Linear programs

10-725 Optimization
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Review: LPs

- LPs: m constraints, n vars
  - A: $R^{m \times n}$ b: R^m c: R^n x: R^n
  - ineq form
    - [min or max] $c^T x$ s.t. $Ax \leq b$
    - $m \geq n$
  - std form
    - [min or max] $c^T x$ s.t. $Ax = b$ $x \geq 0$
    - $m \leq n$

max $2x+3y$ s.t.
$x + y \leq 4$
$2x + 5y \leq 12$
$x + 2y \leq 5$
$x, y \geq 0$

Review: LPs

- Polyhedral feasible set
  - infeasible (unhappy ball)
  - unbounded (where's my ball?)
- Optimum at a vertex (= a 0-face)
- Transforming LPs
  - changing ≥ to ≤ to =
  - getting rid of free vars or bounded vars

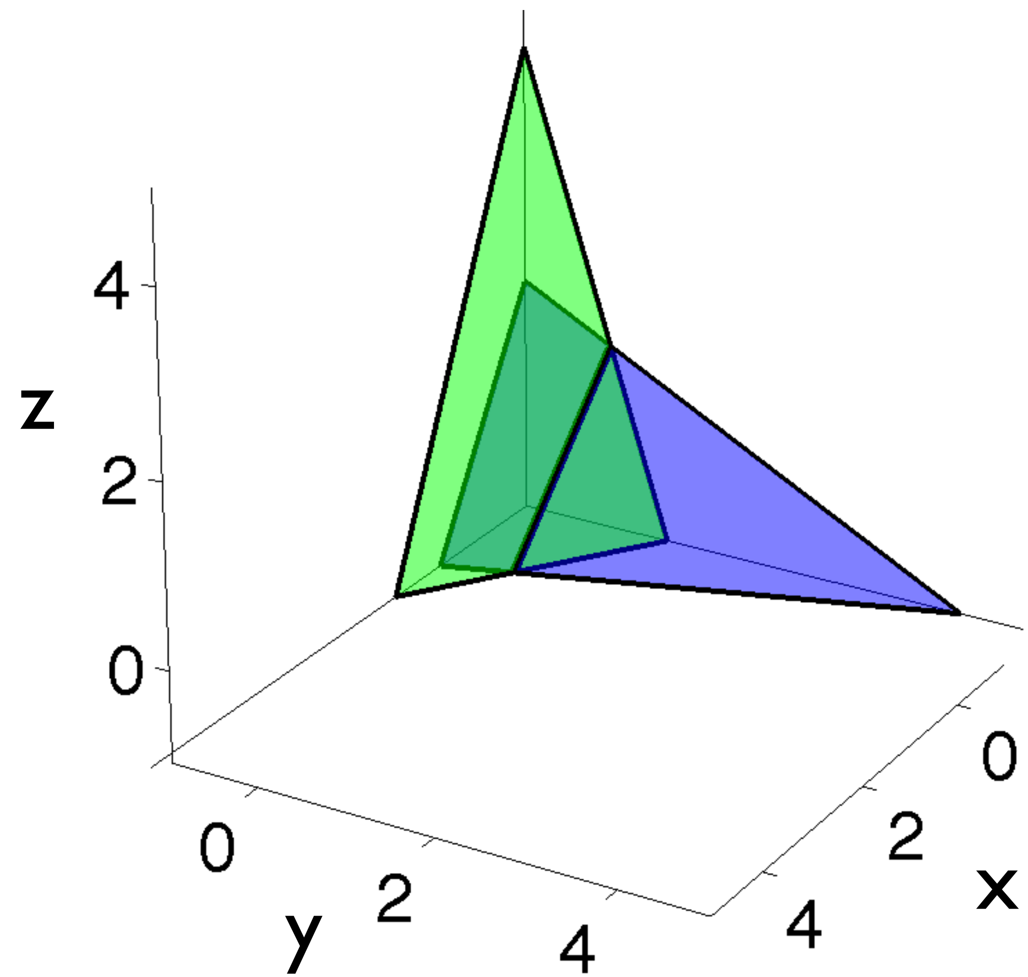
Review: LPs

- Tableau:

x	y	u	v	w	z	RHS
1	1	1	0	0	0	4
2	5	0	1	0	0	12
1	2	0	0	1	0	5
-2	-3	0	0	0	1	0

- Row operations to get equivalent tableaux
- Basis (more or less corresponds to a corner)
  - use row ops to make m×m block of tableau = identity matrix
  - set nonbasic vars = 0: enough constraints to fully specify all other variables (so, a 0-face, if it's feasible)

Ineq form is projected std form



$x, y, z \geq 0$

$A[x;y;z] = b$

3	1	2	|	5
2	3	1	|	5

Three bases

1	0	5/7	10/7
0	1	−1/7	5/7

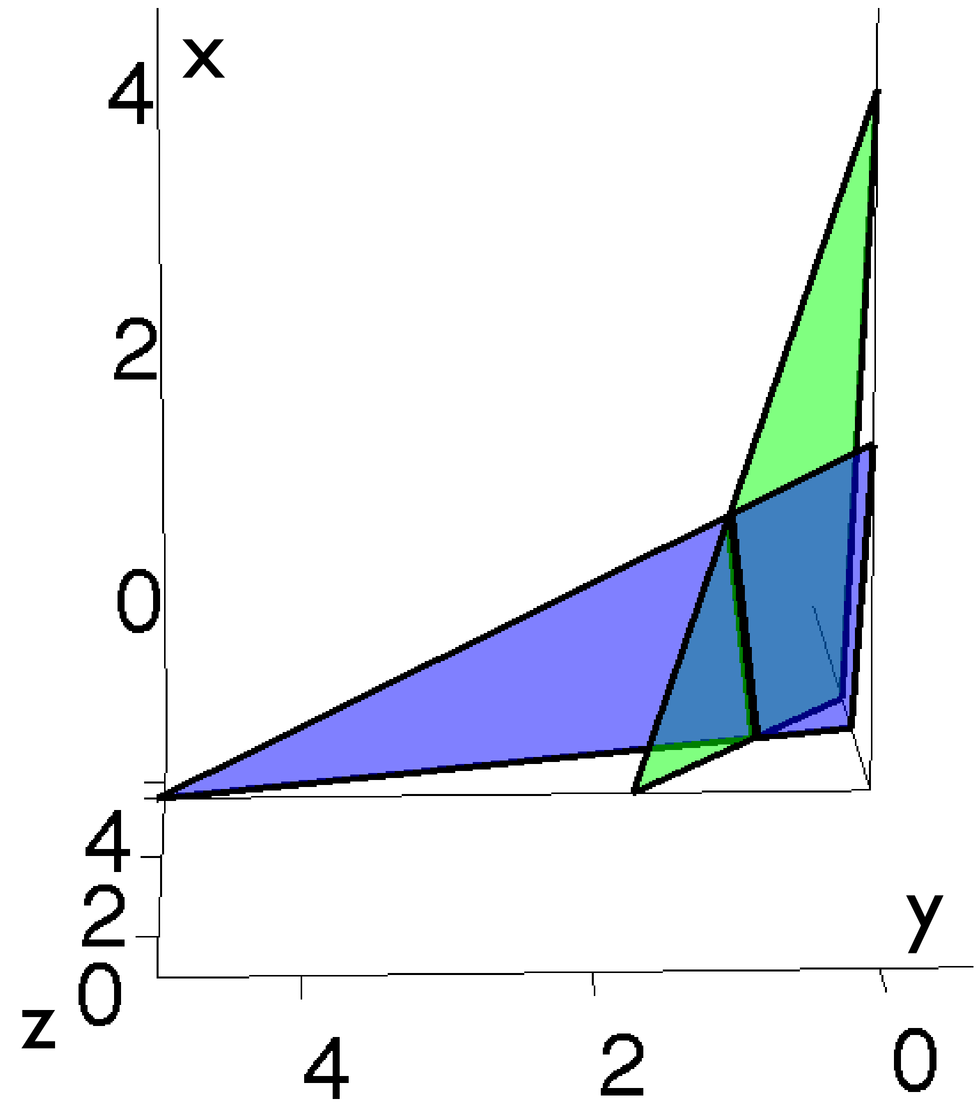
$-5 \le z \le 2$

1	5	0	5
0	−7	1	−5

$5/7 \le y \le 1$

1/5	1	0	1
7/5	0	1	2

$x \le 5 \quad x \le 10/7$



What if we can't pick basis?

- E.g., suppose A doesn't have full row rank
  - can't pick m linearly independent cols
- Ex:
  - $3x + 2y + 1z = 3$
  - $6x + 4y + 2z = 6$

What if we can't pick basis?

- E.g., suppose fewer vars than constraints
  - A taller than it is wide, $m \geq n$
  - can't pick enough cols of A to make a square matrix
- Ex:

Nonsingular

- We can assume
  - $n \geq m$ (at least as many vars as constrs)
  - A has full row rank
- Else, drop rows (maintaining rank) until it's true
- Called ***nonsingular*** standard form LP

Naive (slooooow) algorithm

- Put in nonsingular standard form
- Iterate through all subsets of n vars
  - if m constraints, how many subsets?
- Check each for
  - full rank ("basis-ness")
  - feasibility (RHS ≥ 0)
- If pass both tests, compute objective
- Maintain running winner, return at end

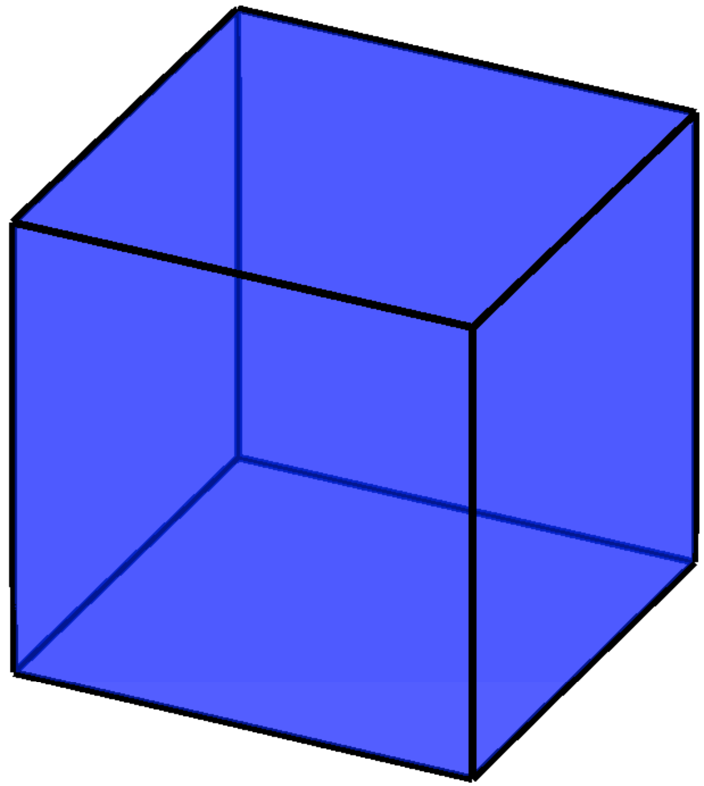
Improving our search

- Naive: enumerate all possible bases
- Smarter: maybe neighbors of good bases are also good?
- Simplex algorithm: repeatedly move to a neighboring basis to improve objective
  - continue to assume nonsingular standard form LP

Neighboring bases

- Two bases are ***neighbors*** if they share (m–1) variables
- Neighboring feasible bases correspond to vertices connected by an edge

x	y	z	u	v	w	RHS
1	0	0	1	0	0	1
0	1	0	0	1	0	1
0	0	1	0	0	1	1



def'n: pivot, enter, exit

Example

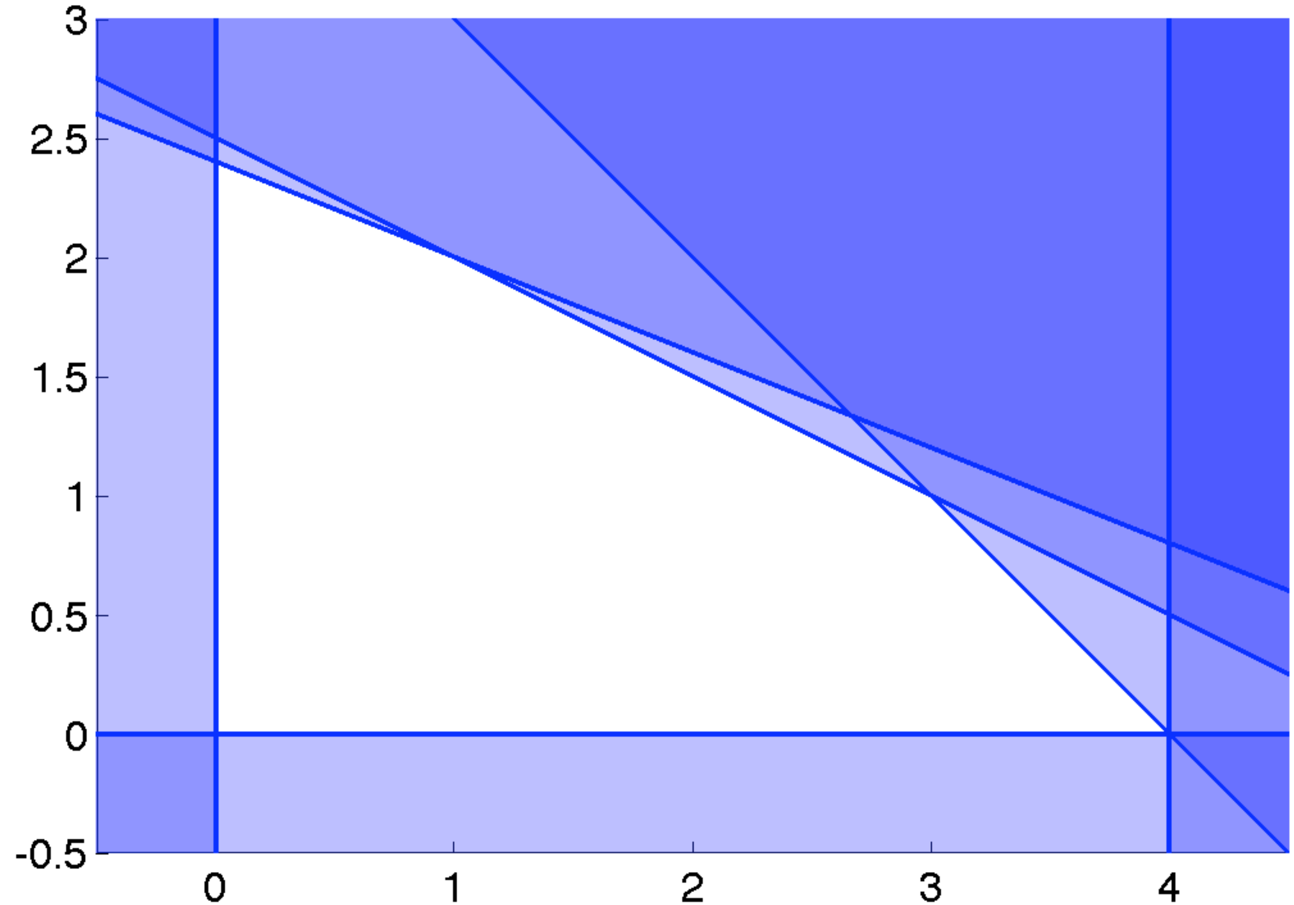
max z = 2x + 3y s.t.

x + y ≤ 4

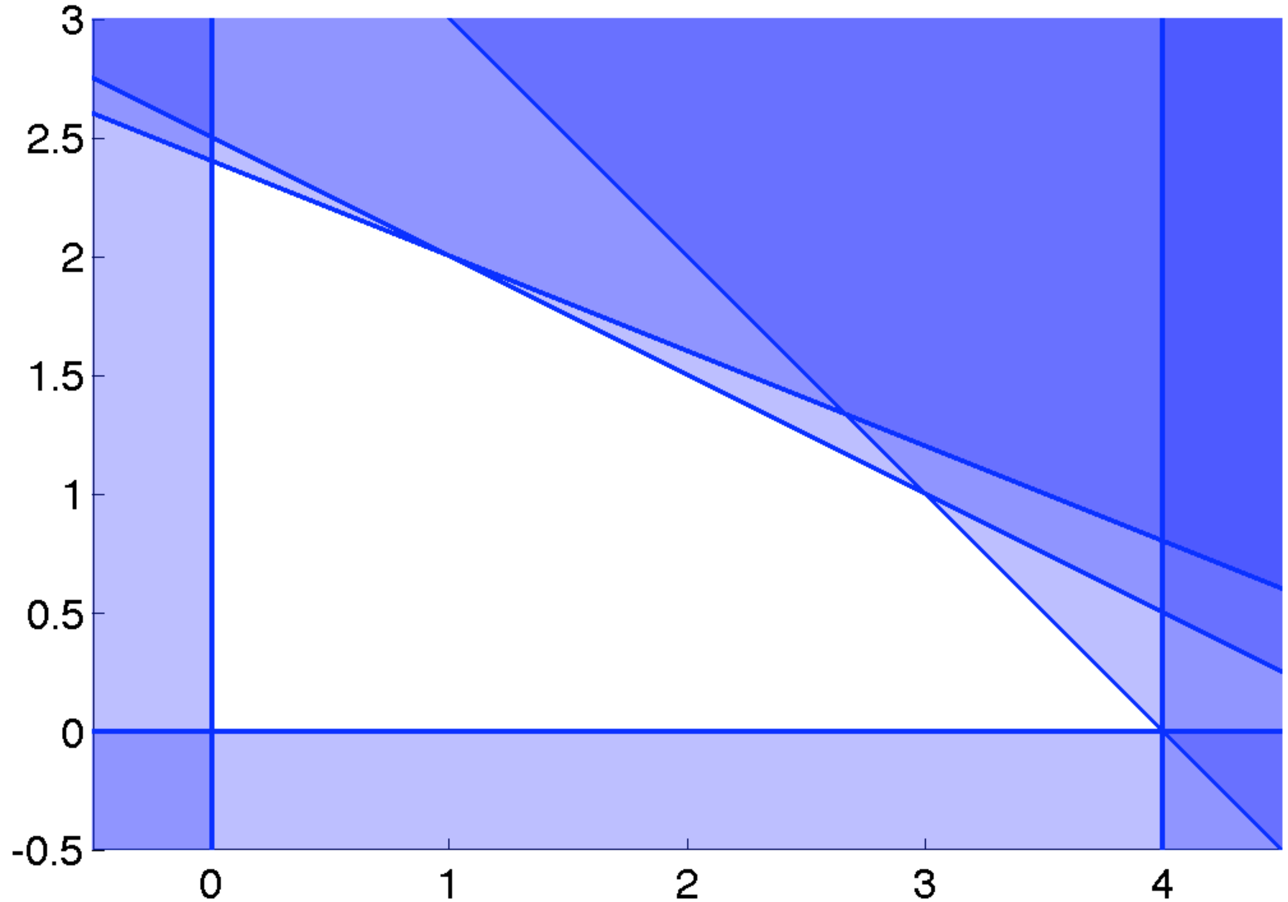
2x + 5y ≤ 12

x + 2y ≤ 5

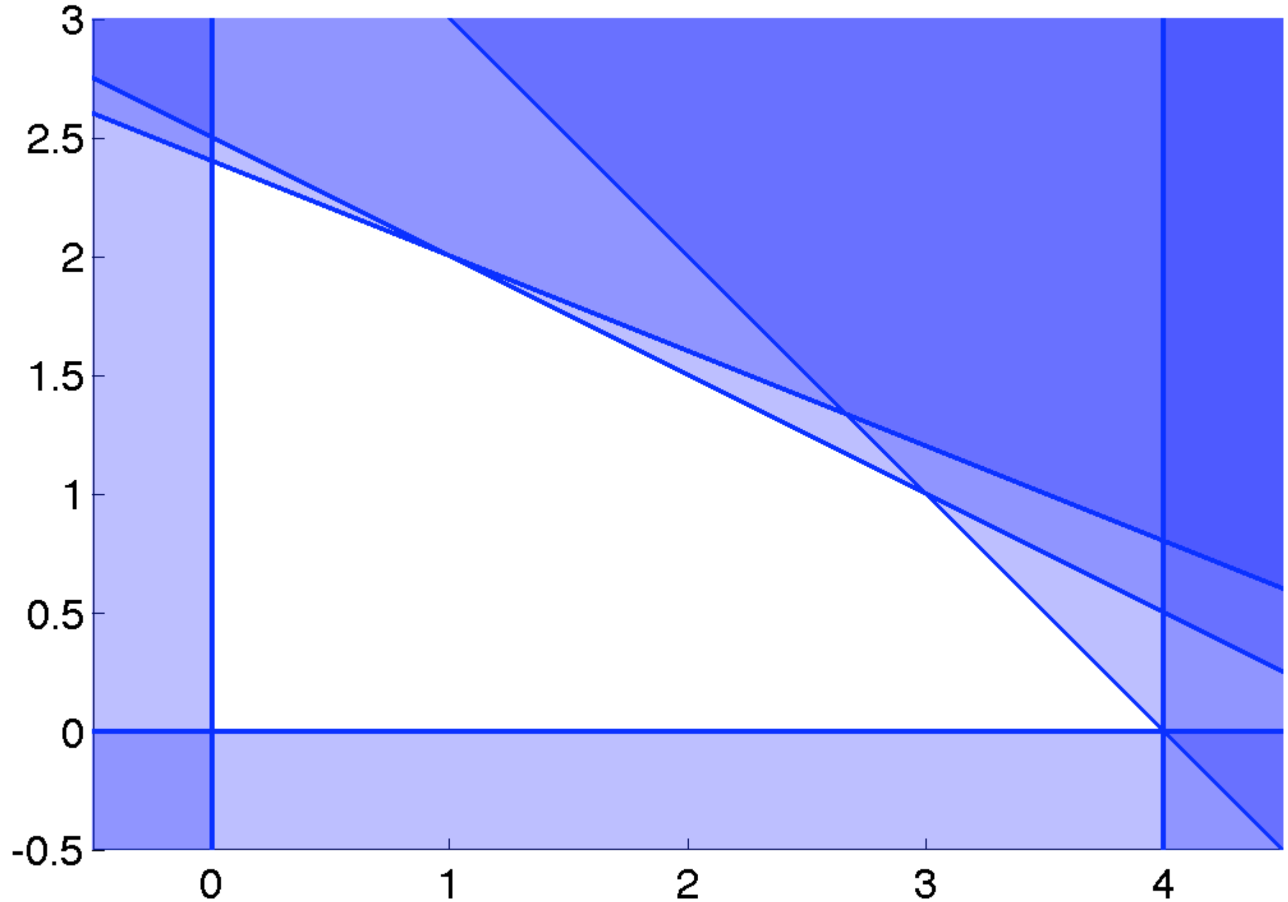
x ≤ 4



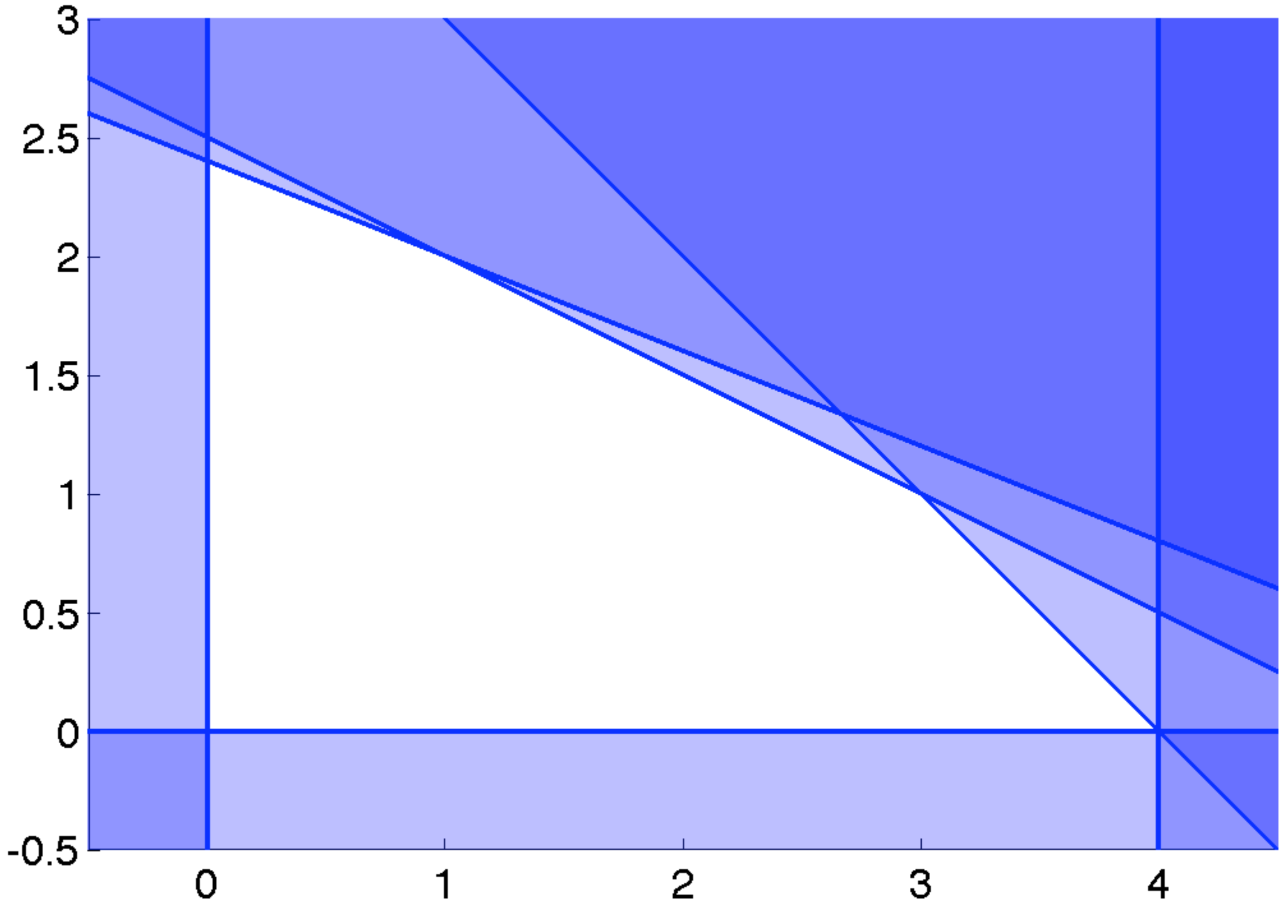
x	y	s	t	u	v	z	RHS
1	1	1	0	0	0	0	4
2	5	0	1	0	0	0	12
1	2	0	0	1	0	0	5
1	0	0	0	0	1	0	4
−2	−3	0	0	0	0	1	0



x	y	s	t	u	v	z	RHS
0.4	1	0	0.2	0	0	0	2.4
0.6	0	1	−0.2	0	0	0	1.6
0.2	0	0	−0.4	1	0	0	0.2
1	0	0	0	0	1	0	4
−0.8	0	0	0.6	0	0	1	7.2

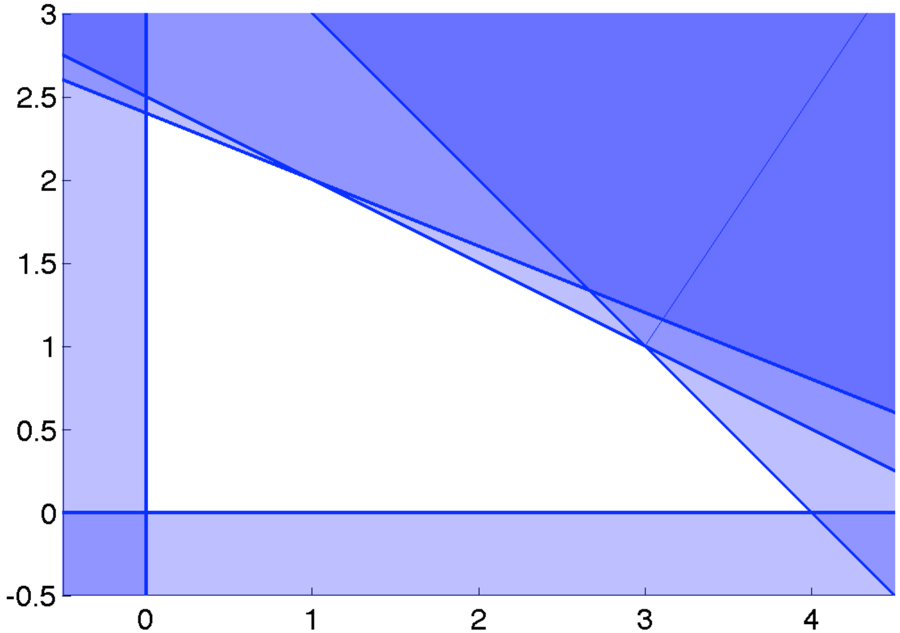


x	y	s	t	u	v	z	RHS
1	0	0	−2	5	0	0	1
0	1	0	1	−2	0	0	2
0	0	1	1	−3	0	0	1
0	0	0	2	−5	1	0	3
0	0	0	−1	4	0	1	8



x	y	s	t	u	v	z	RHS
1	0	2	0	−1	0	0	3
0	1	−1	0	1	0	0	1
0	0	1	1	−3	0	0	1
0	0	−2	0	1	1	0	1
0	0	1	0	1	0	1	9

Initial basis



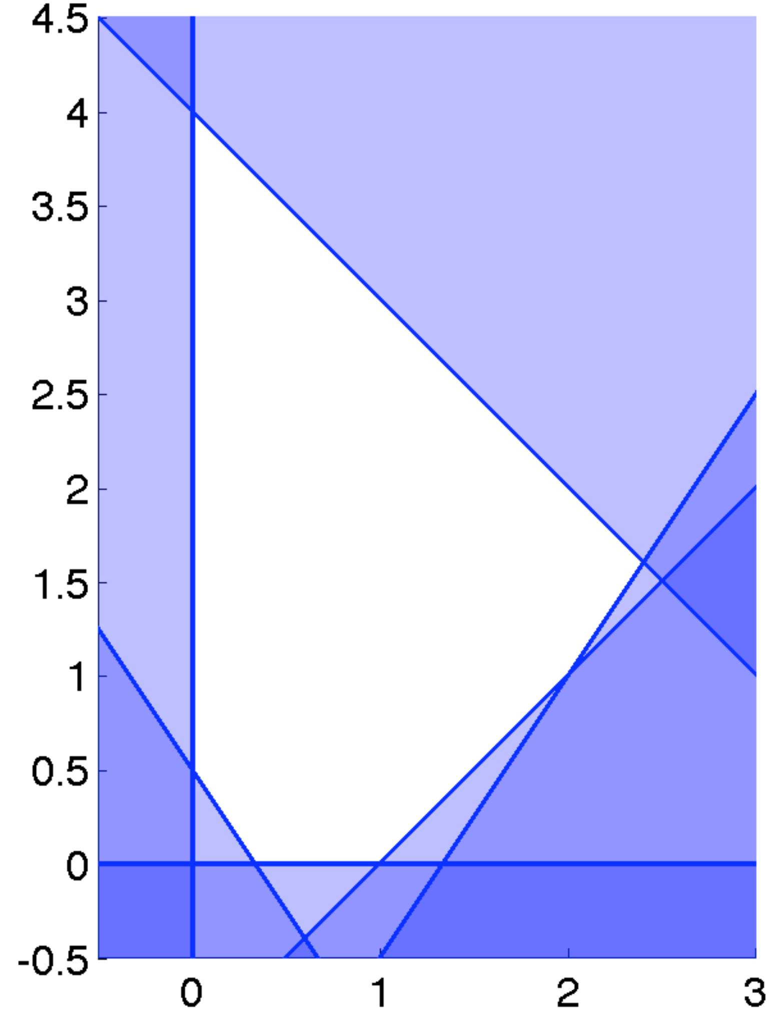
x	y	u	v	w	RHS
1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	5

- So far, assumed we started w/ feasible basic solution—in fact, it was trivial to find one
- Not always so easy in general

Big M

```
0 ≤ x, y, s1..s6
max x - 2y
```

x	y	slacks				z	RHS
1	1	1	0	0	0	0	4
3	-2	0	1	0	0	0	4
1	-1	0	0	1	0	0	1
-3	-2	0	0	0	1	0	-1
-1	2	0	0	0	0	1	0



- Can make it easy: variant of slack trick
  - For each violated constraint, add var w/ coeff –1
  - Penalize in objective; negate constraint

Simplex in one slide

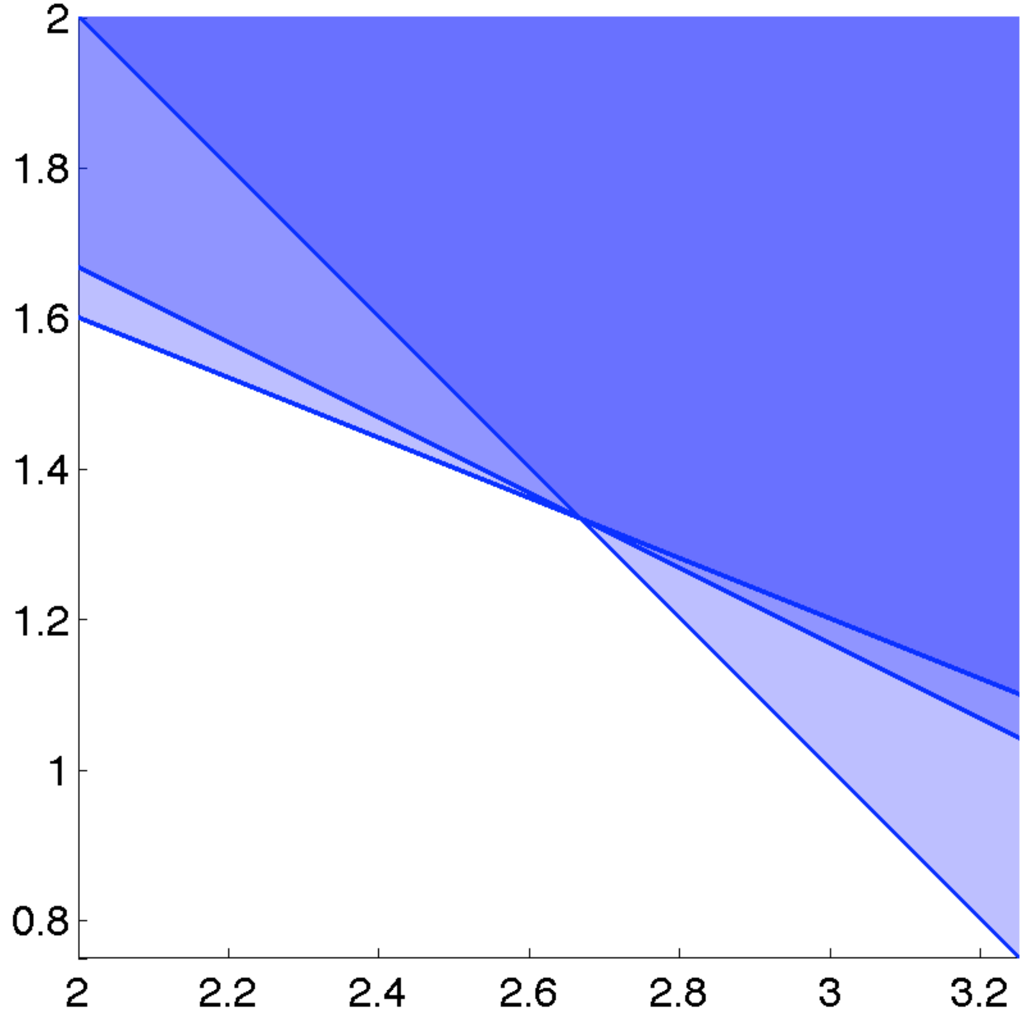
(skipping degeneracy handling)

- Given a nonsingular standard-form max LP
- Start from a feasible basis and its tableau
  - big-M if needed
- Pick non-basic variable w/ coeff in objective ≤ 0
- Pivot it into basis, getting neighboring basis
  - select exiting variable to keep feasibility
- Repeat until all non-basic variables have objective ≥ 0

Degeneracy

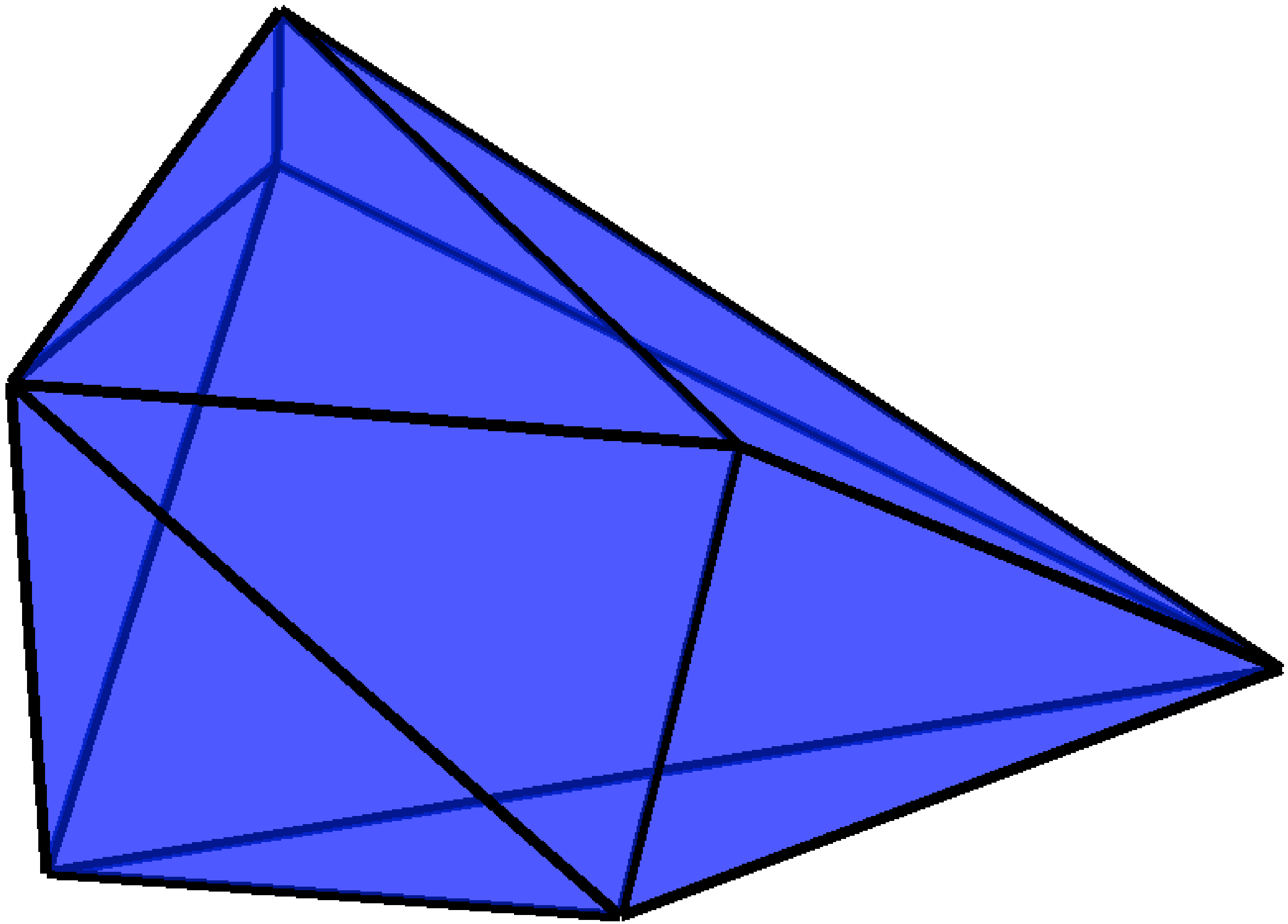
- Not every set of m variables yields a corner
  - some have rank < m (not a basis)
  - some are infeasible
- Can the reverse be true? Can two bases yield the same corner?

Degeneracy



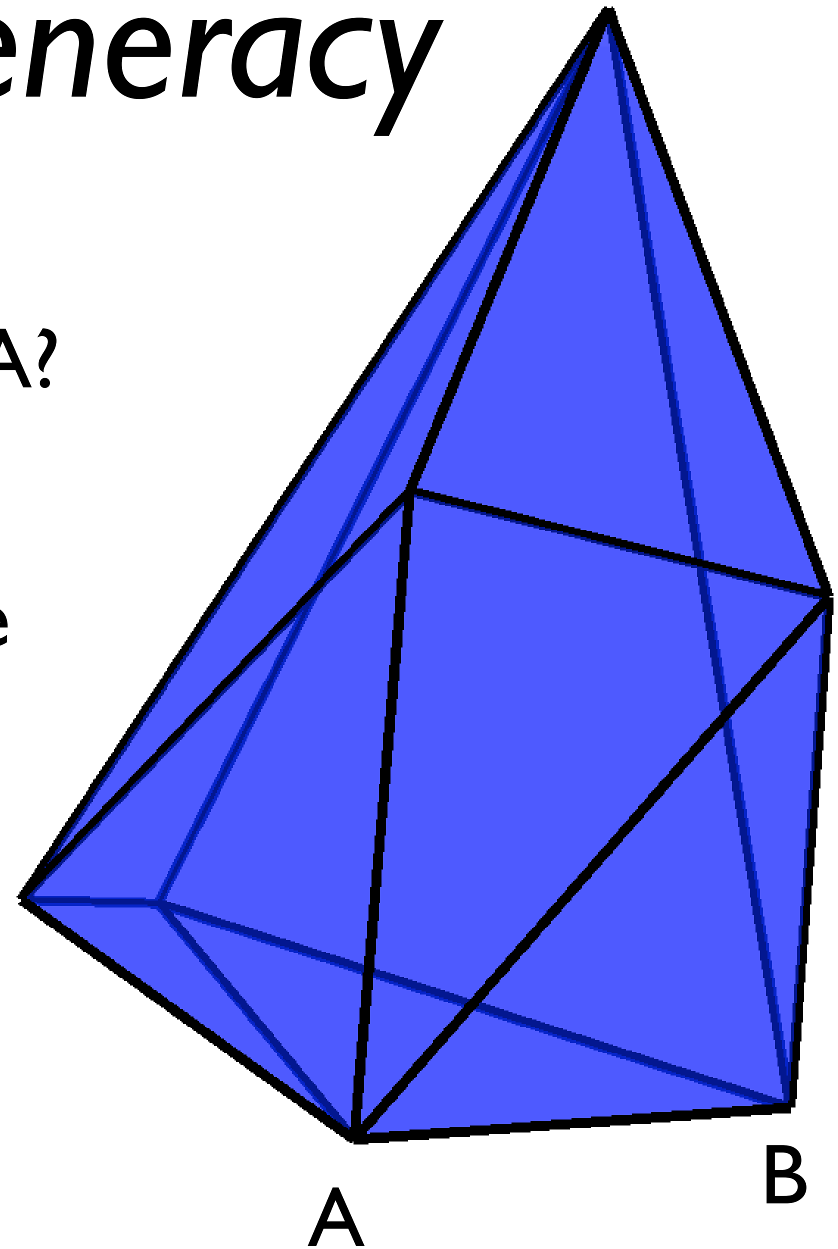
x	y	u	v	w	RHS
1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	16/3
1	0	0	-2	5	8/3
0	1	0	1	-2	4/3
0	0	1	1	-3	0
1	0	2	0	-1	8/3
0	1	-1	0	1	4/3
0	0	1	1	-3	0

Degeneracy in 3D



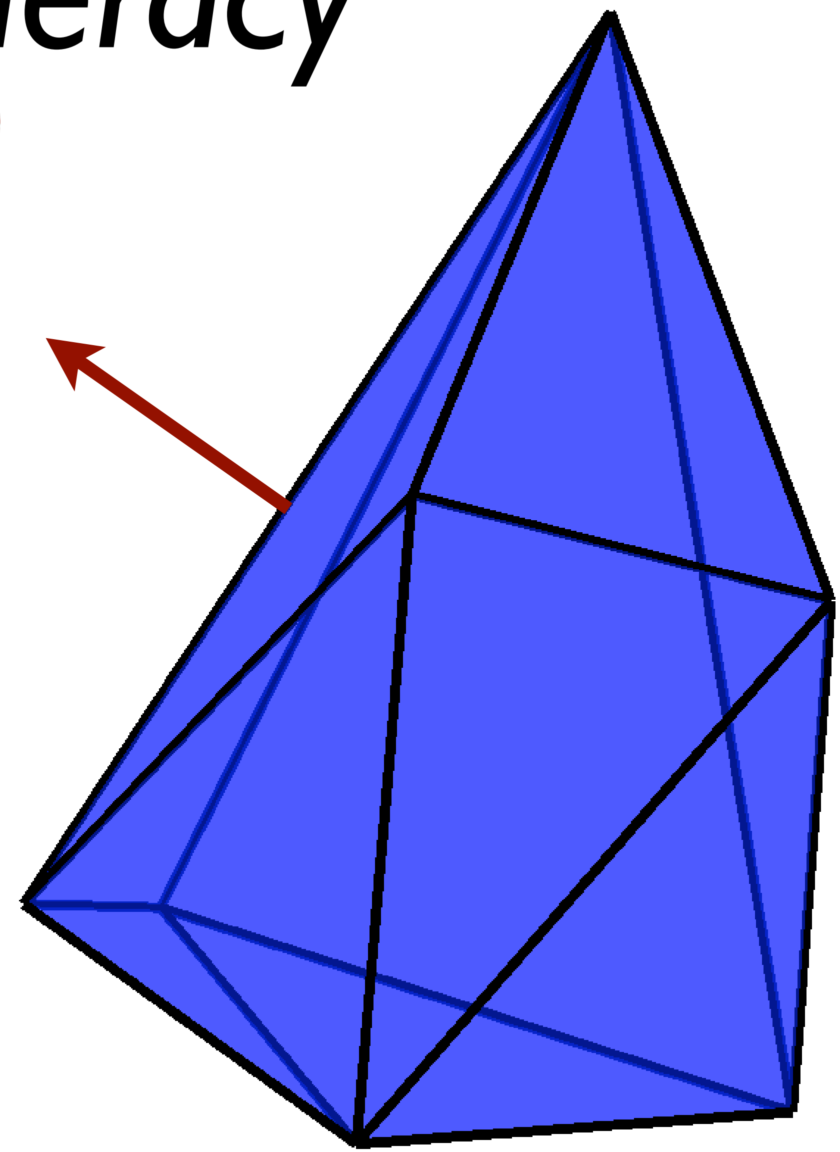
Bases & degeneracy

- How many bases for vertex A?
  -
- Are they all neighbors of one another?
  -
- Are they all neighbors of B?
  -



Dual degeneracy

- More than m entries in objective row = 0
  - so, a nonbasic variable has reduced cost = 0
  - objective orthogonal to a d-face for d ≥ 1



Handling degeneracy

- Sometimes have to make pivots that don't improve objective
  - stay at same corner (exiting variable was already 0)
  - move to another corner w/ same objective (coeff of entering variable in objective was 0)
- Problem of cycling
  - need an anti-cycling rule (there are many…)
  - e.g.: add tiny random numbers to obj, RHS