Primal Cone Program:

$$\min_{x} f(x) \qquad Ax+b \in K$$

or

$$\min_{x,y} f(x) \qquad Ax+b = y, \quad y \in K$$

For now consider:

$$\min \sum_i c_i \cdot x_i \quad (\text{linear})$$

$$\forall i \quad A_i x_i = b_i, \quad x_i \in K_i$$

Some cones:

$\mathbb{R}^n$, dual $\{0\}$

$\mathbb{R}^n_+$, self-dual

S^n_+, self-dual

$\|\cdot\|$-cone $= \{(x,t) : \|x\| \leq t\}$

Dual is $\|\cdot\|_*$-cone

Proof: $(x,t) \in \|\cdot\|$-cone, $\alpha \geq 0$

$\alpha(x,t) = (\alpha x, \alpha t)$

$\|\alpha x\| = \alpha\|x\| \leq \alpha t$ ✓

$(y,u) \in \|\cdot\|$-cone

$\|x+y\| \leq \|x\| + \|y\| \leq t+u$ ✓

Dual cone: $\{(y,u) : x\cdot y + tu \geq 0, \ \forall (x,t) \in \|\cdot\|\text{-cone}\}$

$\forall x$, $|x\cdot y| \leq \|x\|\|y\|_* \leq t\|y\|_*$

if $\|y\|_* \leq u \Rightarrow |x\cdot y| \leq tu$

$\Rightarrow x\cdot y + tu \geq 0$

if $\|y\|_* > u$

$\Rightarrow u < \max_{\|x\|=1} x\cdot y = x^*\cdot y$

Choose $(-x^*, 1) \in \|\cdot\|$-cone

$-x^*\cdot y + ut = -x^*\cdot y + u < -u+u = 0$

$\Rightarrow (y,u) \notin K^*$

$\|\cdot\|_2$-cone is second-order cone, self-dual

Convex Cone:

$\forall x \in K, \ \alpha \geq 0, \quad \alpha x \in K$

$\forall x,y \in K \quad x+y \in K$

Dual Cone:

$$K^* = \{y : x\cdot y \geq 0, \ \forall x \in K\}$$

Inequalities to Cone constraints:

$$Ax \leq b$$

$$\Leftrightarrow Ax - b - y = 0, \quad y \in \mathbb{R}^m_+$$

(convex) Quadratic constraint to Cone constraint:

$$x^T H x + b^T x + c \leq 0$$

$$\Leftrightarrow x^T L L^T x + b^T x + c \leq 0 \quad (\text{Cholesky Factorization})$$

Claim: $\Leftrightarrow \|y\| \leq u$

$$u = (1 - b^T x - c)/2$$

$$y = \begin{pmatrix} (1 + b^T x + c)/2 \\ L^T x \end{pmatrix}$$

Proof: $\|y\| \leq u$

$$\Leftrightarrow \|y\|^2 \leq u^2$$

$$\Leftrightarrow \frac{(1+b^T x + c)^2}{4} + xLL^T x \leq \frac{(1-b^T x - c)^2}{4}$$

$$\Leftrightarrow xLL^T x + \frac{\cancel{1-1} + 2b^T x + 2b^T x + 2c + 2c}{4} + \frac{\cancel{(b^T x)^2 - (b^T x)^2} + \cancel{2(b^T x)c - 2(b^T x)c}}{4} + \frac{\cancel{c^2 - c^2}}{4} \leq 0$$

$$\Leftrightarrow xLL^T x + b^T x + c \leq 0$$

LASSO:

$$\min_{w} \tfrac{1}{2}\|Xw - y\|^2 + \lambda \|w\|_1$$

$$= \min_{z,w,\alpha,\beta} \alpha/2 + \lambda\beta$$

$\|z\|^2 \leq \alpha \quad (z \cdot z \leq \alpha)$

$\|w\|_1 \leq \beta$

$Xw - y = z$

$$= \min_{v,z,w,\alpha,\beta,t} \alpha/2 + \lambda\beta$$

$\|v\| \leq t \quad (v,t) \in \|\cdot\|_2\text{-cone}$

$\|w\|_1 \leq \beta \quad (w,\beta) \in \|\cdot\|_1\text{-cone}$

$Xw - y = z$

$t = \dfrac{1+\alpha}{2}$

$v = \begin{pmatrix} (1-\alpha)/2 \\ z \end{pmatrix}$

SVM: $\min_{w,b,\xi} \tfrac{1}{2}\|w\|^2 + C\sum \xi_i$

$\forall i \quad y_i(w \cdot x_i + b) \geq 1 - \xi_i$

$\xi_i \geq 0$

$$= \min_{w,b,\xi,t,z} t/2 + C\sum \xi_i$$

$\|w\|^2 \leq t$

$\forall i \quad y_i(w \cdot x + b) + \xi_i - z_i = 1$

$z_i, \xi_i \geq 0$

$$= \min_{v,u,w,b,\xi,t,z} t/2 + C\sum \xi_i$$

$\forall i \quad y_i(w \cdot x + b) + \xi_i - z_i = 1$

$\xi_i, z_i \geq 0$

$\|v\| \leq u \quad (v,u) \in \|\cdot\|_2\text{-cone}$

$\xi, z \in \mathbb{R}^m_+$

$u = \dfrac{1+t}{2}$

$v = \begin{pmatrix} (1-t)/2 \\ w \end{pmatrix}$

Dantzig Selector:

$$\min_{w} \|w\|_1$$

$$\|X^T(y - Xw)\|_\infty \leq \varepsilon$$

$$= \min_{w,z,t,\alpha} t$$

$\|w\|_1 \leq t \quad (w,t) \in \|\cdot\|_1\text{-cone}$

$\|z\|_\infty \leq \alpha \quad (z,\alpha) \in \|\cdot\|_\infty\text{-cone}$

$\alpha = \varepsilon$

$X^T(y - Xw) = z$

Solving with Interior Point Methods:

Need a barrier function for our cones: and interior point

$\mathbb{R}^n, \{0\}$ - None!

$\mathbb{R}^n_+ = -\sum_i \log x_i \quad x = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$S^n_+ = -\log\det(X) \quad X = I$

$\|\cdot\|\text{-cone} = -\log(t - \|x\|) \quad t = 1,\ x = 0$

$\Rightarrow$ Handle inequality constraints explicitly using infeasible start Newton Method

$$\min_x f(x) - \varphi(x)$$

$$Ax = b$$

Solving with Gradient Methods:

Need to project onto cone:

$\mathbb{R}^n, \{0\}$ easy

$\mathbb{R}^n_+$ - truncate

S^n_+ - truncate Eigenvalues

$$\|\cdot\|_2\text{-cone} - \begin{cases} (x,t) & \|x\| \leq t \\ c(x, \|x\|) & -\|x\| \leq t \leq \|x\| \\ (0,0) & t \leq -\|x\| \end{cases}$$

$c = \dfrac{\|x\| + t}{2\|x\|}$

But, inequality constraints are hard...

Idea take the dual!

$$\min f(x)$$
$$Ax+b=y$$
$$y \in K$$

$$\min_{x,y} \max_{\substack{u,v \\ v\in K^*}} f(x) - u^T(y-Ax-b) - y^T v$$

$$= \max_{\substack{u,v \\ v\in K^*}} \min_{x,y} f(x) - u^T(y-Ax-b) - y^T v$$

$$-u-v=0$$

$$= \max_{v\in K^*} \min_{x} f(x) + v^T b + v^T Ax$$

$$= \max_{v\in K^*} v^T b - f^*(A^T v) \qquad = \max_{v\in K^*} v^T b - \frac{\|A^T v - c\|^2}{2\alpha} \qquad \nabla = b - \frac{A(A^T v - c)}{\alpha}$$

Use an accelerated gradient method, if f^* is smooth

If its not, we can approximate it with a smooth function.

Lemma: if $f(x)$ is α-strongly convex then $f^*(y)$ has $\frac{1}{\alpha}$-Lipschitz gradient (Cauchy Schwarz)

e.g. $f(x) = c^T x$. $f(x) = c^T x + \frac{\alpha}{2}\|x\|^2$

$f^*(y) = I(y=c)$; $f^*(y) = \max_x x^T y - c^T x - \frac{\alpha}{2}\|x\|^2$

$$y-c-\alpha x = 0$$

Recover primal solution $\rightarrow$ $\boxed{x = \frac{y-c}{\alpha}}$

$$= \frac{(y-c)^T(y-c)}{2\alpha} = \frac{\|y-c\|^2}{2\alpha}$$

Can solve, again, with accelerated gradient, but trade-off between closeness of approximation and smoothness of dual.