Canonical Optimization Problem: $f^* = \min_{x\in X} f(x)$; $X = \{x : h_i(x) \leq 0,\ \ell_j(x) = 0\}$

Lagrangian: $L(x,\lambda,\gamma) = f(x) + \sum_i \lambda_i h_i(x) + \sum_j \gamma_j \ell_j(x)$

Note: $\min_{x\in X} f(x) = \min_{x\in\mathbb{R}^n} \max_{\substack{\lambda\geq 0\\ \gamma\in\mathbb{R}}} L(x,\lambda,\gamma)$

Lagrange dual function: $g^* = g(\lambda,\gamma) = \min_{x\in\mathbb{R}^n} L(x,\lambda,\gamma)$

Dual problem: $\max_{\substack{\lambda\geq 0\\ \gamma\in\mathbb{R}}} g(\lambda,\gamma) := \max_{\substack{\lambda\geq 0\\ \gamma}} \min_{x\in\mathbb{R}^n} L(x,\lambda,\gamma)$

$\min_x x$

$x^2 \leq \varepsilon^2$

$L(x,\lambda) = x - \lambda(\varepsilon^2 - x^2)$

$g(\lambda) = \min_x x - \lambda(\varepsilon^2 - x^2)$

$1 + 2\lambda x = 0$

$x = \frac{-1}{2\lambda}$

$= \frac{-1}{4\lambda} - \lambda\varepsilon^2$

$\lambda = \frac{1}{4\varepsilon}$, $x = \frac{2\varepsilon}{\lambda}$

$\forall x \in X, \lambda \geq 0 \quad f(x) \geq L(x,\lambda,\gamma)$

$\forall\lambda\geq 0$

$\Rightarrow f^* \geq \min_{x\in X} L(x,\lambda,\gamma) \geq \min_{x\in\mathbb{R}^n} L(x,\lambda,\gamma) = g(\lambda,\gamma)$

$\Rightarrow f^* \geq g^*$ (weak duality)

Strong duality if holds at equality.

Slater's condition: convex problem $\exists\, x\in X$ feasible, strictly feasible for non-affine constraints

$\Rightarrow$ strong duality holds

Matrix games: $A\in\mathbb{R}^{m\times n}$, A_{ij} is payoff to row player if he plays i and she plays j

$-A_{ij}$ " column " "

$\Rightarrow x, y \in \Delta = \{p : p \geq 0, \sum_{i=1}^n p_i = 1\}$ are randomized strategies, then $x^T A y$ is x's expected payoff

Model as an optimization problem:

x goes first: $\max_{x\in\Delta} \min_{y\in\Delta} x^T A y$ or y goes first: $\min_{y\in\Delta} \max_{x\in\Delta} x^T A y$

$= \max_{\substack{x\in\Delta\\ t}} t$ (P)

$A^T x \geq te$

$\min_{y\in\Delta} q$

$Ay \leq qe$ (D)

$\max_{x,t} \min_{\substack{y,u\geq 0\\ q}} t - (te - A^T x)^T y + u^T x + q(1 - e^T x)$

$t(1 - y^T e) + (Ay + u - qe)^T x + q$

$\min_{y,u\geq 0} q$, $y^T e = 1$, $Ay + u - qe = 0$

Strong Duality holds by Slater's condition

$\Rightarrow$ No advantage to going "second"

KKT conditions:

$$\nabla f(x^*) + \sum_i \lambda_i^* h_i(x^*) + \sum_j \gamma_j^* l_j(x^*) = 0 \qquad \text{(stationary)}$$

$$\left.\begin{array}{ll} h_i(x^*) \leq 0 & \forall i \\ l_j(x^*) = 0 & \forall j \end{array}\right. \qquad \text{(primal feasible)}$$

$$\lambda^* \geq 0 \qquad \text{(dual feasible)}$$

$$\lambda_i^* h_i(x^*) = 0 \quad \forall i \qquad \text{(complementary slackness)}$$

Strong Duality $\Rightarrow$ KKT

x^* is primal opt, $\lambda^* \gamma^*$ are dual opt

$$f(x^*) = g(\lambda^*, \gamma^*) = \min_{x \in \mathbb{R}^n} f(x) + \sum_i \lambda_i^* h_i(x) + \sum_j \gamma_j^* l_j(x)$$

$$\leq f(x^*) + \underbrace{\sum_i \lambda_i^* h_i(x^*)}_{\leq 0} + \sum_j \gamma_j^* \underbrace{l_j(x^*)}_{=0} \qquad \text{~~(primal feasible)~~}$$

$$\leq f(x^*) \qquad \text{(primal feasible)}$$

$$\Rightarrow f(x^*) = f(x^*) + \sum_i \lambda_i^* h_i(x^*) + \sum_j \gamma_j^* l_j(x^*)$$

$$\Rightarrow \lambda_i^* h_i(x^*) = 0 \quad \forall i \qquad \text{(complementary slackness)}$$

$$\Rightarrow x^* = \operatorname{argmin}_{x \in \mathbb{R}^n} f(x) + \sum \lambda_i^* h_i(x) + \sum_j \gamma_j^* l_j(x)$$

$$\Rightarrow \nabla f(x^*) + \sum_i \lambda_i^* \nabla h_i(x^*) + \sum_j \gamma_j^* l_j(x^*) = 0 \quad \text{(stationarity)}$$

KKT + convex $\Longrightarrow$ strong duality

$$g^* = g(\lambda^*, \gamma^*) = \min_{x \in \mathbb{R}} L(x, \lambda^*, \gamma^*)$$

$$= L(x^*, \lambda^*, \gamma^*) \qquad \text{(stationarity)}$$

$$= f(x^*) - f^* \qquad \text{(primal feasible, complementary slackness)}$$