Linear programs

10-725 Optimization
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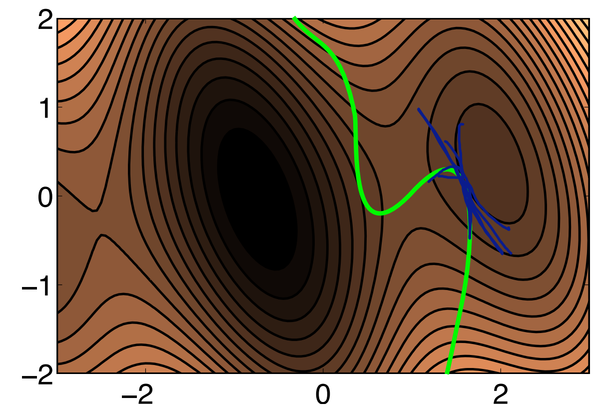
Review



- Newton w/ equality constraints
- Examples:
  - bundle adjustment
  - MLE in exponential families
- Convergence: damped phase, quadratic phase
- Compare: Newton, FISTA, (stoch) (sub)gradient
- Variations: trust region, quasi-Newton, Gauss-Newton, Levenberg-Marquardt

Variations: Fisher scoring

- Recall Newton in exponential family

$$\overset{H}{\mathrm{Var}[x \mid \theta]}\, d\theta = \overbrace{E[x \mid \theta] - \bar{x}}^{-g}$$

- Can use this formula in place of Newton, even if not an exponential family
  - descent direction, even w/ no regularization
  - "Hessian" is independent of data
  - often a wider radius of convergence than Newton
  - can be superlinearly convergent

Administrivia

- HW2 due now
- HW3 out tonight (hopefully)
- Final project update
  - project milestone report requirements on web site
  - final poster session (3:30–6:30 12/12 NSH Atrium, 3PM setup)
  - set up meetings w/ TA mentors

Linear programs

- n variables: $x = (x_1, x_2, \ldots, x_n)^T$
  - ranges: $[l_i, u_i]$

$l_i \to -\infty$, $u_i \to +\infty$

- Objective: min or max $\sum_{i=1}^{n} c_i x_i = c^T x$
- m constraints (equality or inequality):

$$\sum_{i=1}^{n} a_{ij} x_i = b_j \qquad (j = 1 \ldots m)$$

$\hookrightarrow \geq \leq$

- Example: max $2x + 3y$ s.t.

$x, y \in \mathbb{R}$

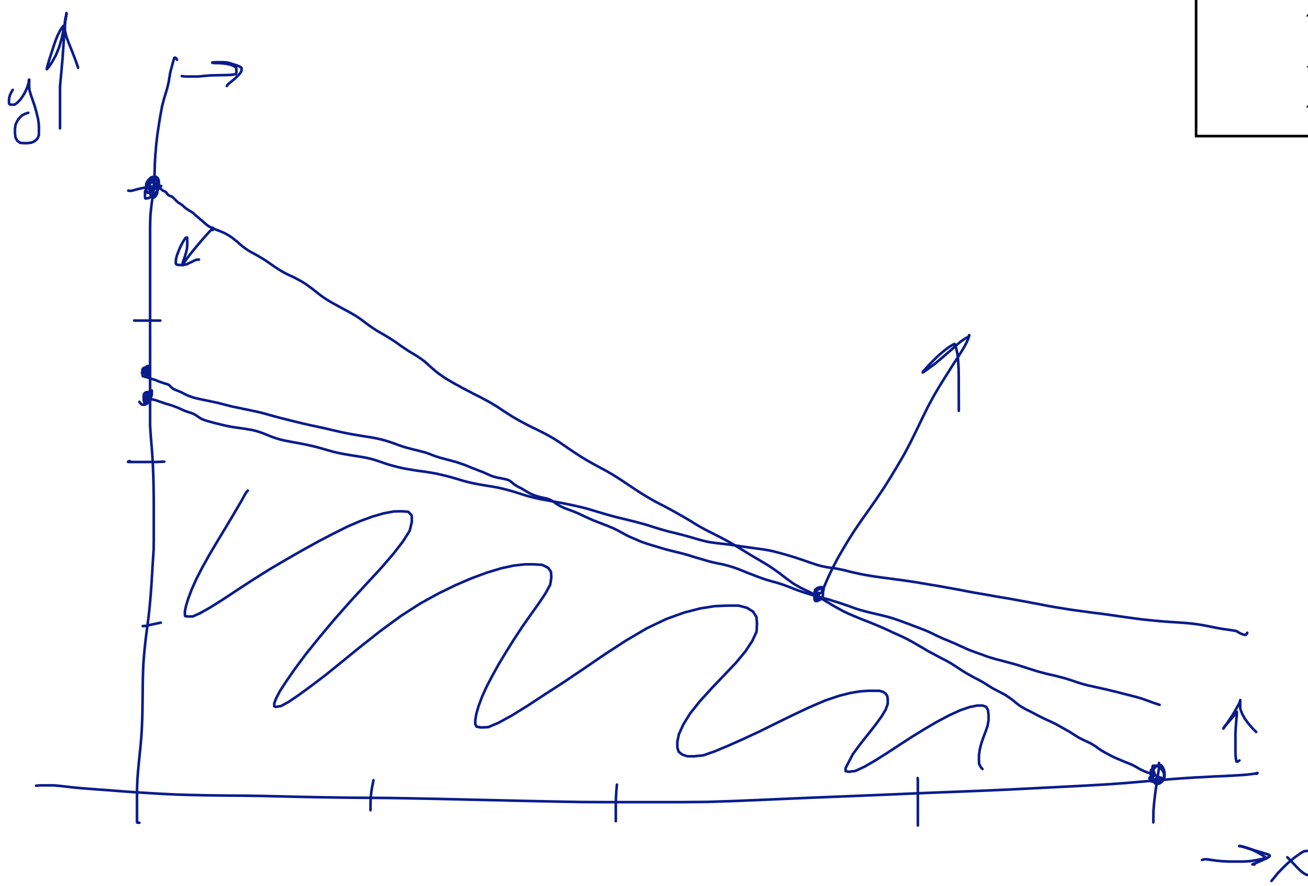
$x + y \leq 4$ $\quad$ $x + 2y \leq 5$

$2x + 5y \leq 12$ $\quad$ $x, y \geq 0$

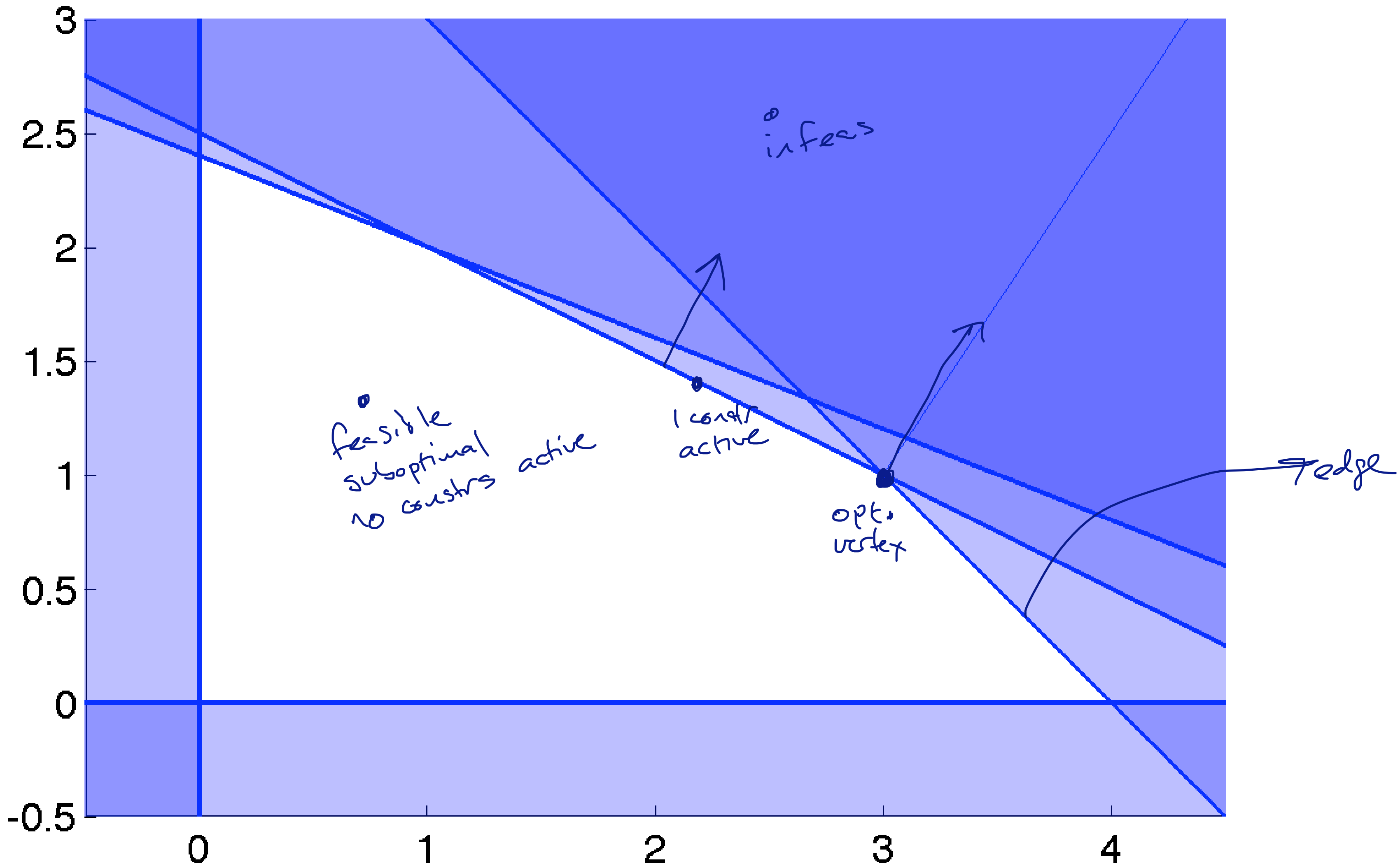
Sketching an LP

max $2x+3y$ s.t.

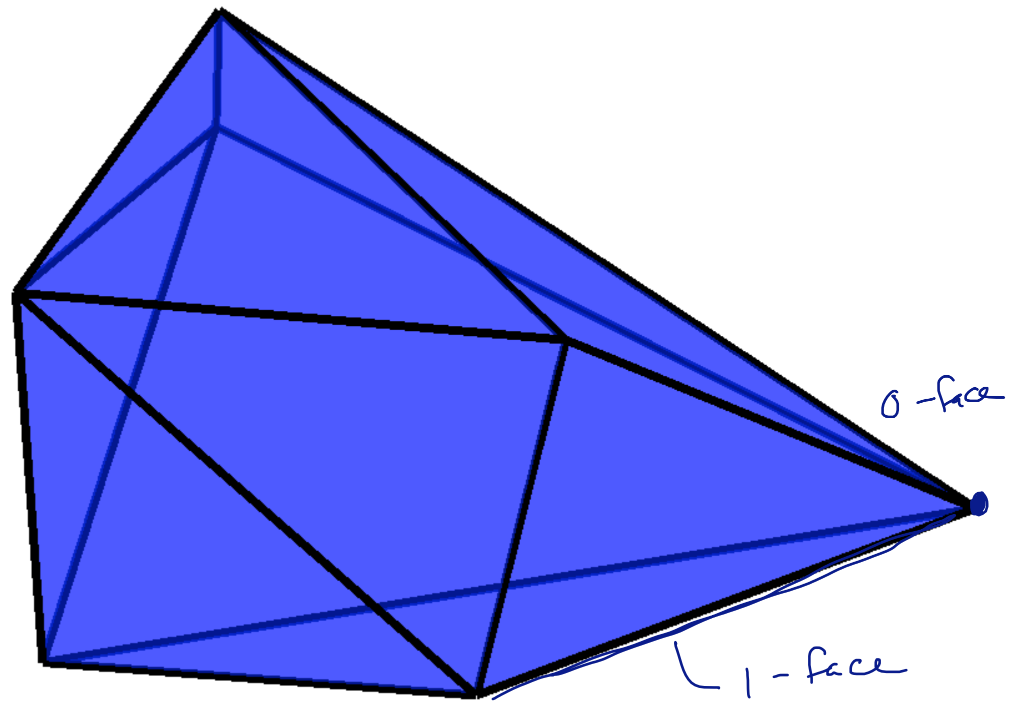
$$x + y \leq 4$$
$$2x + 5y \leq 12$$
$$x + 2y \leq 5$$
$$x, y \geq 0$$



Did the prof get it right?



Polyhedra



- hull({points}) or ∩({halfspaces})
- Vertices, edges, faces
  - in general: d-faces
    - n vars: d-face = set of feasible points that make $n-d$ independent halfspace constrs tight
    - therefore, dimensionality = $n-(n-d) \rightarrow d$ dimens. set
  - n vars, m≥n halfspaces: can have 0-faces thru n-faces

n-1 face "facet"

Matrix notation

- For a vector of variables v $\begin{pmatrix} x \\ y \end{pmatrix}$ and a constant matrix A and vector b,
  - Av ≤ b [componentwise]
- Objective: $c^T v$
- E.g.: max $c^T v$ s.t. $Av \le b$
  - $A = \begin{pmatrix} 1 & 1 \\ 2 & 5 \\ 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$ $b = \begin{pmatrix} 4 \\ 12 \\ 5 \\ 0 \\ 0 \end{pmatrix}$ $c = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

max 2x+3y s.t.
$x + y \le 4$
$2x + 5y \le 12$
$x + 2y \le 5$
$x, y \ge 0$

Finding the optimum

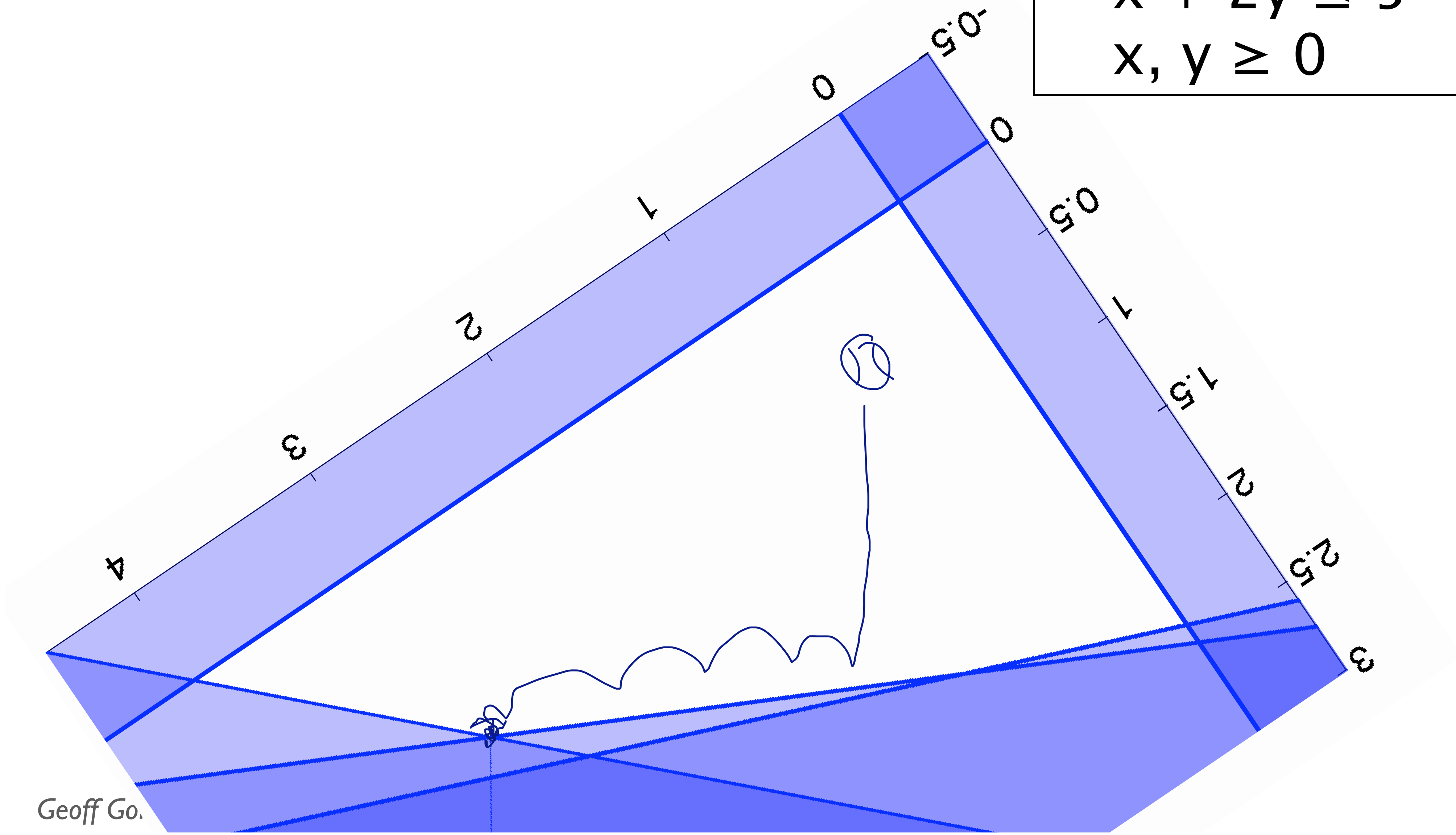
max $2x+3y$ s.t.

$x + y \le 4$

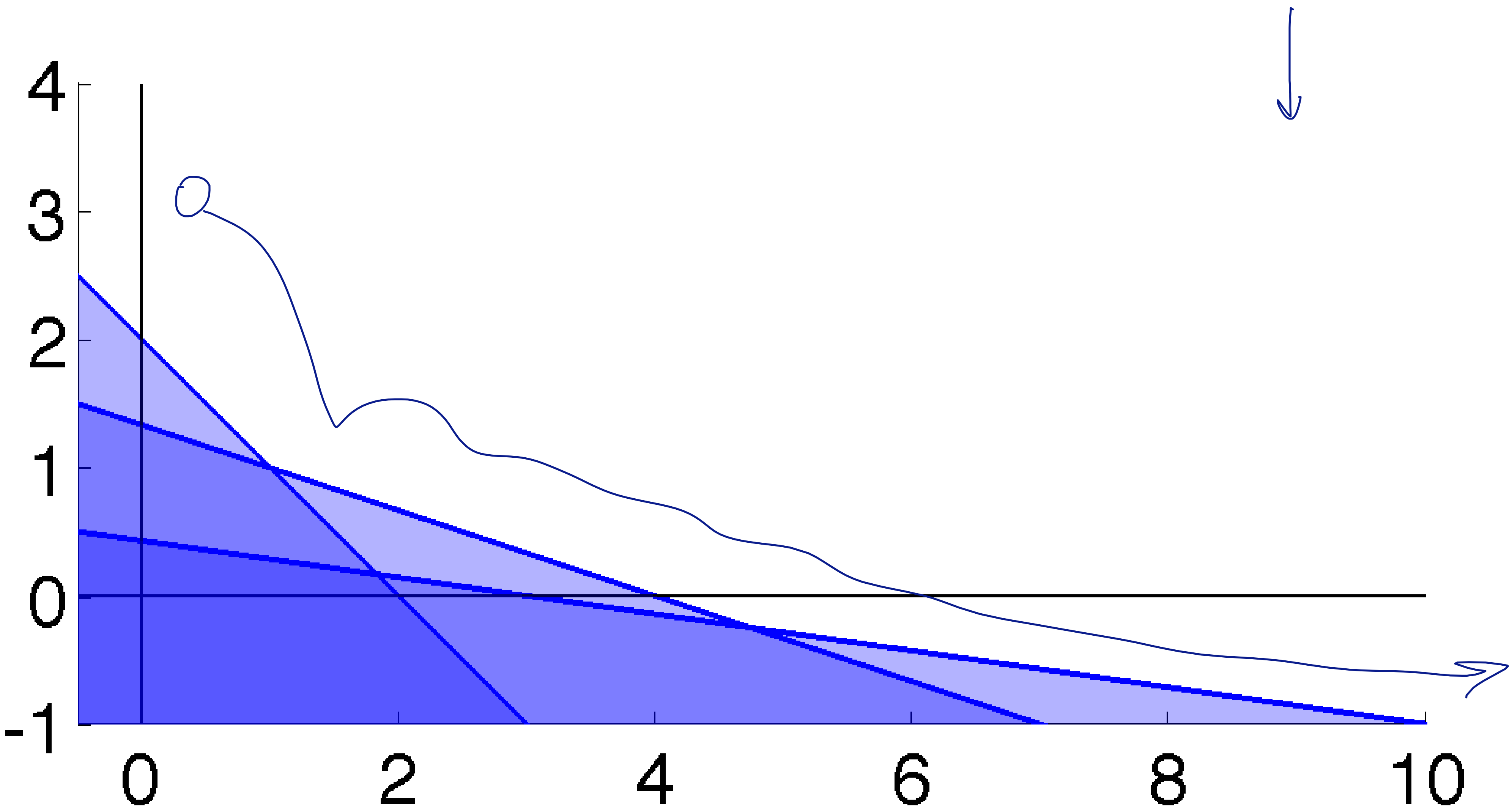
$2x + 5y \le 12$

$x + 2y \le 5$

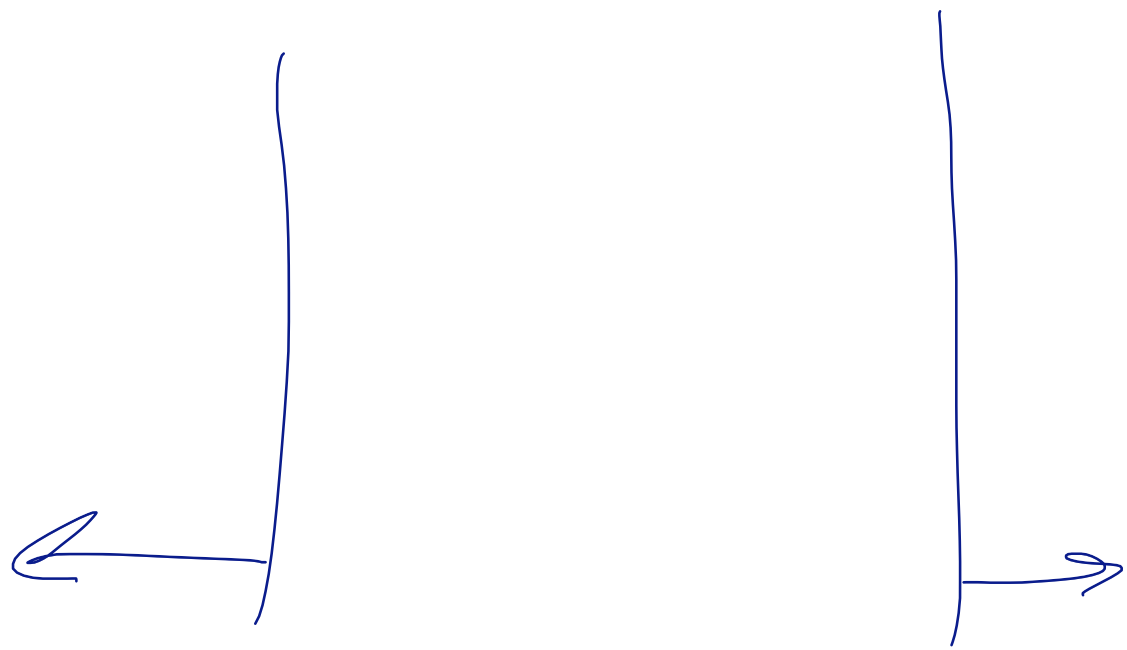
$x, y \ge 0$



Where's my ball?



Unhappy ball



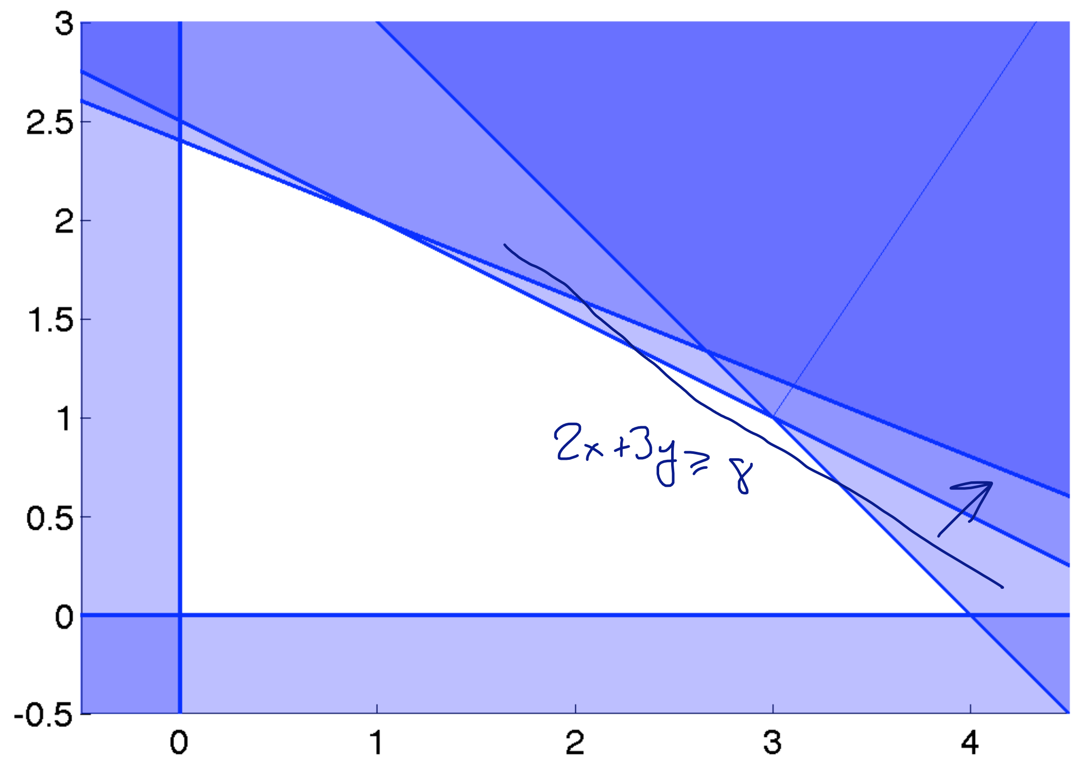
- max $2x + 3y$ subject to
- $x \geq 5$
- $x \leq 1$

Convention

- min over empty set = $+\infty$
- max over empty set = $-\infty$
- Adding an element always
  - decrease min
  - incr. max

Linear feasibility

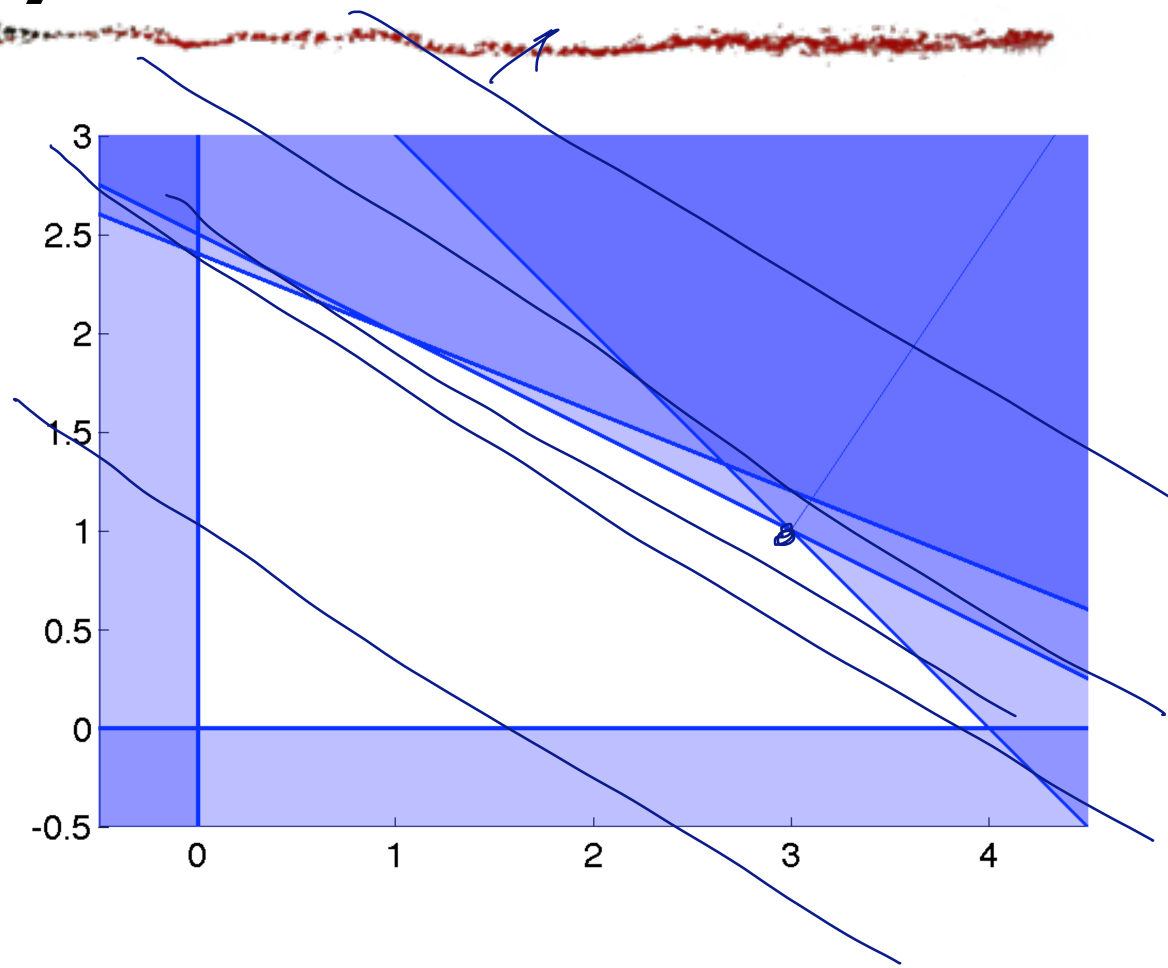
- find (x, y) s.t.
  - $x + y \leq 4$
  - $2x + 5y \leq 12$
  - $x + 2y \leq 5$
  - $x, y \geq 0$



- Any easier than LP? no

Binary search

- find (x, y) s.t.
  - $x + y \leq 4$
  - $2x + 5y \leq 12$
  - $x + 2y \leq 5$
  - $x, y \geq 0$



vs. max $2x+3y$ s.t. ↑

Transforming LPs

- Getting rid of inequalities (except variable bounds)

$$x + y \le 4$$

$x + y + s = 4$ $\quad s \in [0, \infty]$

- Getting rid of equalities

$$x + 2y = 4$$

$x + 2y \le 4$

$x + 2y \ge 4$

Transforming LPs

- Getting rid of free vars $x \in \mathbb{R}$ $y \in \mathbb{R}$

$$\max x + y \text{ s.t.}$$
$$2x + y \leq 3$$
$$y \geq 0$$

$x = a - b$ $a \geq 0$ $b \geq 0$

- Getting rid of bounded vars

$x \in [2, 5]$ $\rightarrow$ $x \in \mathbb{R}$ $2 \leq x \leq 5$

Standard form LP

max 2x+3y s.t.
x + y ≤ 4
2x + 5y ≤ 12
x + 2y ≤ 5
x, y ≥ 0

$x + y + u = 4$

- all variables are nonnegative
- all constraints are equalities
- E.g.: max $c^T q$ s.t. $Aq = b$, $q \geq 0$
  - $q = [x\ y\ u\ v\ w]^T$

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 12 \\ 5 \end{pmatrix}$$

$$c = \begin{pmatrix} 2 & 3 & 0 & 0 & 0 \end{pmatrix}^T$$

tableau

Objective in tableau

$$\begin{aligned} &\max\ 2x+3y \text{ s.t.}\\ &x + y \le 4\\ &2x + 5y \le 12\\ &x + 2y \le 5\\ &x, y \ge 0 \end{aligned}$$

- Add an extra variable z
  - constrain it to = the objective

x	y	u	v	w	z	RHS
1	1	1	0	0	0	4
2	5	0	1	0	0	12
1	2	0	0	1	0	5
−2	−3	0	0	0	1	0

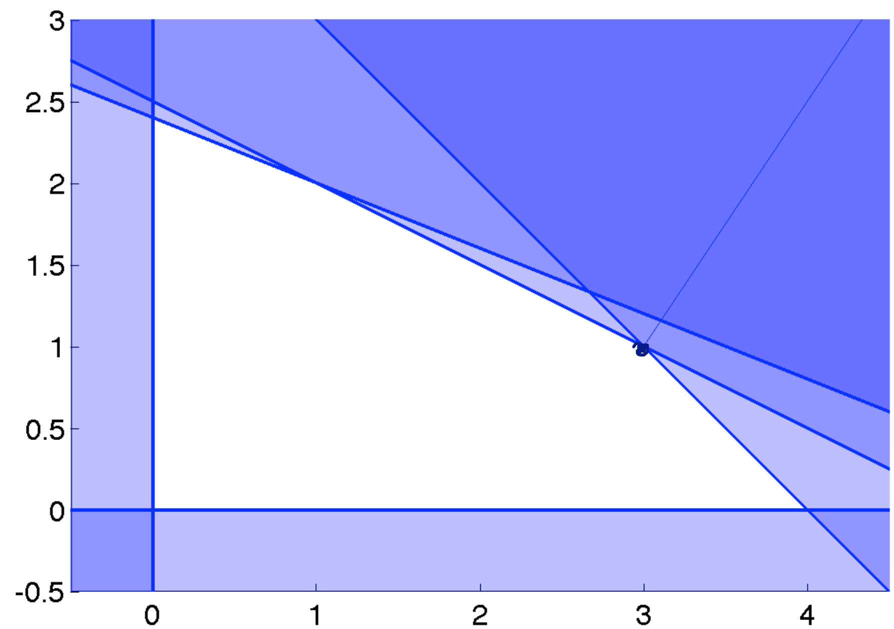
→ no constr on range of z

Standard v. inequality forms

- max $2x+3y$ s.t.
  - $x + y \le 4$
  - $2x + 5y \le 12$
  - $x + 2y \le 5$
  - $x, y \ge 0$
- or s.t.
  - $x + y + u = 4$
  - $2x + 5y + v = 12$
  - $x + 2y + w = 5$
  - $x, y, u, v, w \ge 0$

m = 3

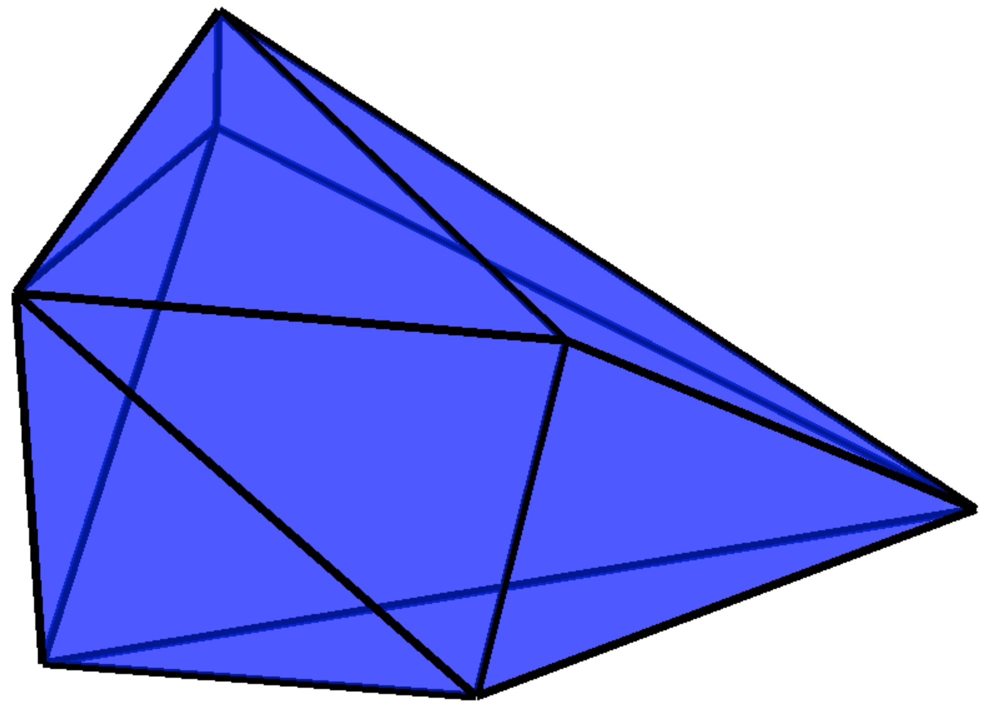
n = 5



m ≤ n

if std fm has n vars, m eqns
then ineq form has n–m vars
and $m+(n-m)=n$ ineqs
(here $m = 3$, $n = 5$)

Faces in standard form



- Inequality form
  - n vars, m≥n halfspaces: can have 0-faces thru n-faces
  - d-face makes n–d inequalities tight
- Standard form
  - n nonneg. vars, m≤n equalities: 0 -faces thru (n-m)-faces

Why is standard form useful?

- Can take linear combinations of constraints
- E.g., $x + 2y = 4$ & $2x + 3y = 5$

$\cdot - \cdot : \quad x + y = 1$

- Easy to manipulate via row operations
- Easy to find corners by Gaussian elimination

Example

$$\begin{array}{cccccc} x & y & u & v & w & \text{RHS} \\ 1 & 1 & 1 & 0 & 0 & 4 \\ 2 & 5 & 0 & 1 & 0 & 12 \\ 1 & 2 & 0 & 0 & 1 & 5 \end{array}$$

set x, y = 0

$u = 4 \quad v = 12 \quad w = 5$

$$\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 4 \\ 2 & 5 & 0 & 1 & 0 & 12 \\ 1 & 2 & 0 & 0 & 1 & 5 \end{array}$$

set v, w = 0

$(x, y, u) = (1, 2, 1)$

$$\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 4 \\ 2 & 5 & 0 & 1 & 0 & 12 \\ 1 & 2 & 0 & 0 & 1 & 5 \end{array}$$

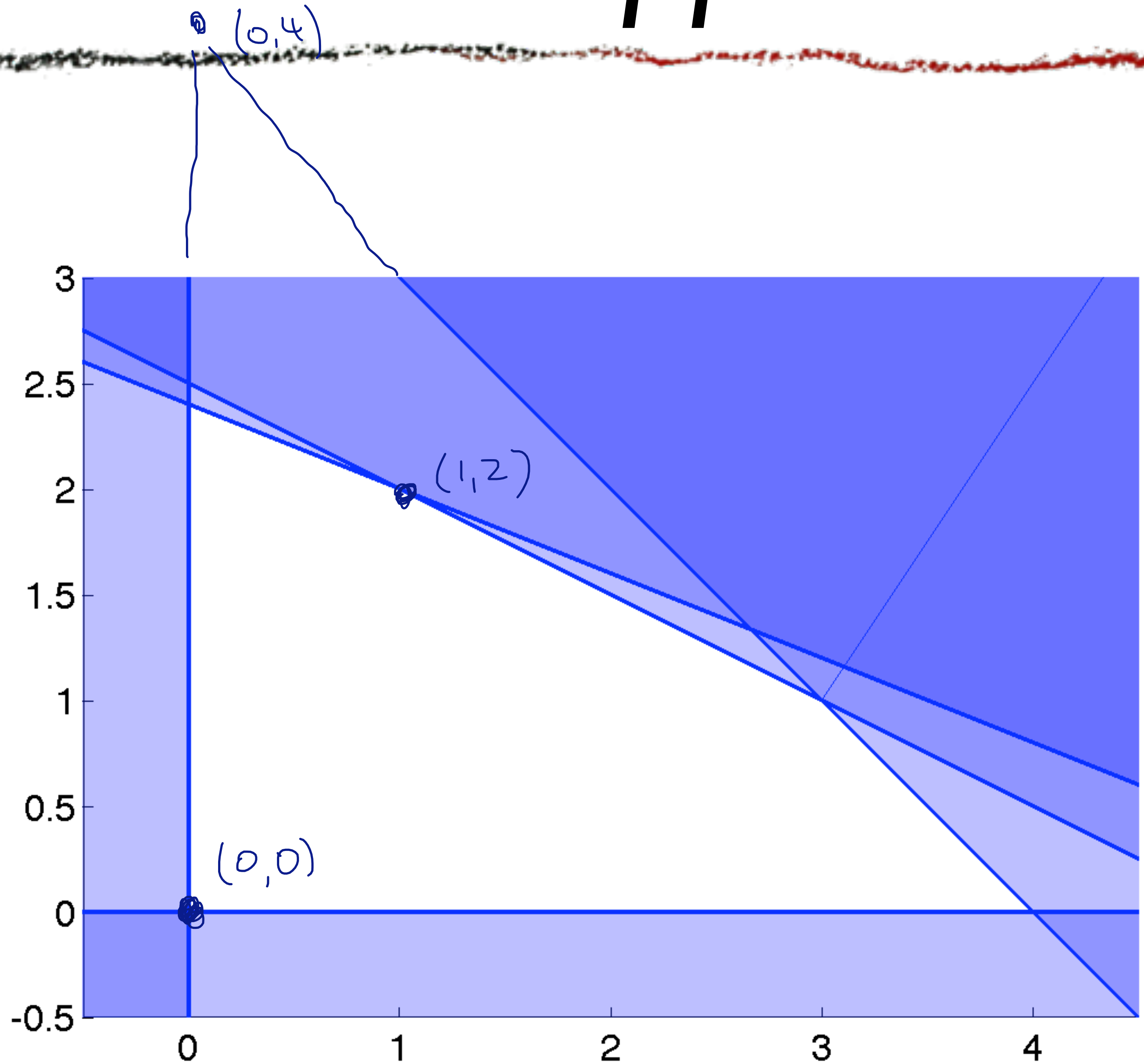
set x, u = 0

$y = 4$

$v = -8$

$w = -3$

What happened?



Row operations

- Can replace any row with linear combination of existing rows
  - as long as we don't lose independence
- Eliminate x from 2nd and 3rd rows

$$
\begin{array}{cccccc}
A & & & & & b \\
1 & 1 & 1 & 0 & 0 & 4 \\
\cancel{2} & \cancel{5} & \cancel{0} & \cancel{1} & \cancel{0} & \cancel{12} \\
\cancel{1} & \cancel{2} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{5} \\
\\
0 & 3 & -2 & 1 & 0 & 4 \\
0 & 1 & -1 & 0 & 1 & 1
\end{array}
$$

Presto change-o

x	y	u	v	w	RHS
1	1	1	0	0	4
0	3	−2	1	0	4
0	1	−1	0	1	1

- Which are the slacks now?
  - x, u, w
  - vars appearing in 1 constr
- Eliminating x from all but one constr: → x slack
- Constraint we used to eliminate x:

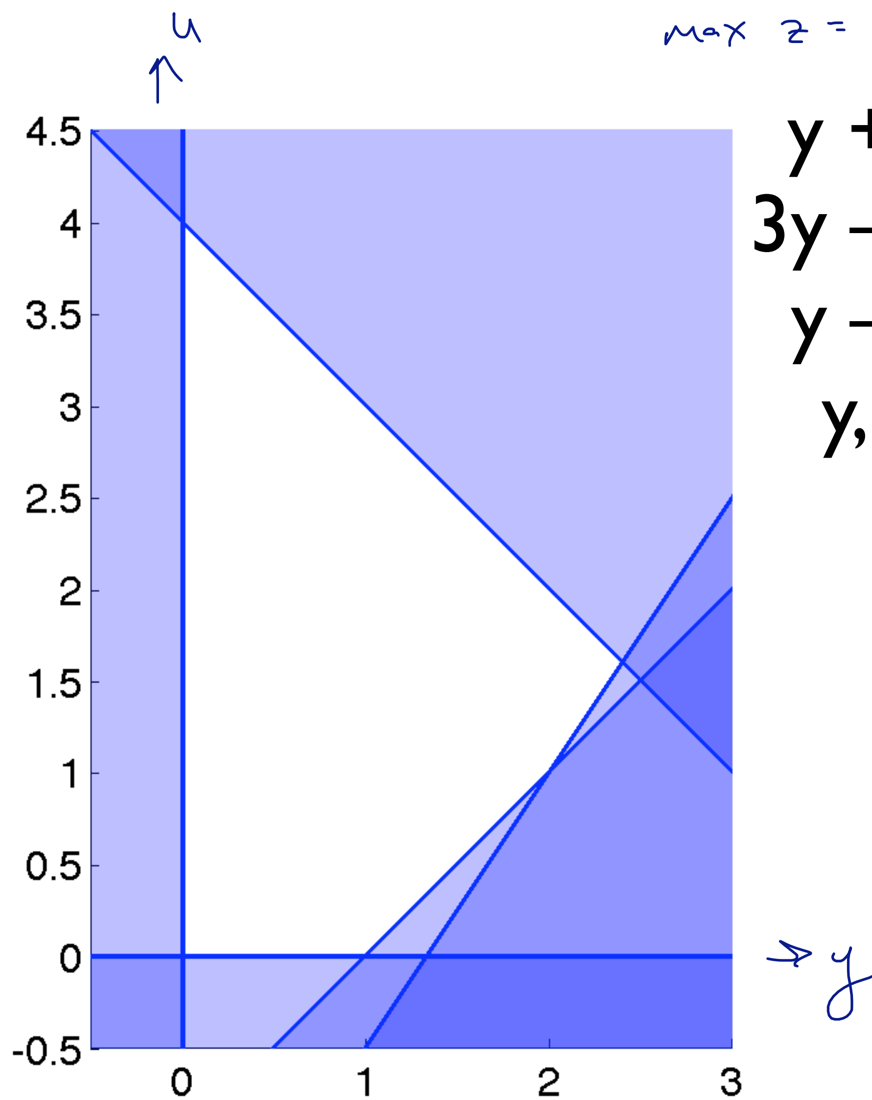
its slack (u) → not a slack

The "new" LP

objective: was $z = 2x + 3y$

max $z = y - 2u$ $(+8)$

x	y	u	v	w	z	RHS
1	1	1	0	0	0	4
0	3	−2	1	0	0	4
0	1	−1	0	1	0	1
~~−2~~	~~−3~~	~~0~~	~~0~~	~~0~~	~~1~~	~~.~~ (0)
0	−1	2	0	0	1	8



$y + u \leq 4$

$3y - 2u \leq 4$

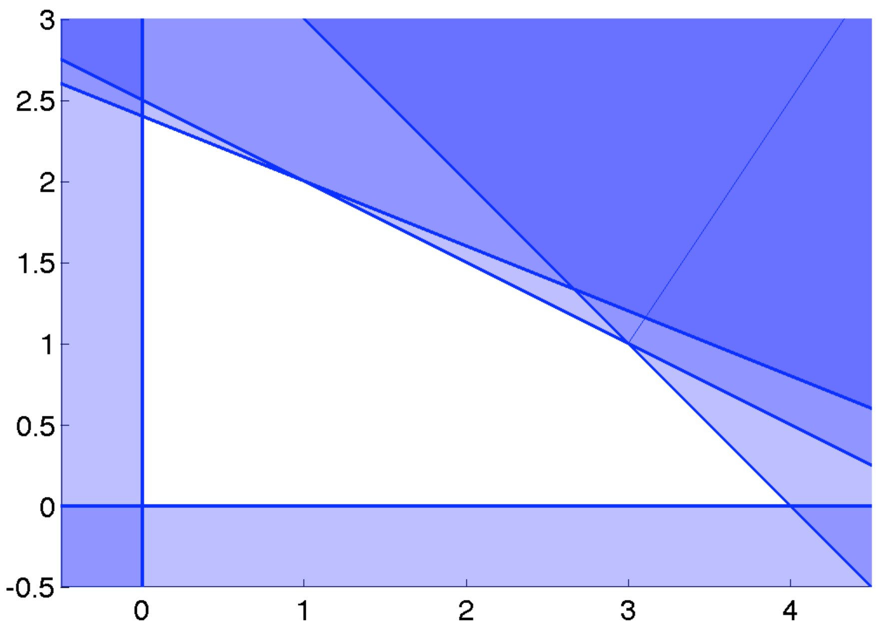
$y - u \leq 1$

$y, u \geq 0$

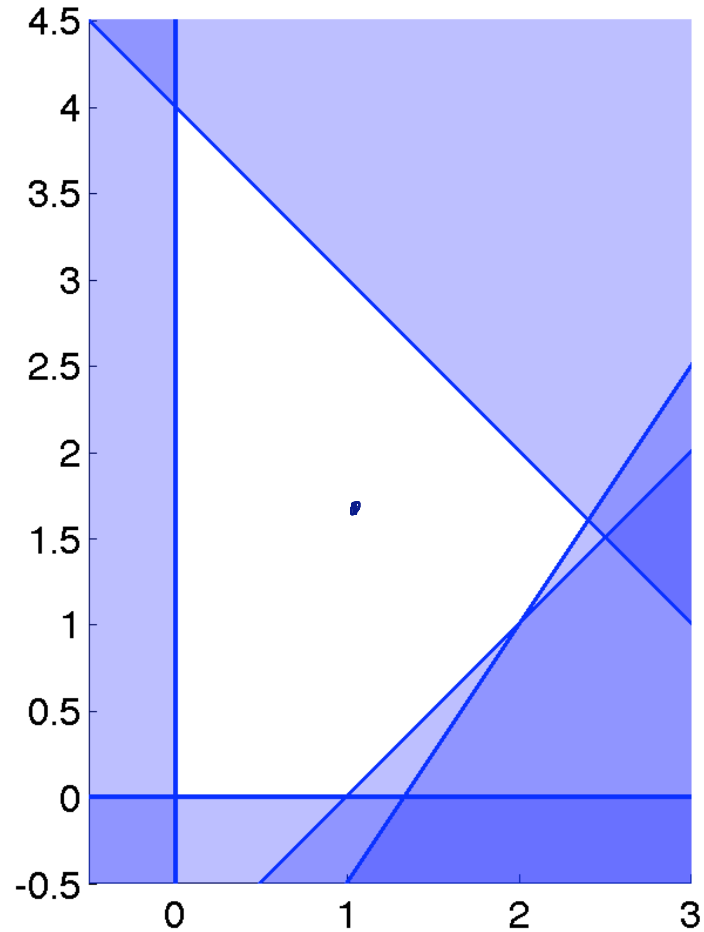
Sketching standard form

- Drop the slacks, sketch in inequality form
- May be several ways

1	1	1	0	0	4
0	3	-2	1	0	4
0	1	-1	0	1	1



1	1	1	0	0	4
2	5	0	1	0	12
1	2	0	0	1	5



What if there aren't slacks?

- Use row ops to ***make*** some:
  - $u, v, w \geq 0$
  - $u + 2v + w = 3$
  - $\cancel{3u + v - w = 5}$

    $0 \quad -5v \quad -4w = -4$

Matlab version

```
A = [1  2  1;
     3  1 -1];
b = [3;5];
A(:,1:2) \ A
ans =
    1.0000         0   -0.6000
         0    1.0000    0.8000

A(:,1:2) \ b
ans =
    1.4000
    0.8000
```

Matlab with z

max z=2x+3y s.t.
x + y + u = 4
2x + 5y + v = 12
x + 2y + w = 5
x, y, u, v, w ≥ 0

- always pick z's column (here, col 6)
- remember z is unconstrained

A and b

x	y	u	v	w	z	RHS
1	1	1	0	0	0	4
2	5	0	1	0	0	12
1	2	0	0	1	0	5
−2	−3	0	0	0	1	0

result = A(:,[1 4 5 6]) \ [A b]

x	y	u	v	w	z	RHS
1	1	1	0	0	0	4
0	3	−2	1	0	0	4
0	1	−1	0	1	0	1
0	−1	2	0	0	1	8

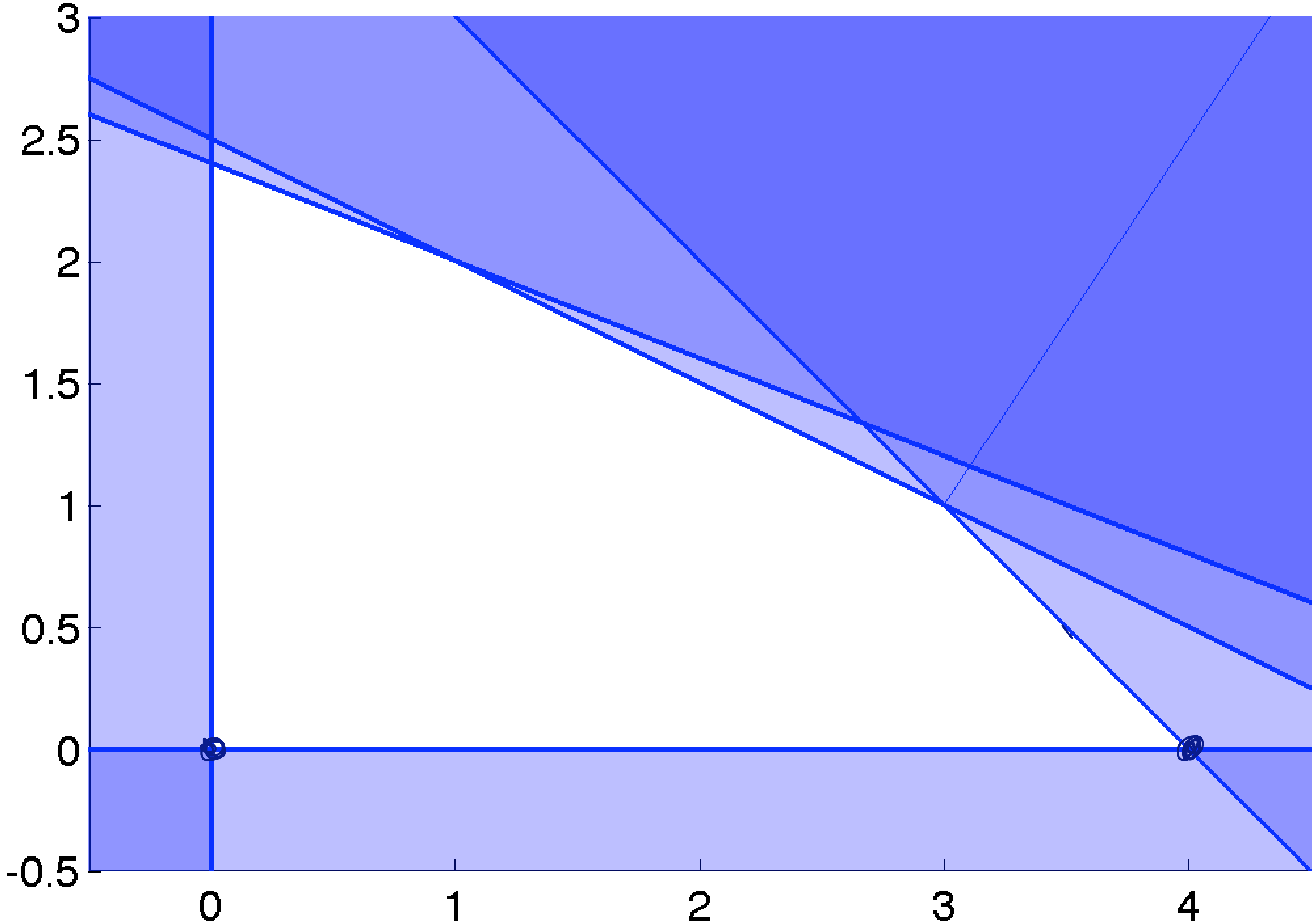
Basis

- {u, v, w, z} and {x, v, w, z} are ***bases***
  - elements = basic variables (always m of them)
  - easy to write values of basic variables in terms of non-basic ones
  - e.g., set x=y=0
  - e.g., set y=u=0

x	y	u	v	w	z	RHS
1	1	1	0	0	0	4
2	5	0	1	0	0	12
1	2	0	0	1	0	5
−2	−3	0	0	0	1	0

x	y	u	v	w	z	RHS
1	1	1	0	0	0	4
0	3	−2	1	0	0	4
0	1	−1	0	1	0	1
0	−1	2	0	0	1	8

Basic solutions



Bases $\leftrightarrow$ corners

- Std form: n vars, m equations
  - fix ineq form: n–m vars, n ineqs
- Pick a basis B for std form
  - m basic vars (≥ 0), n–m nonbasic (set to 0)
- Each nonbasic var yields a tight inequality
  - var is either a slack or explicit in ineq fm
    - explicit: one of n–m "trivial" ($x \geq 0$) ineqs tight
    - slack: one of m "real" ineqs tight
- Ineq fm: n–m vars and n–m tight ineqs → corner