Meshing in Fixed Dimension in near Optimal Work and Time Sequential and Parallel

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Gary Miller
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Ohio
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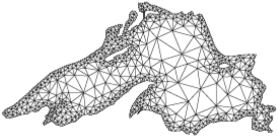

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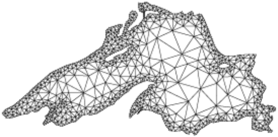

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- Geometric data structure – Intel chip

Who Uses Meshes?

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Who Uses Meshes?

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- Many people go to amazing ends not to mesh.

What do People say about the meshing problem?

- Half the people say the problem is solved.

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- Half the people say the problem is solved.
- The other half say the problem is impossible

The Static Meshing Problem

Meshing Algorithm Requirements:

- Guarantees on Element Quality

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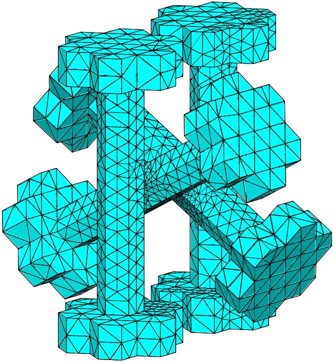
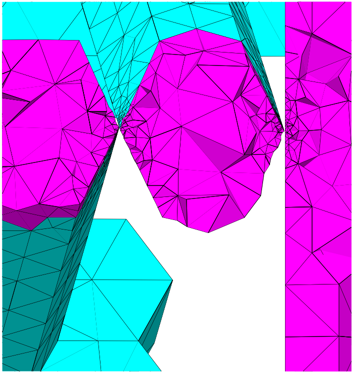
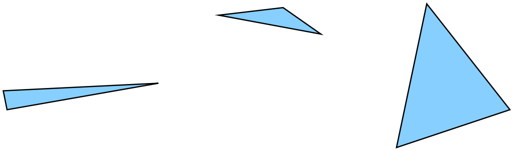
- Efficient Runtime and Space Usage

A Simple Example





Skinny Elements Bad, Round Elements Better



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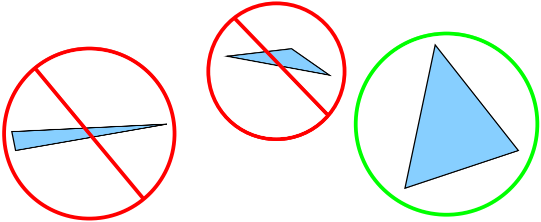

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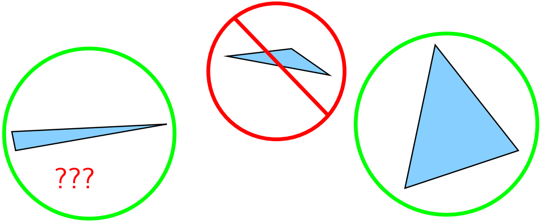
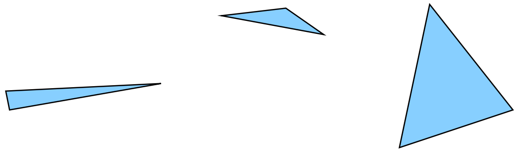


Determining Element Quality



- What does it mean to be round?

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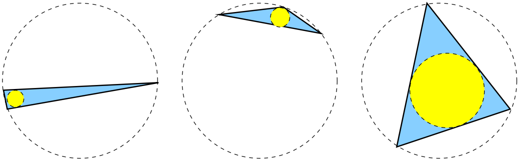

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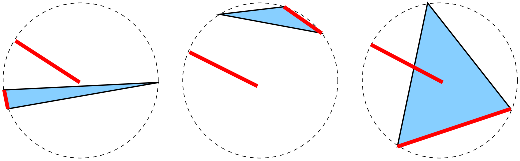

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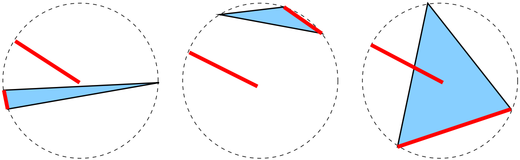

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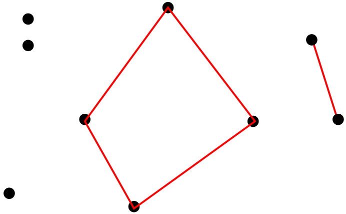
- Bounded Aspect Ratio
- Bounded Radius-Edge Ratio
- Input Parameter Determines "Good"

Topologically Conforming Meshes



- In 2 Dimensions, Features are Vertices and Edges

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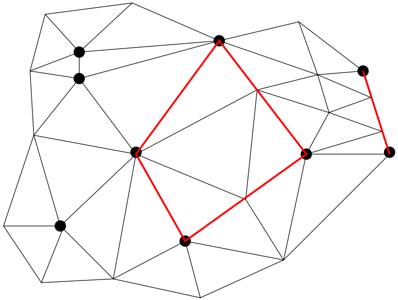
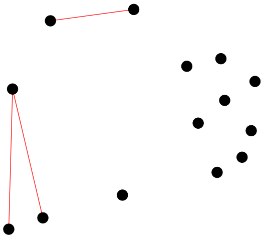


- In 2 Dimensions, Features are Vertices and Edges
- Mesh Must Contain Features (Topologically Conform)

Local Feature Size (lfs)



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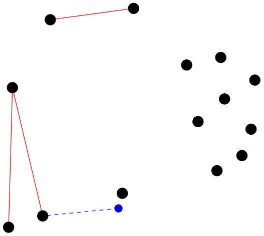

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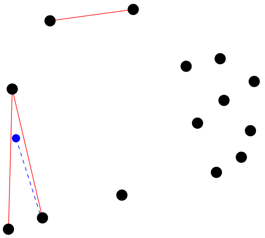

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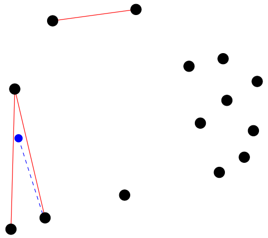

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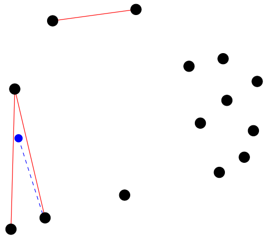

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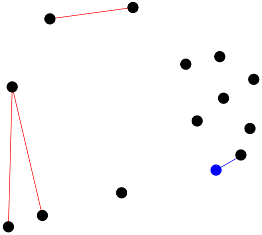

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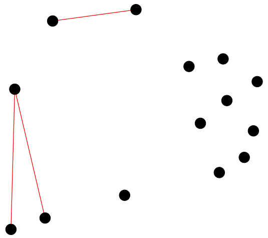

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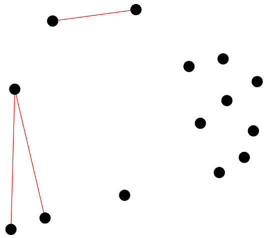

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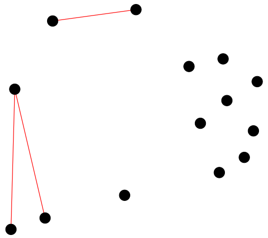

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- $|E| \in O(\mathrm{lfs}(V_1)), O(\mathrm{lfs}(V_2))$ or perhaps just $|E| \in O(\mathrm{lfs})$
- Critical definition for analysis.

Mesh Size Lower Bound

Theorem: Given a set of input features, *any* geometrically conforming mesh with good with bounded aspect ratio elements, the number of vertices must be:

$$\Omega\left(\int_D \frac{1}{\mathrm{lfs}^d(\mathbf{x})} d\mathbf{x} \right)$$

Note: A bounded radius-edge mesh maybe smaller

$O(1)$-Approximations to Optimal Size

In general, if we guarantee that:

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After post processing to remove slivers (Li & Teng)

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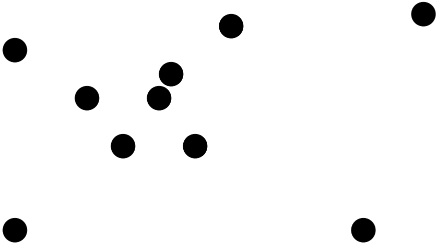

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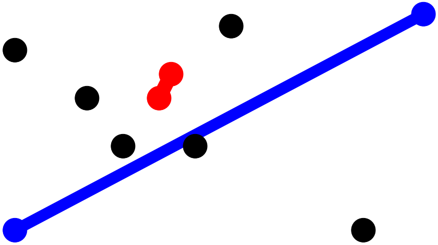

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- More like $O(d!(n \log L/s + m))$, maybe $O(k^d(n \log L/s + m))$

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- Bird's-Eye View of Runtime Proof

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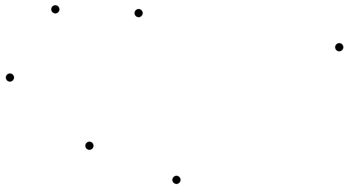
- This is all assuming $L/s \in poly(n)$

Main Result

Theorem

Bounded aspect ratio meshing in any fixed dimension in $O(n \log L/s + m)$ work and parallel time $O(\log n \log L/s)$.

The Delaunay Mesh



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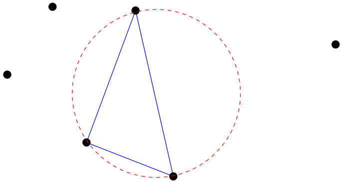

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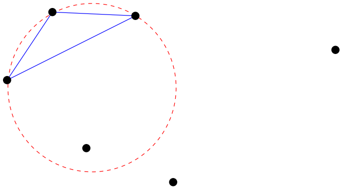

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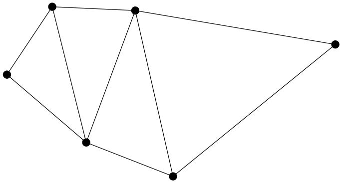

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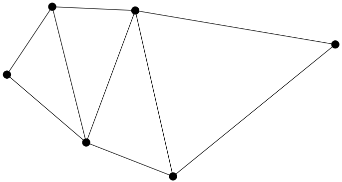

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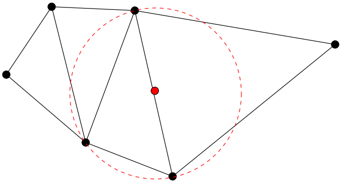

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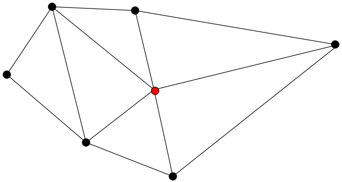

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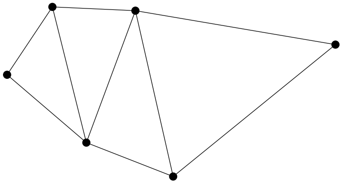

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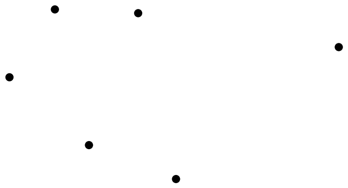

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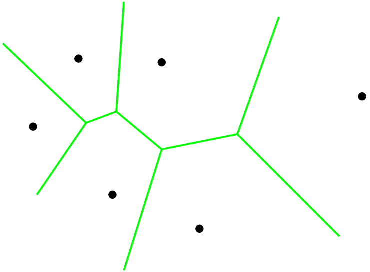

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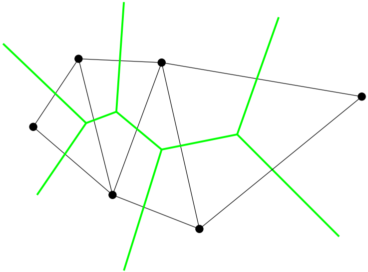

- Nearest Neighbor Partition
- Dual to the Delaunay Triangulation

Delaunay Refinement Algorithm

- Obtain the Delaunay Triangulation
- While there are poor elements (`Clean` move)
  - Destroy a Poor Quality Element by Inserting the Circumcenter
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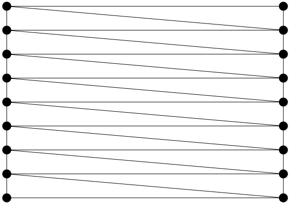

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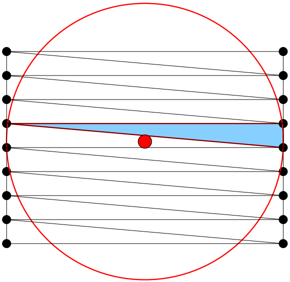

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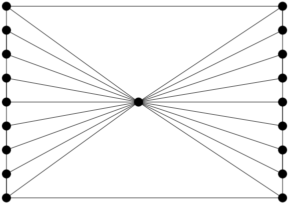

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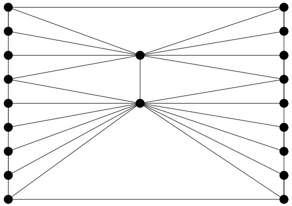

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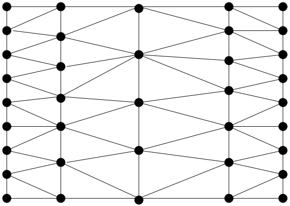

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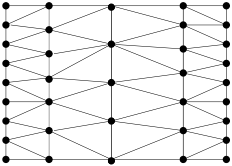
Incremental Delaunay Refinement Algorithms

- Skinny triangles really happen in real examples!

Ruppert's Algorithm Guarantees



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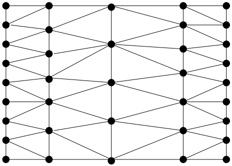

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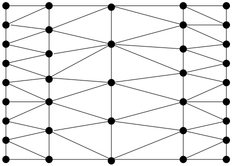
- By Design: All output elements have quality guarantees

Ruppert's Algorithm Guarantees



- Theorem (Ruppert): This terminates with

$$|E| \in \Omega(\mathrm{lfs})$$

- By Design: All output elements have quality guarantees
- Nontrivial Fact: The output size is $O(1)$-Optimal.

Runtime Concerns

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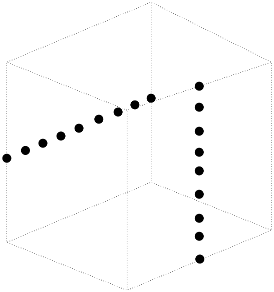
- Thus 3-D space/time is $\Omega(n^2)$

$\Theta(n^2)$ Configurations Can Happen in Practice



- Arises due to skew edges

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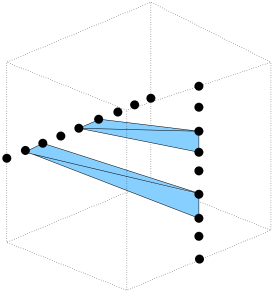

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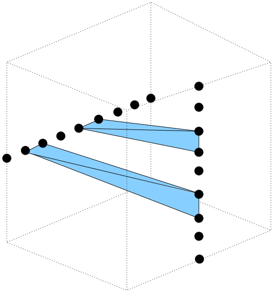

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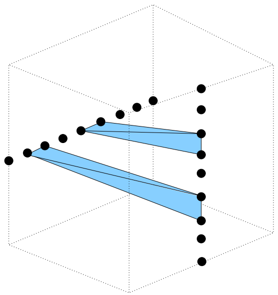
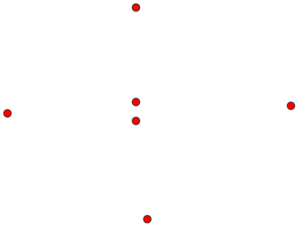


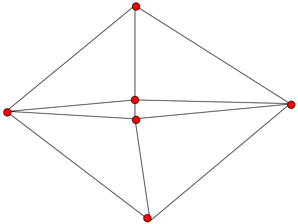
- Arises due to skew edges
- Delaunay Connectivity has all Vertical/Horizontal pairs: $(n/2)^2$
- Never actually contained in Final Output Mesh
  - *How can we avoid creating such intermediate structures?*

Two Competing Goals



- Opposing Goals of Quality and Conformity Create Work

Two Competing Goals



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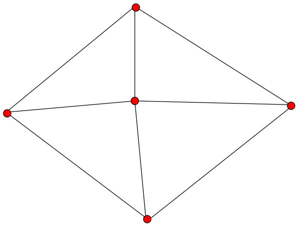

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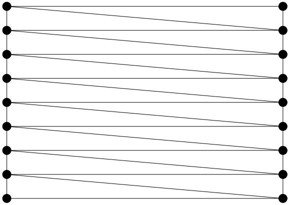

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- Ruppert's Algorithm: Always Conforming, Gradually Quality

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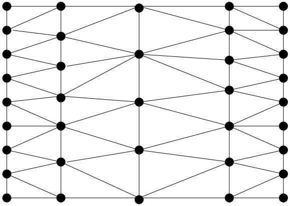

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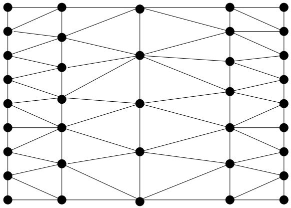
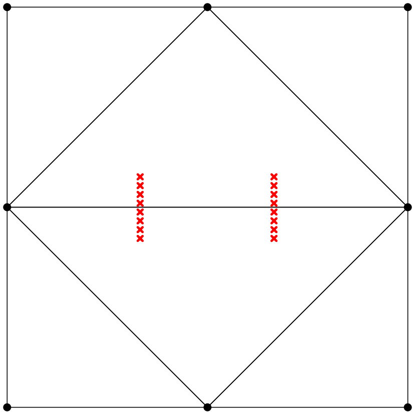


- Opposing Goals of Quality and Conformity Create Work
- Ruppert's Algorithm: Always Conforming, Gradually Quality
- **SVR Main Idea:** Always Quality, Gradually Conforming

SVR in Abstract

- Outer Loop Invariant: Mesh Is Quality
- While Mesh is not Conforming
    - Try to Conform a Little Bit More
    - *While* Mesh is not Quality
        - Destroy Poor Quality Element (Insert it's CC, Update Delaunay)

SVR in Action



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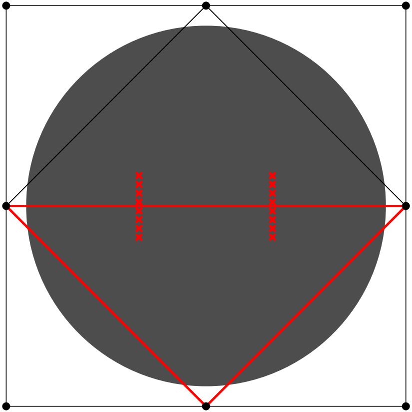

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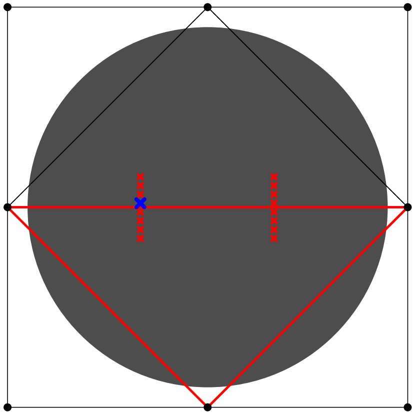

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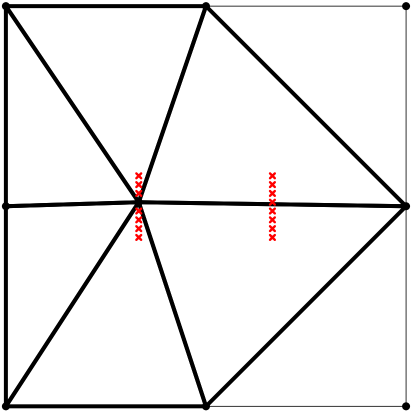

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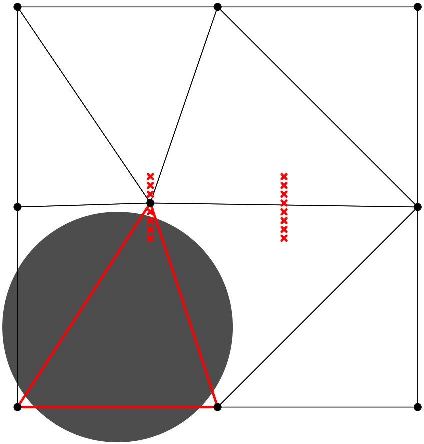

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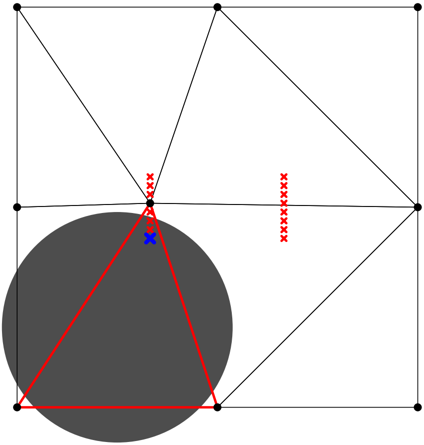

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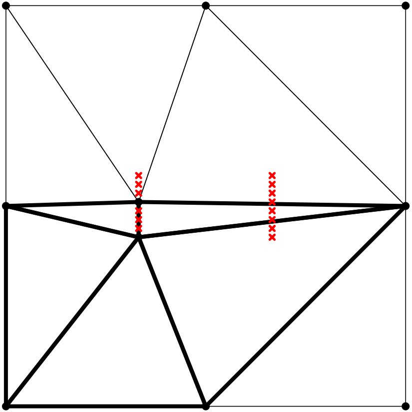

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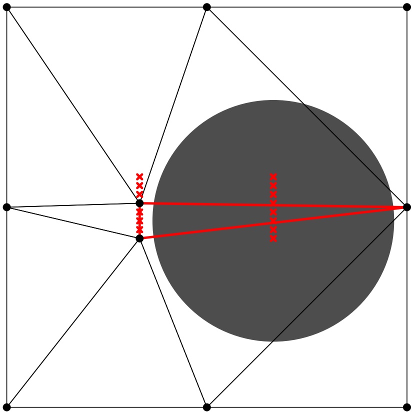

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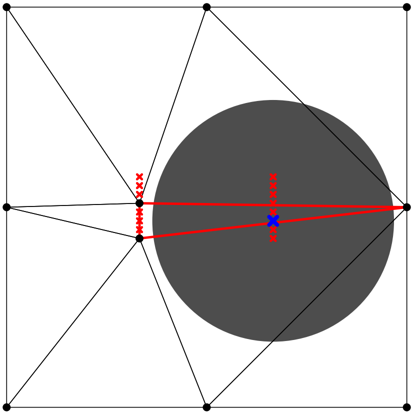

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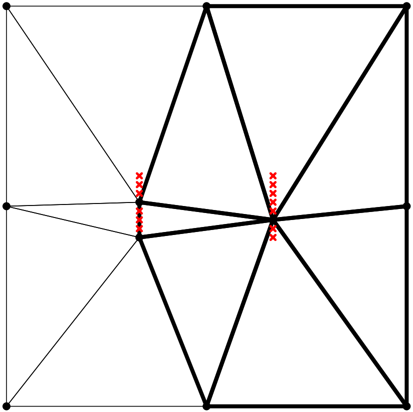

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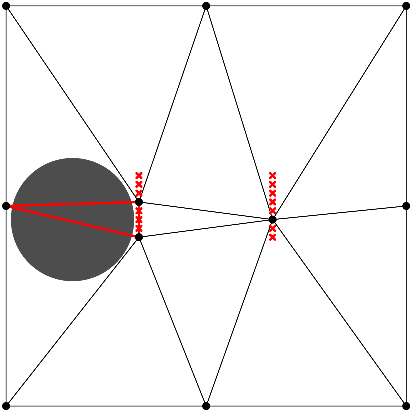

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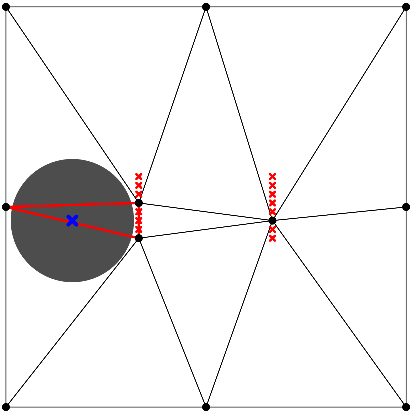

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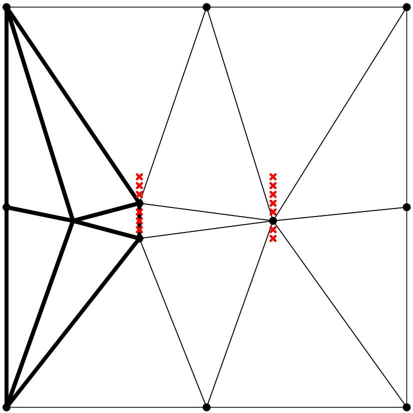

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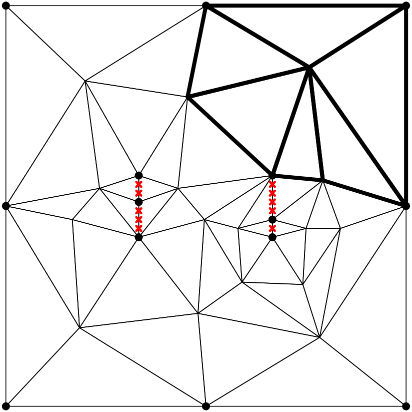

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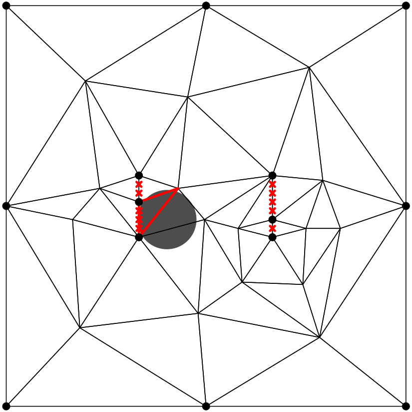

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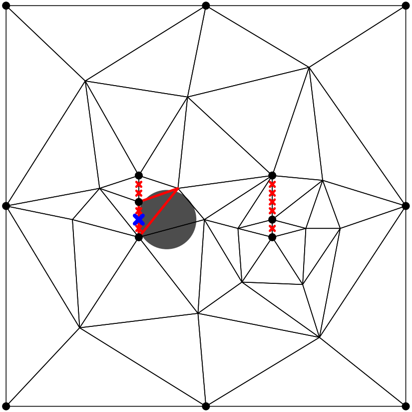

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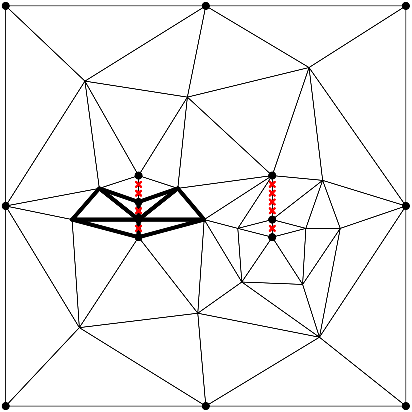

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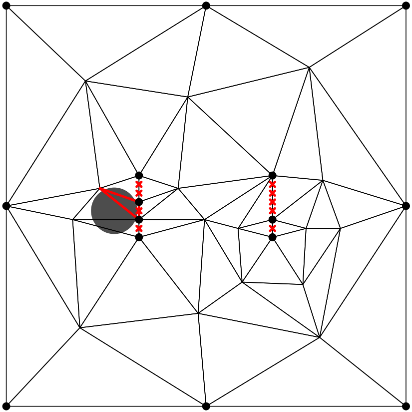

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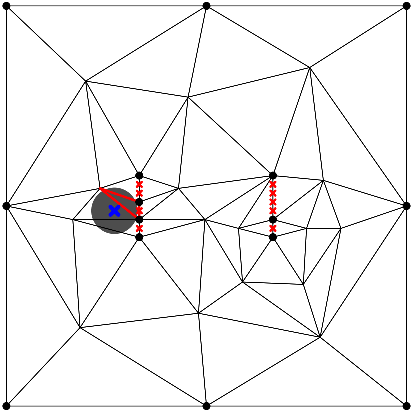

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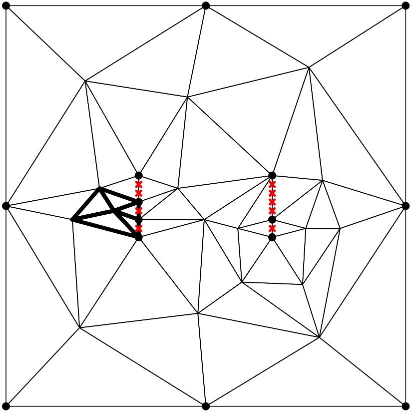

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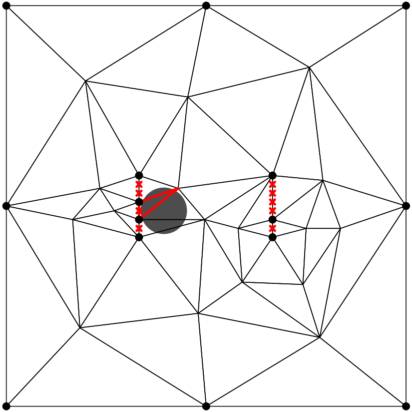

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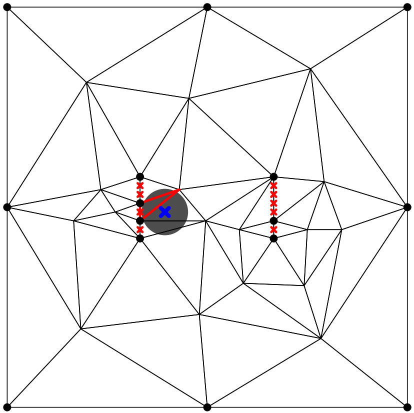

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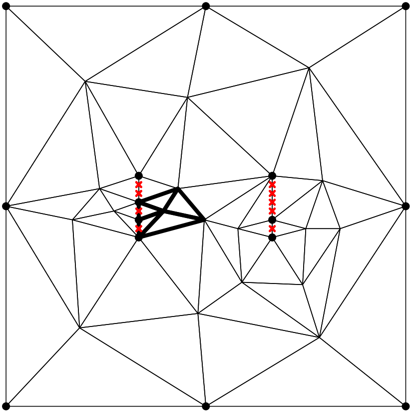

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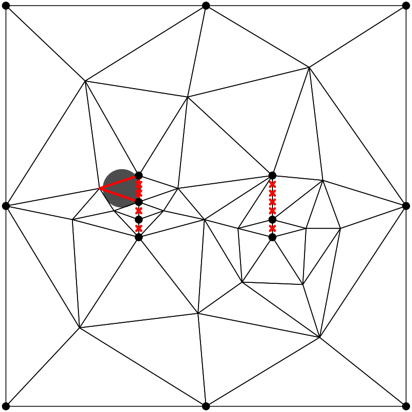

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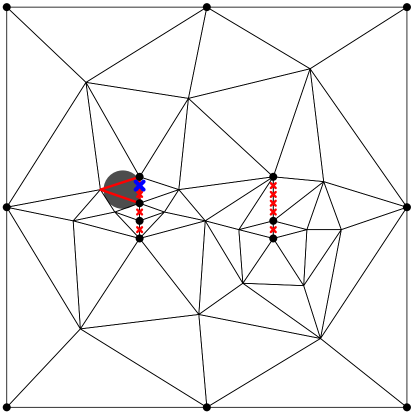

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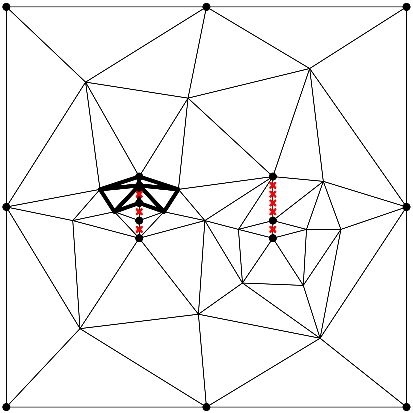

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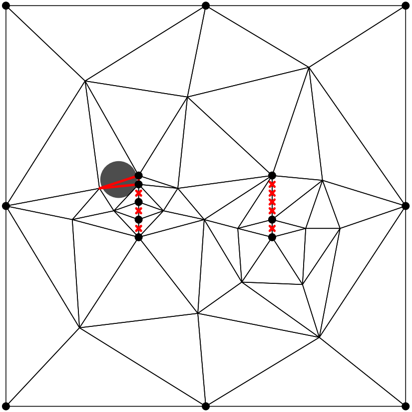

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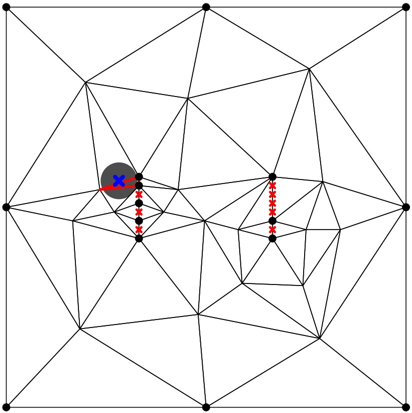

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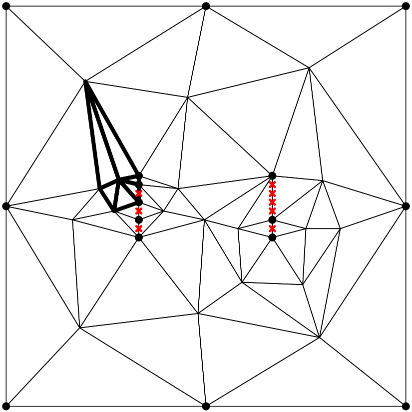

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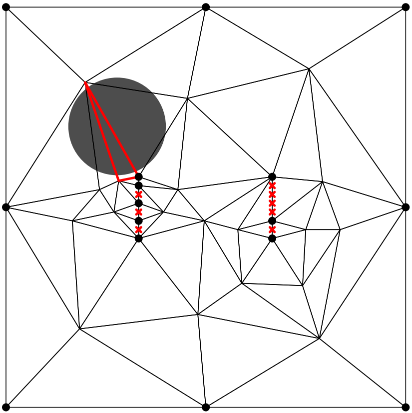

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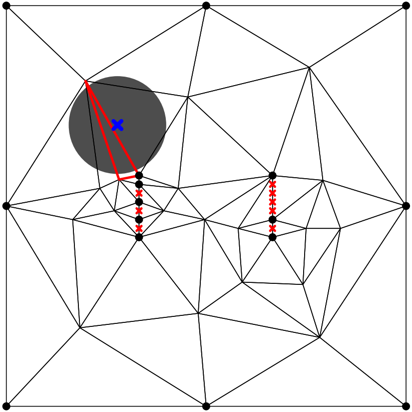

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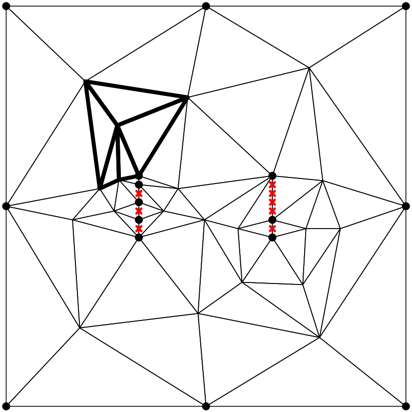

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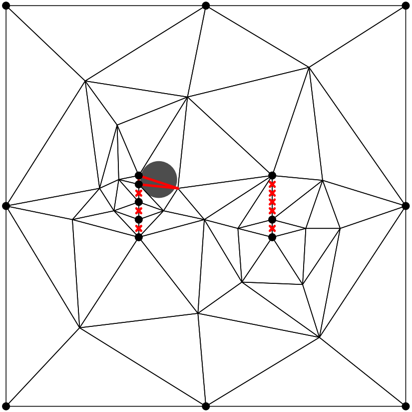

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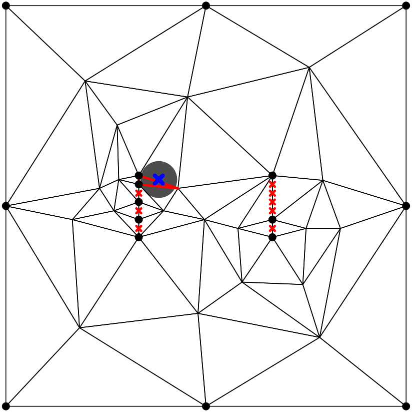

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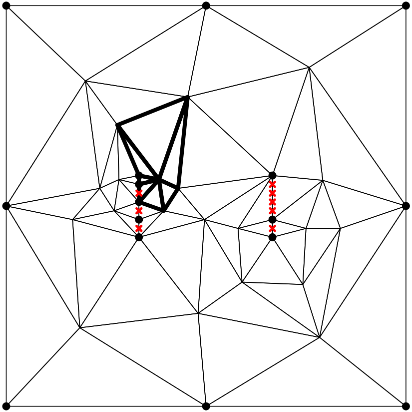

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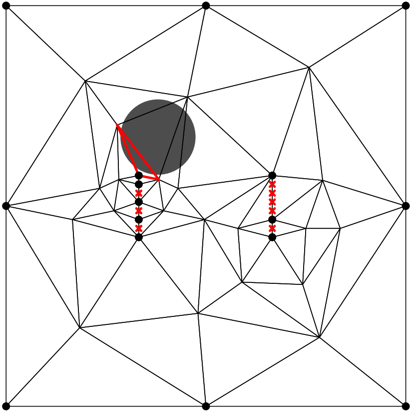

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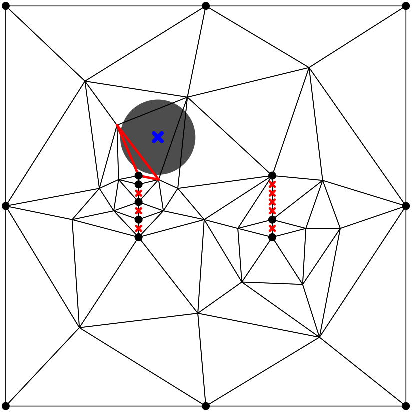

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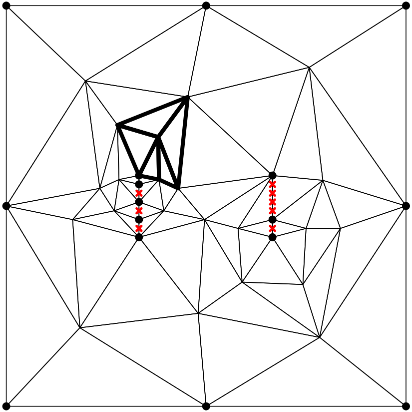

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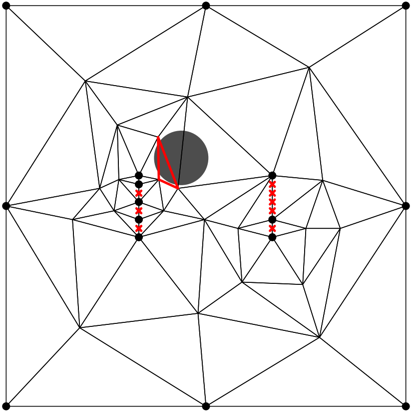

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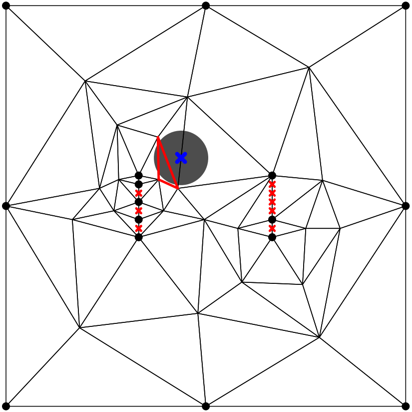

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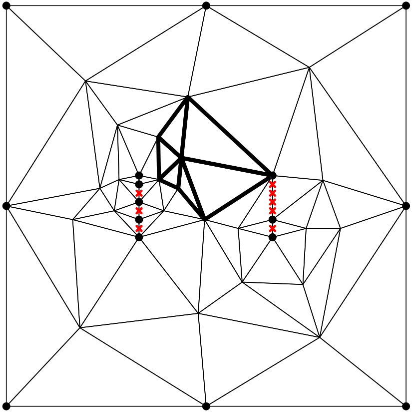

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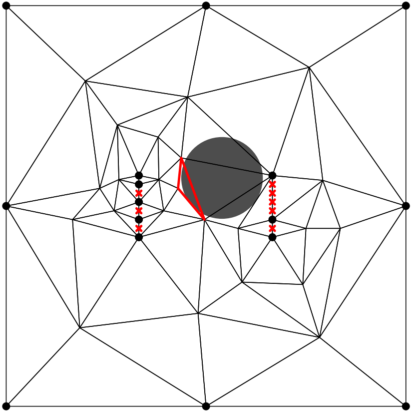

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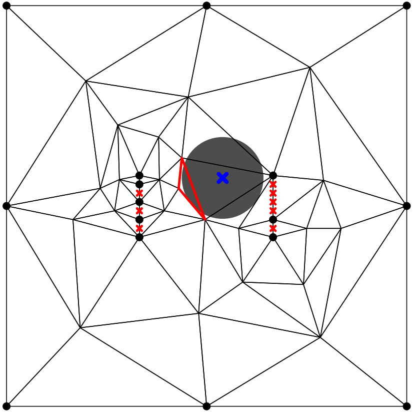

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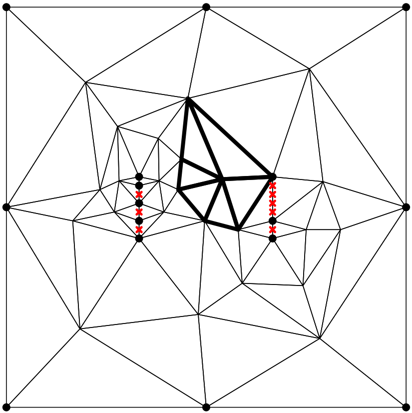

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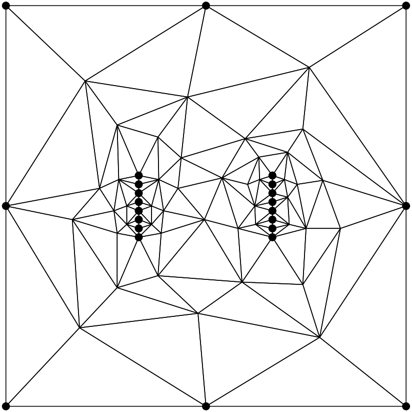

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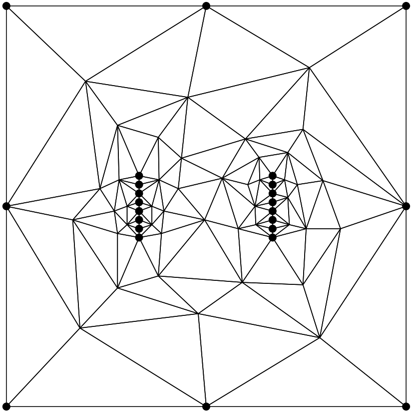
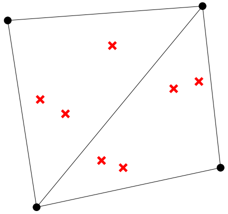


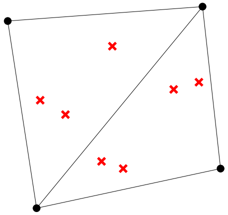
Maintaining Quality, Gradually Conform
Gradual Mesh Size Decrease

Try to Conform a Little Bit More . . .



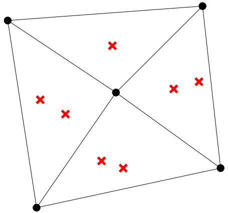
- **Break** Move

Try to Conform a Little Bit More . . .



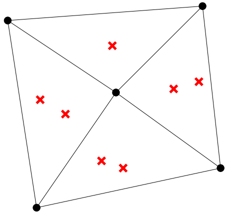
- **Break** Move
- Pick some cell that contains uninserted points still doesn't conform

Try to Conform a Little Bit More . . .



- **Break** Move
- Pick some cell that contains uninserted points still doesn't conform
- Try to insert furthest corner of the cell

Try to Conform a Little Bit More . . .



- **Break** Move
- Pick some cell that contains uninserted points still doesn't conform
- Try to insert furthest corner of the cell
- **Eagerly** keep track of where I still need to conform:

The Priority Queue for SVR

- Cell-Queue (Tet)

The Priority Queue for SVR

- Cell-Queue (Tet)
- Cells in Queue

The Priority Queue for SVR

- Cell-Queue (Tet)
- Cells in Queue
  - Bad-Aspect-Ratio Cells (Clean Move)

The Priority Queue for SVR

- Cell-Queue (Tet)
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  - Cells containing uninserted points (Break Move)

The Priority Queue for SVR

- Cell-Queue (Tet)
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  - Cells containing uninserted points (Break Move)
- Process Cell in Cell-Queue with
  TRY-TO-INSERT(furthest point of Cell)

The Priority Queue for SVR

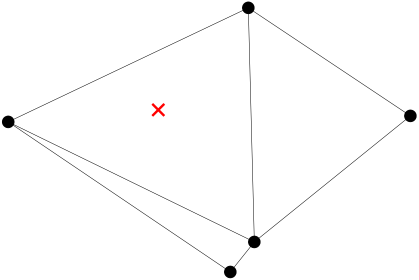
- Cell-Queue (Tet)
- Cells in Queue
  - Bad-Aspect-Ratio Cells (Clean Move)
  - Cells containing uninserted points (Break Move)
- Process Cell in Cell-Queue with **TRY-TO-INSERT(furthest point of Cell)**
- Priority clean moves first

Inserting Points

TRY-TO-INSERT(P) IF $\exists$ "nearby" uninserted point Q THEN add Q ELSE P

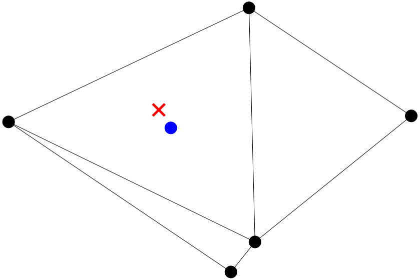
Priority Queue: Clean before Breaks

Conflicts Between Goals



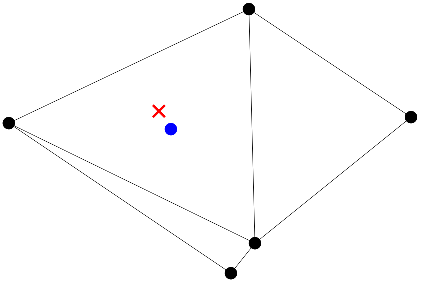
- Notice the Break Move need not do any conforming!

Conflicts Between Goals



- Notice the Break Move need not do any conforming!
- Whenever we *Destroy Element*, we might need to **yield**

Conflicts Between Goals



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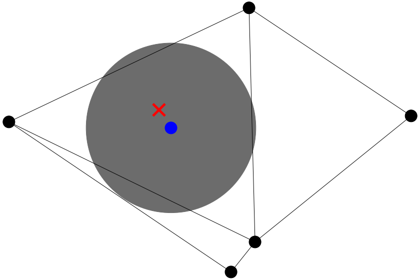

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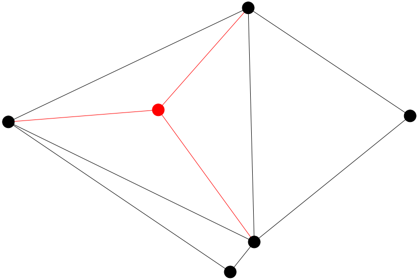

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Were we successful in avoiding the bad intermediate stages?

Sparse Voronoi Refinement

- A Re-Scheduled Version of a Traditional Incremental Meshing Algorithm.

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Insuring Conforming by Maintaining empty Balls

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Balls and Multiple Meshes

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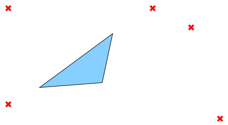

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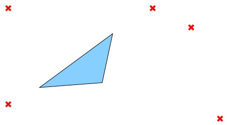

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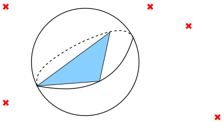

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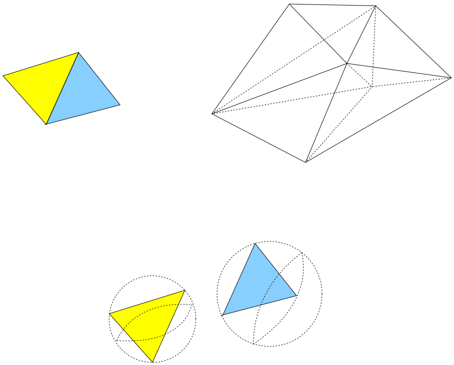
- Lower Dimensional Meshes Recursively have their own conformity queues.

Handling Features



3D Mesh, Queue of Uninserted Features, 2D Mesh

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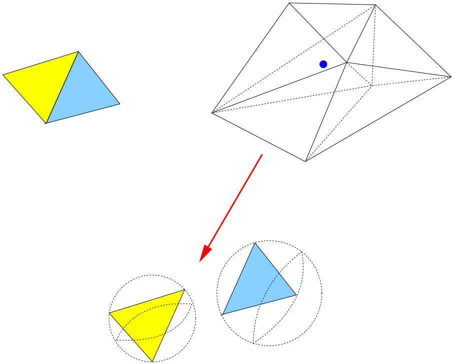

3D Mesh Wants to Insert a Point

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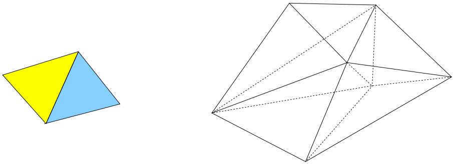
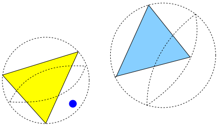
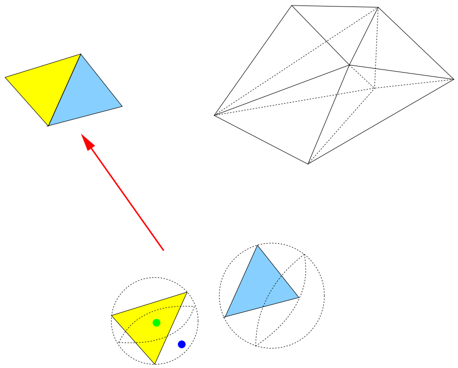




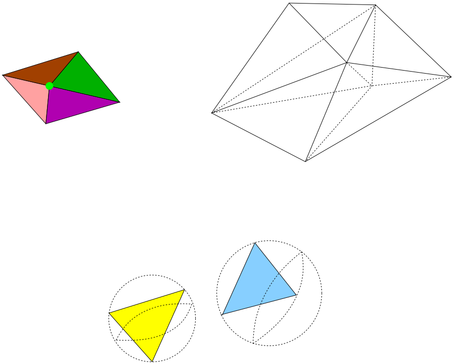
Does it Encroach on Any Balls on the Queue?

Handling Features



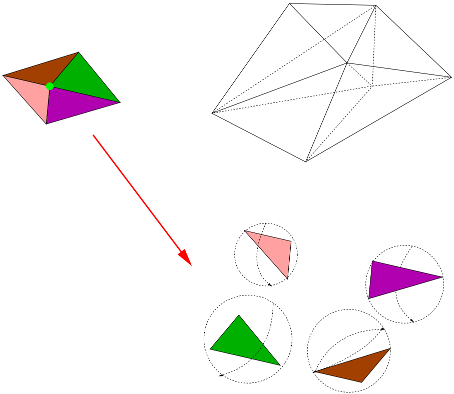
Yield to a lower Dimensional Insertion

Handling Features



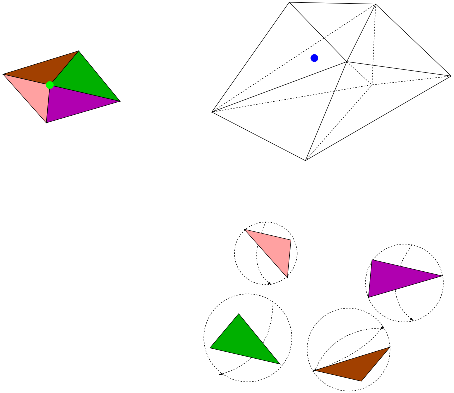
Perform an Insertion in the Lower Dimensional Mesh

Handling Features



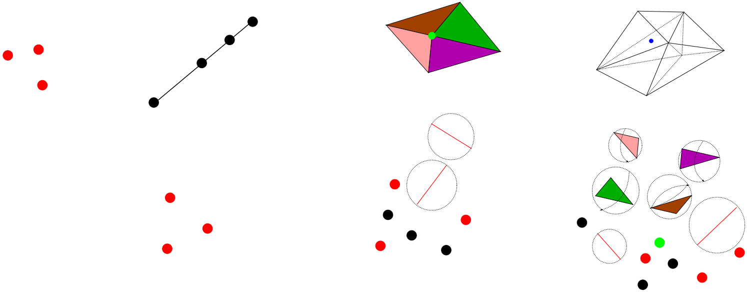
Update the Higher Dimensional Queue

Handling Features



Try Again

Handling Features



In General, Meshes and Queues at Every Level

Cells Points and Balls

Abstract Objects:

- Cell: A Voronoi cell of an inserted point

Structures:

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Always Quality Mesh

- Outer Loop Invariant: Mesh Is Quality
- Until Mesh is Conforming
  - Try to Conform a Little Bit More
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- In Fact, we always have a "Weak-Quality" bound.

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- Split:
  $O(m)$ time Building/Maintaining the mesh
  $O(n \log L/s)$ time maintaining the Conformity Queue

Quality Gives Degree Bound

- **Theorem:** *[MTTW96]* Every vertex in a good radius-edge mesh has constant degree.

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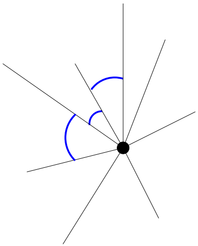

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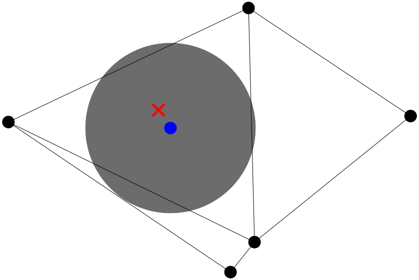
Point Location Events



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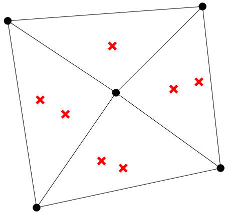
Point Location Events



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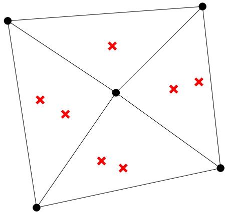

- Two types of Events:
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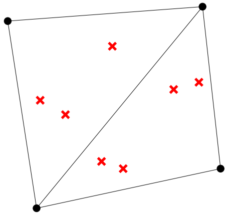
- Two types happen at the "same time" with the "same cost"

Work Per Event



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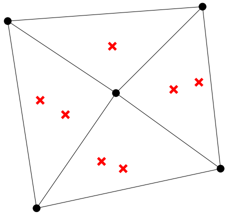

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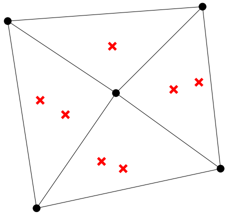

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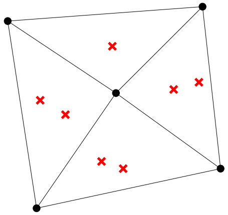

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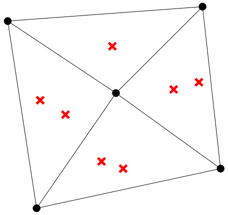

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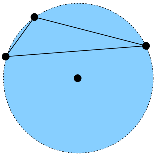
Bounding k



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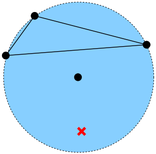

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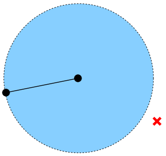

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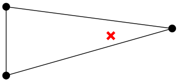

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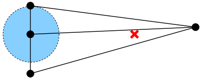

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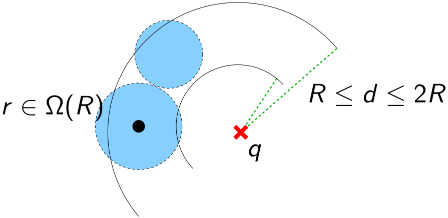

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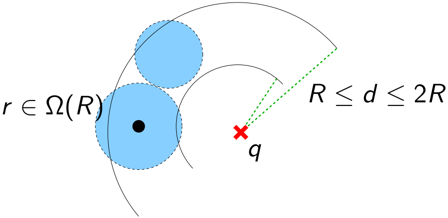

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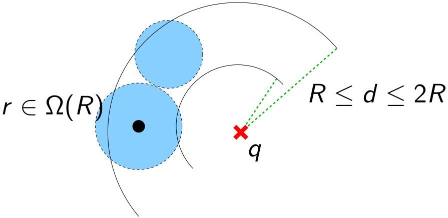

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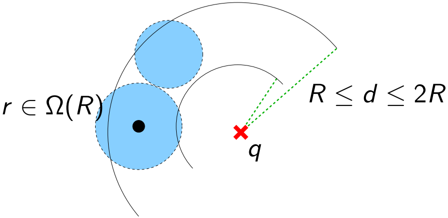

- Radius Doubling Annulus around q
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- $O(1)$ Events per Annulus affecting q

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- How many total annulii around q? Largest is L, smallest?

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- $O(n \log L/s)$ total work to maintain Conformity Queue

Intersection Sizes

Theorem

Suppose $\mathcal{V}$ bded aspect ratio Voronoi diagram and B is a ball with no points of $\mathcal{V}$ in its interior then B intersects a bded number cells.

False: Need center of B is in convex closure of points of $\mathcal{V}$.

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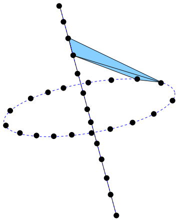
- Notice: $O(m)$ Optimal Space Usage because of Sparsity

Research Implementation (3D Point Sets)

Quadratic Delaunay Example
Problem Size: $n = 1000$



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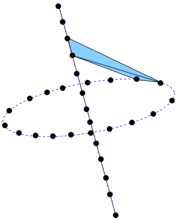

Algorithm	Lifetime	MaxTets	Worst Degree	Output
Pyramid	1.3M Tets	1.0M	1000 (84 avg.)	14K V, 87K Tets
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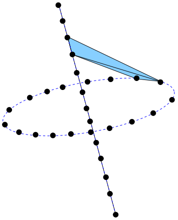

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$n = 10000$, Pyramid runs out of Memory (1.25 Gig Laptop)

SVR outputs 774K Tets, sees 2.7M Lifetime, worst case degree is 41

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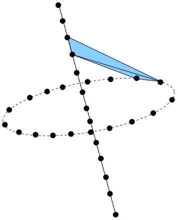

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SVR	308K Tets	81K	39 (24 avg.)	13K V, 81K Tets

$n = 10000$, Pyramid runs out of Memory (1.25 Gig Laptop)
SVR outputs 774K Tets, sees 2.7M Lifetime, worst case degree is 41

Conclusions

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  Element Shape / Output Size / Conformity Guarantees.

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- Replacing runtime term $\log L/s$ with $\log n$

Thanks!