Simplifying rational functions

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This document describes how to use synthetic division and partial fraction expansion to reduce a rational function to its canonical form. Synthetic division and partial fraction expansion are implemented in Matlab's `residue` function, which is a good way to experiment with them.

1 Partial fractions

Suppose we have a rational function

$$\frac{B(s)}{A(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_0}{a_m s^m + a_{m-1} s^{m-1} + \ldots + a_0}$$

We would like to represent it in a simpler form. It turns out that any rational function can be decomposed in the *partial fraction expansion*

$$\frac{B(s)}{A(s)} = K(s) + \frac{C_1(s)}{P_1(s)} + \frac{C_2(s)}{P_2(s)} + \ldots$$

Here $K(s)$ is a polynomial, and $C_i(s)$ and $P_i(s)$ are "simple" polynomials. $K(s)$ is called the *direct term*; it is necessary only if $n \geq m$, and if it exists it has degree $(n - m)$.

The denominator polynomials P_i depend on the roots of A, which are also called *poles* of the rational function. (Roots of B are called *zeros* of the rational function.) Each isolated root x_i of A results in a denominator polynomial of the form $P_i(s) = s - x_i$; each complex conjugate pair of roots $x_i \pm y_i\mathbf{i}$ gives a denominator of the form $P_i(s) = s^2 + 2x_i s + x_i^2 - y_i^2$. Multiple roots result in terms of higher degree; for example a real root x_i with multiplicity k gives a denominator $P_i(s) = (s - x_i)^k$.

To determine the direct term we can use *synthetic division* (see below). So for now let us assume $n < m$. In this case we can use the Heaviside method (also called the cover-up method) to determine the coefficient polynomials C_i.

The simplest case is an isolated root x_i of $A(s)$. In this case, C_i is a constant, and we have

$$(s - x_i)\frac{B(S)}{A(S)} = (s - x_i)\left(K(s) + \frac{C_1(s)}{P_1(s)} + \frac{C_2(s)}{P_2(s)} + \ldots\right)$$

$$\left[(s-x_i)\frac{B(S)}{A(S)}\right]_{s=x_i} = C_i$$

where the second equation holds because every term on the right-hand side contains a factor $(s-x_i)$ except for the term $(s-x_i)C_i/(s-x_i)$. So, we can determine C_i by deleting one of the factors $(s-x_i)$ of A_i from our rational function, and evaluating the result at x_i.

For example, suppose we have

$$\frac{A(s)}{B(s)} = \frac{1}{(s^2+1)(s-2)}$$

We will then have a term in our expansion

$$\frac{a}{s-2}$$

To determine a, we evaluate $1/(s^2+1)$ at $s=2$. This tells us that $a=1/5$, so our term is

$$\frac{1/5}{s-2}$$

This way of determining coefficients gives the method its name: we "covered up" the factor $1/(s-2)$ of $B(s)/A(s)$ and evaluated the remaining expression at $s=2$.

If our denominator has a repeated root or a complex conjugate pair of roots (or even a repeated conjugate pair), then we will have a factor $P_i(s)$ in the denominator which has degree $d>1$. This factor will result in a term $C_i(s)/P_i(s)$ in our expansion, where degree$(C_i) < d$. In this case we can determine the coefficients of C_i by evaluating $P_i(s)B(s)/A(s)$ at the d points where P_i is zero; this will result in d equations in the d unknown coefficients.

For example, consider again the rational function

$$\frac{A(s)}{B(s)} = \frac{1}{(s^2+1)(s-2)}$$

The factor (s^2+1) leads to a term in our expansion

$$\frac{as+b}{s^2+1}$$

To determine a and b, we evaluate $1/(s-2)$ at the two points at which (s^2+1) is 0, namely $\pm\mathbf{i}$. This gets us two equations,

$$a\mathbf{i}+b = \frac{1}{\mathbf{i}-2} = -\frac{\mathbf{i}+2}{5} \qquad -a\mathbf{i}+b = \frac{1}{-\mathbf{i}-2} = \frac{\mathbf{i}-2}{5}$$

Solving these equations gives $a=-1/5$ and $b=-2/5$; combining the new term with our previous result tells us that our final expansion is

$$\frac{B(s)}{A(s)} = \frac{1/5}{s-2} - \frac{s/5+2/5}{s^2+1}$$

2 Synthetic division

We are given a rational function $B(s)/A(s)$ with numerator degree n and denominator degree m. If $n \geq m$, we can pull out a *quotient* term $K(s)$, leaving a *remainder* term $R(s)$ with degree$(R) < m$, so that

$$\frac{B(s)}{A(s)} = K(s) + \frac{R(s)}{A(s)}$$

The process is analogous to long division, and is called *synthetic division*. We will illustrate it by example: suppose we start with

$$\frac{B(s)}{A(s)} = \frac{s^3 - s^2 + s + 1}{s^2 - 4s + 3}$$

We are looking for $K(s)$ and $R(s)$, with degree$(R) < 2$, so that

$$B(s) = K(s)A(s) + R(s)$$

To get the highest-order term of $B(s)$ (namely s^3) right, we can see that we have to multiply $A(s)$ by s. If we set $K_1(s) = s$, we have

$$R_1(s) = B(s) - K_1(s)A(s) = (s^3 - s^2 + s + 1) - (s^3 - 4s^2 + 3s) = 3s^2 - 2s + 1$$

This gets us partway to our goal: $R_1(s)$ has a smaller degree than $B(s)$ did, but not small enough. But, we can repeat the process: to get rid of the leading term of $R_1(s)$ (namely $3s^2$), we can multiply $A(s)$ by 3. Setting $K_2(s) = s + 3$, we have

$$R_2(s) = B(s) - K_2(s)A(s) = (s^3 - s^2 + s + 1) - (s^3 - 4s^2 + 3s) - (3s^2 - 12s + 9)$$

Cancelling terms gives $R_2(s) = 10s - 8$, which has sufficiently low degree, so we can take $R = R_2$ and $K = K_2$.