$$\min_{x \in C} f(x) \qquad \max_{x \in C} \tilde{f}(x)$$

$$\iff -\min_{x \in C} \underline{-\tilde{f}(x)}$$

primal LP
feasible then strong duality

dual LP
feasible then strong duality.

$f(x) - g(u,v)$ Duality gap.

$f(x^\star) \geq g(u,v)$ $\forall$ $\overset{\text{dual}}{\text{feasible}}$ u,v

$f(x) \geq g(u^\star, v^\star)$ $\forall$ primal feas x

$$\underbrace{f(x) - f^\star \leq f(x) - g(u,v)}_{\varepsilon}$$

$$x \in \mathbb{R}^n, \quad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

$\|\cdot\|$, dual $\|\cdot\|_*$

$|z^T x|$ $\qquad \frac{z}{\|z\|}$

$$\|x\|_* = \max_{\|u\| \leq 1} u^T x \geq \frac{z^T x}{\|z\|}$$

$$z^T x \leq \|z\| \|x\|_*$$

$x^\star, \quad u^\star, v^\star$ optimal.

duality gap is zero $\quad f(x^\star) = g(u^\star, v^\star)$

$$f(x^\star) = g(u^\star, v^\star)$$
$$= \min_x \underbrace{f(x) + \sum u_i^\star h_i(x) + \sum v_j^\star \ell_j(x)}$$
$$\longrightarrow = f(x^\star) + \sum u_i^\star h_i(x^\star) + \sum v_j^\star \ell_j(x)$$
$$\longrightarrow = f(x^\star)$$

$$0 \ni \partial f(x^\star) + \sum u_i^\star \partial h_i(x^\star) + \sum v_j^\star \partial \ell_j(x^\star)$$

stationarity

$$\sum u_i^\star h_i(x^\star) = 0$$
$$u_i^\star h_i(x^\star) = 0 \quad \text{for all } i$$

comp. slackness

$$g(u^*, v^*) = f(x^*) + \sum u_i^* h_i(x^*) + \sum v_j l_j(x^*)$$

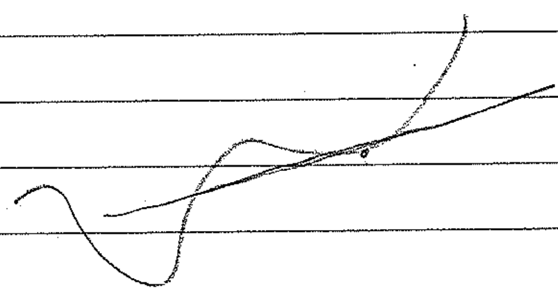
$$= L(x^*, u^*, v^*)$$

$$= f(x^*)$$

$$g(u^*, v^*) = \min_x \; f(x) + \sum u_i^* h_i(x) + \sum v_j^* l_j(x)$$

$$\overset{\text{stationarity}}{=} f(x^*) + \sum u_i^* \cancel{h_i(x^*)} + \sum v_j^* \cancel{l_j(x^*)}$$

$$\overset{\text{complementary slackness}}{=} f(x^*)$$



$$\min \ \tfrac{1}{2}x^TQx + c^Tx$$
$$\text{s.t.} \quad Ax = 0 \qquad u$$

(E.g. Newton's step $\quad Ax^{(k)} = b \qquad x^{(k+1)} = x^{(k)} + x$)

$$0 = Qx + c + A^Tu$$
$$Ax = 0$$

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} -c \\ 0 \end{bmatrix}$$

$$\min \ \tfrac{1}{2}\|y - Ax\|^2 + \lambda\|x\|_1$$

$$-A^T(y - Ax) + \lambda s = 0, \quad s \in \partial\|x\|_1$$

$$A^T(y - Ax) = \lambda s$$

$$s \in \begin{cases} \{-1\} & x_i < 0 \\ \{1\} & > 0 \\ [-1, 1] & = 0. \end{cases}$$

$$\left|A_i^T(y - Ax)\right| < \lambda \implies x_i = 0.$$

$$\left|z_j^T(y - Ax)\right| < \lambda \qquad [A \ z]$$

$$A = [A_{(1)} \cdots A_{(p)}]$$

$$x = [x_{(1)} \cdots x_{(p)}]$$

$$\min_{x} \frac{1}{2}\|y - Ax\|^2 + \lambda \sum \|x_{(i)}\|_2 \sqrt{p_{(i)}}$$

$$A_{(i)}^T (y - Ax) = \lambda \sqrt{p_{(i)}}\, s_{(i)}$$

$$s_{(i)} \in \partial \|x_{(i)}\|_2$$

$$\in \begin{cases} \left\{ \dfrac{x_i}{\|x_i\|_2} \right\} & x_{(i)} \neq 0 \\ \{ z : \|z\|_2 \leq 1 \} & x_{(i)} = 0 \end{cases}$$

$$\| A_{(i)}^T (y - Ax) \|_2 < \lambda \sqrt{p_{(i)}} \Rightarrow x_{(i)} = 0$$

$$A_{(i)}^T (y - Ax) = \lambda \sqrt{p_{(i)}} \frac{x_{(i)}}{\|x_{(i)}\|_2}$$

$$x_{(i)} = \left(A_{(i)}^T A_{(i)} + \frac{\lambda \sqrt{p_{(i)}}\, I}{\|x_{(i)}\|_2} \right)^{-1} A_{(i)}^T r_{-(i)}$$

$$\min \; L(x,\lambda) = f(x) + \underline{\lambda}(h(x) - t)$$

$$\Longleftrightarrow \min \; f(x) + \lambda h(x)$$

$$\min \; f(x) + \lambda h(x)$$

$$\underline{t = h(x^*)}$$

$$\min \; f(x) \quad \text{st} \quad h(x) \leq t$$

$$\min \; f(x) + \lambda(h(x) - t)$$

$$\Longleftrightarrow \min \; f(x) + \lambda h(x)$$

$$\lambda(h(x^*) - t) = 0$$

$$f(x) + \lambda(h(x) - t)$$

$$f(x) + \lambda h(x)$$

$$\frac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_1$$

x^* is unique

\# of nonzeros in $x^* \leq \min\{n, p\}$

$$A^T \nabla f(Ax) + \lambda s = 0$$

$$-A^T \nabla f(Ax) = \lambda s$$

$$s \in \partial \|x\|_1$$

$$\in \begin{cases} \{\text{sign}(x_i)\} & x_i \neq 0 \\ [-1, 1] & = 0 \end{cases}$$

$$S = \{j : |A_j^T \nabla f(Ax)| = \lambda\}$$

$$x_i = 0 \quad \text{for all } i \notin S$$

$$\text{rank}(A_S) < |S|$$

$$A_i = \sum_{\substack{j \in S \\ j \neq i}} c_j A_j$$

$$1 = \sum_j \underbrace{c_j s_j s_i}_{a_j}$$

$s_i A_i = \sum a_j s_j A_j$ where a_j satisfy

$s_i A_i \in \text{aff}\{ s_j A_j,\ j \in S \setminus \{i\} \}$

$\Rightarrow$ A entries density over $\mathbb{R}^{n p}$
then this can't happen wp 1

wp 1 $\quad \text{rank}(A_S) = |S|$
$\leq \min\{n, p\}$

Reduced problem

$$\min \ f(A_S x_S) + \lambda \|x_S\|_1$$

~~strictly~~ convex $\quad A_S^T \nabla^2 f(A_S x_S) A_S$

$\Rightarrow$ unique solution