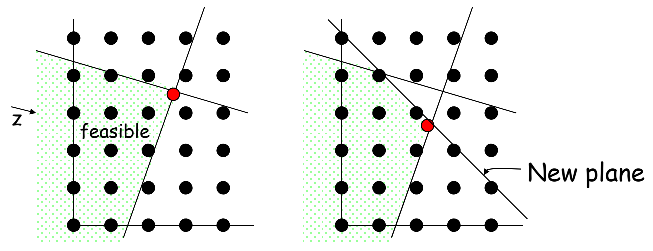
Cutting Plane: general algorithm

minimize: $z = c^T x$, subject to $x \in S$

```
function CP(R, c)
        // R a relaxed set of constraints $Ax \leq b$
           s.t. $S \subset$ polytope($Ax \leq b$)
   repeat:
      x = LP(R,c)
      if $x \in S$ return x
      find an inequality r satisfied by S,
           but violated by x (r separates x from S)
      R = R $\cup$ {r}
```

Can add multiple inequalities on each iteration

Cutting Plane



Note that we are removing a corner, and no integer solutions are being excluded.

Picking the Plane

Method 1: Gomory cuts (1958)

- Cuts are generated from the LP Tableau
  Each row defines a potential cut
- Guaranteed to converge on solution
- General purpose, but inefficient in practice

Method 2: problem specific cuts (templates)

- Consider the problem at hand and generate cuts based on its structure
- A **template** is a problem specific **set** of cuts (probably of exponential size) which S satisfies. Each round picks a cut from this set.

Templates for the TSP problem

We consider some example templates used in solutions of the Traveling Salesman Problem.

Recall that x_{ij} indicates the edge from v_i to v_j

Assume the symmetric TSP: $x_{ij} = x_{ji}$

Consider subsets of vertices $U \subset V$.

define: $\delta_x(U) = \sum x_{ij}$, $v_i \in U$, $v_j \in V-U$
(i.e. the number of times path crosses into/outof U)

Degree Constraints: $\delta_x(\{v_i\}) = 2$, $1 \leq i \leq n$

Subtour Constraints: $\delta_x(U) \geq 2$, $U \subset V$ **A template**

There are an exponential number of these

Templates for the TSP problem

A set of contraints (a template) is **facet-defining** for S if each constraint is on a facet of the convex hull of S.

We would like templates which are facet defining since, intuitively, they will more quickly constrain us to the boundary of S.

The subtour template is facet defining.

In practice the subtour inequalities are not enough to contrain the solution to integral solutions.

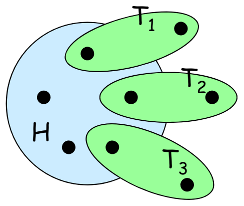
Are there other sets of facet defining constraints?

Templates for the TSP problem

Blossom inequalities (Edmonds 1965):

Defined by H (handle) and $T_1, \ldots, T_k$ (teeth) satisfying:
$k \geq 3$ and odd, $|T_i| = 2$

$T_i \cap T_j = \varnothing$, $|H \cap T_i| = 1$, $|T_i \setminus H| = 1$

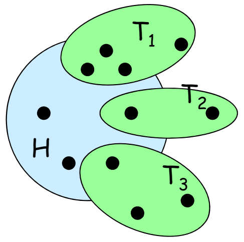


$$\delta_x(H) + \Sigma_{i=1}^{k} \delta_x(T_i) \geq 3k + 1$$

Templates for the TSP problem

Comb inequalities (Grotschel 1977)

Just generalizes T_i to be any size. At least one element of each T has to be in and out of H.



$$\delta_x(H) + \Sigma_{i=1}^{k} \delta_x(T_i) \geq 3k + 1$$

The art of Templates

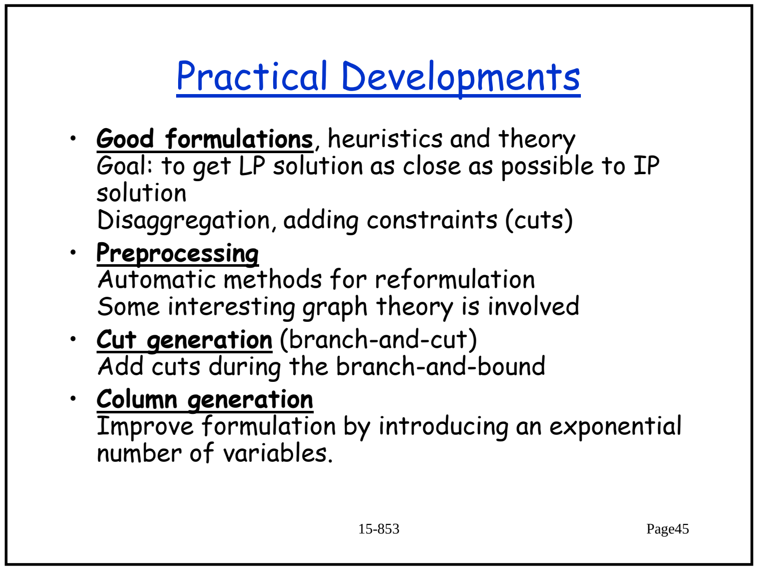
Picking the right set of templates, and applying them in the right way is the art of solving NP-hard problems with integer programming.

Different problems have different templates.

One needs to find good algorithms for selecting a member of a template that separates x from S (can be quite complicated on its own).

Cutting planes often used in conjunction with branch and bound.

Can interleave template cuts with Gomory cuts (e.g. use Gomory cuts when the set of template cuts "dries out").



Practical Developments

- **Good formulations**, heuristics and theory
  Goal: to get LP solution as close as possible to IP solution
  Disaggregation, adding constraints (cuts)
- **Preprocessing**
  Automatic methods for reformulation
  Some interesting graph theory is involved
- **Cut generation** (branch-and-cut)
  Add cuts during the branch-and-bound
- **Column generation**
  Improve formulation by introducing an exponential number of variables.

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