# Lecture 6
# Linear Processing

ch. 5 of *Machine Vision* by Wesley E. Snyder & Hairong Qi

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# Linear Operators

“If and only if”

- $D$ is a linear operator iff:

$$D(\alpha f_1 + \beta f_2) = \alpha D(f_1) + \beta D(f_2)$$

Where $f_1$ and $f_2$ are images,

and $\alpha$ and $\beta$ are scalar multipliers

- Not a linear operator (why?):

$$g = D(f) = af + b$$

# Kernel Operators

- Kernel ($h$) = "small image"
  - Often 3x3 or 5x5
- Correlated with a "normal" image ( $f$ )

| | | |
|---|---|---|
| $h_{-1,-1}$ | $h_{0,-1}$ | $h_{1,-1}$ |
| $h_{-1,0}$ | $h_{0,0}$ | $h_{1,0}$ |
| $h_{-1,1}$ | $h_{0,1}$ | $h_{1,1}$ |

| | | | | |
|---|---|---|---|---|
| $f_{0,0}$ | $f_{1,0}$ | $f_{2,0}$ | $f_{3,0}$ | $f_{4,0}$ |
| $f_{0,1}$ | $f_{1,1}$ | $f_{2,1}$ | $f_{3,1}$ | $f_{4,1}$ |
| $f_{0,2}$ | $f_{1,2}$ | $f_{2,2}$ | $f_{3,2}$ | $f_{4,2}$ |
| $f_{0,3}$ | $f_{1,3}$ | $f_{2,3}$ | $f_{3,3}$ | $f_{4,3}$ |
| $f_{0,4}$ | $f_{1,4}$ | $f_{2,4}$ | $f_{3,4}$ | $f_{4,4}$ |

- Implied correlation (sum of products) makes a kernel an operator. A *linear* operator.
- Note: This use of correlation is often mislabeled as convolution in the literature.
- **Any linear operator applied to an image can be approximated with correlation.**

# Kernels for Derivatives

- Task: estimate partial spatial derivatives
- Solution: numerical approximation
  - $[\, f(x+1) - f(x)\, ]/1$
    - Really Bad choice: not even symmetric
  - $[\, f(x+1) - f(x-1)\, ]/2$
    - Still a bad choice: very sensitive to noise
  - We need to blur away the noise (only blur orthogonal to the direction of each partial):

$$\frac{\partial f}{\partial x} = \frac{1}{6}\left( \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \otimes f \right) \quad \text{or} \quad \frac{\partial f}{\partial x} = \frac{1}{8}\left( \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \otimes f \right)$$

Correlation (sum of products)

The *Sobel* kernel is center-weighted

# Derivative Estimation #2: Use Function Fitting

- Think of the image as a surface
  - The gradient then fully specifies the orientation of the tangent planes at every point, and vice-versa.
- So, fit a plane to the neighborhood around a point
  - Then the plane gives you the gradient
- The concept of fitting occurs frequently in machine vision. Ex:
  - Gray values
  - Surfaces
  - Lines
  - Curves
  - Etc.

# Derivative Estimation: Derive a 3x3 Kernel by Fitting a Plane

- If you fit by minimizing squared error, and you use symbolic notation to generalize, you get:
  - A headache
  - The kernel that we intuitively guessed earlier:

$$\frac{1}{6}\begin{array}{|c|c|c|}\hline -1 & 0 & 1 \\ \hline -1 & 0 & 1 \\ \hline -1 & 0 & 1 \\ \hline\end{array}$$

# Vector Representations of Images

- Also called lexicographic representations
- Linearize the image
  - Pixels have a single index (that starts at 0)

| | | | |
|---|---|---|---|
| $f_{0,0}$ | $f_{1,0}$ | $f_{2,0}$ | $f_{3,0}$ |
| $f_{0,1}$ | $f_{1,1}$ | $f_{2,1}$ | $f_{3,1}$ |
| $f_{0,2}$ | $f_{1,2}$ | $f_{2,2}$ | $f_{3,2}$ |
| $f_{0,3}$ | $f_{1,3}$ | $f_{2,3}$ | $f_{3,3}$ |

0 is the Lexicographic index

| | | | |
|---|---|---|---|
| $F_0$ | $F_1$ | $F_2$ | $F_3$ |
| $F_4$ | $F_5$ | $F_6$ | $F_7$ |
| $F_8$ | $F_9$ | $F_{10}$ | $F_{11}$ |
| $F_{12}$ | $F_{13}$ | $F_{14}$ | $F_{15}$ |

Change of coordinates

$F_0=7$

| | | | |
|---|---|---|---|
| 7 | 4 | 6 | 1 |
| 3 | 5 | 9 | 0 |
| 8 | 1 | 4 | 5 |
| 2 | 0 | 7 | 2 |

$$F = \begin{bmatrix} 7 \\ 4 \\ 6 \\ 1 \\ 3 \\ 5 \\ 9 \\ 0 \\ 8 \\ 1 \\ 4 \\ 5 \\ 2 \\ 0 \\ 7 \\ 2 \end{bmatrix}$$

Vector listing of pixel values

# Vector Representations of Kernels

This is HUGE ($N^2$)

- Can also linearize a kernel
- Linearization is unique for each pixel coordinate and for each image size.
  - For pixel coordinate (1,2) (i.e. pixel $F_9$) in our image:

| | | | |
|---|---|---|---|
| $F_0$ | $F_1$ | $F_2$ | $F_3$ |
| $F_4$ | $F_5$ | $F_6$ | $F_7$ |
| $F_8$ | $F_9$ | $F_{10}$ | $F_{11}$ |
| $F_{12}$ | $F_{13}$ | $F_{14}$ | $F_{15}$ |

$h=$

| | | |
|---|---|---|
| -3 | 1 | 2 |
| -5 | 4 | 6 |
| -7 | 9 | 8 |

$$H_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & -3 & 1 & 2 & 0 & -5 & 4 & 6 & 0 & -7 & 9 & 8 & 0 \end{bmatrix}^T$$

$$H_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -3 & 1 & 2 & 0 & -5 & 4 & 6 & 0 & -7 & 9 & 8 \end{bmatrix}^T$$

- Can combine the kernel vectors for each of the pixels into a single lexicographic kernel matrix ($H$)
- H is *circulant* (columns are rotations of one another). Why?

$$H = \begin{bmatrix} \cdots & -3 & \cdots & 0 & 0 & \cdots \\ & 1 & & 0 & 0 & \\ & 2 & & 0 & 0 & \\ & 0 & & 0 & 0 & \\ & -5 & & -3 & 0 & \\ & 4 & & 1 & -3 & \\ & 6 & & 2 & 1 & \\ & 0 & & 0 & 2 & \\ & -7 & & -5 & 0 & \\ & 9 & & 4 & -5 & \\ & 8 & & 6 & 4 & \\ & 0 & & 0 & 6 & \\ & 0 & & -7 & 0 & \\ & 0 & & 9 & -7 & \\ & 0 & & 8 & 9 & \\ & 0 & & 0 & 8 & \end{bmatrix}$$

$H_5$ $H_9$ $H_{10}$

# Convolution in Lexicographic Representations

- Convolution becomes matrix multiplication!
- Great conceptual tool for proving theorems
- $H$ is almost never computed or written out

# Basis Vectors for (Sub)Images

- Carefully choose a set of basis vectors (image patches) on which to project a sub-image (window) of size (x,y)
  - Is this lexicographic?
- The basis vectors with the largest coefficients are the most like this sub-image.
- If we choose meaningful basis vectors, this tells us something about the sub-image

Cartesian Basis Vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix}^T$$

$$\mathbf{u}_2 = \begin{bmatrix} 0\,1\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix}^T$$

$$\vdots$$

$$\mathbf{u}_9 = \begin{bmatrix} 0\,0\,0\,0\,0\,0\,0\,0\,1 \end{bmatrix}^T$$

Frei-Chen Basis Vectors

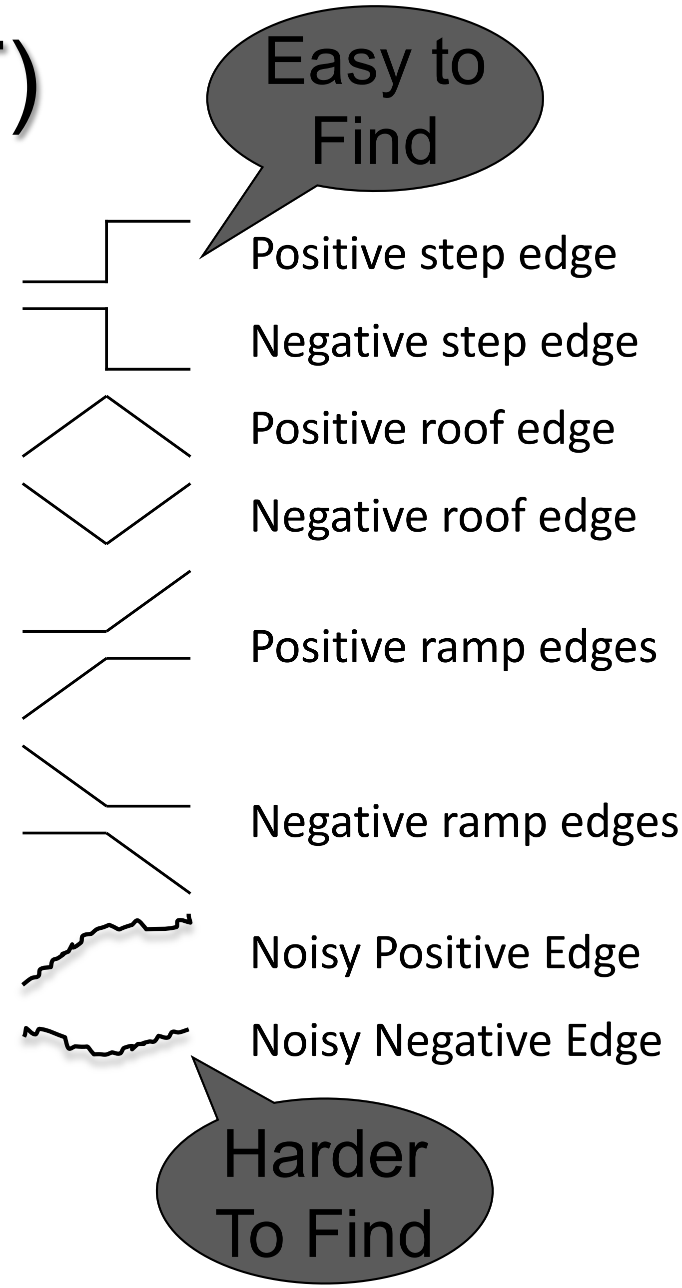
$$\mathbf{u}_1 = \begin{bmatrix} 1 & \sqrt{2} & 1 \\ 0 & 0 & 0 \\ -1 & -\sqrt{2} & -1 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 1 & 0 & -1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & 0 & -1 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0 & -1 & \sqrt{2} \\ 1 & 0 & -1 \\ -\sqrt{2} & 1 & 0 \end{bmatrix}$$

$$\mathbf{u}_4 = \begin{bmatrix} \sqrt{2} & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \quad \mathbf{u}_5 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{u}_6 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

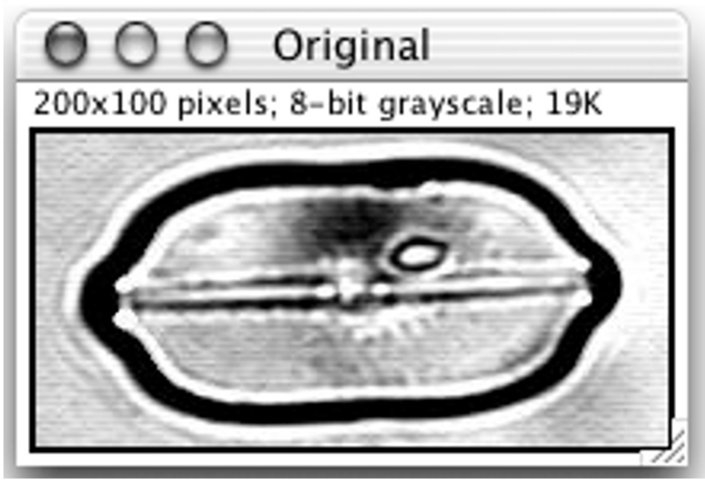
$$\mathbf{u}_7 = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \quad \mathbf{u}_8 = \begin{bmatrix} -2 & 1 & -2 \\ 1 & 4 & 1 \\ -2 & 1 & -2 \end{bmatrix} \quad \mathbf{u}_9 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

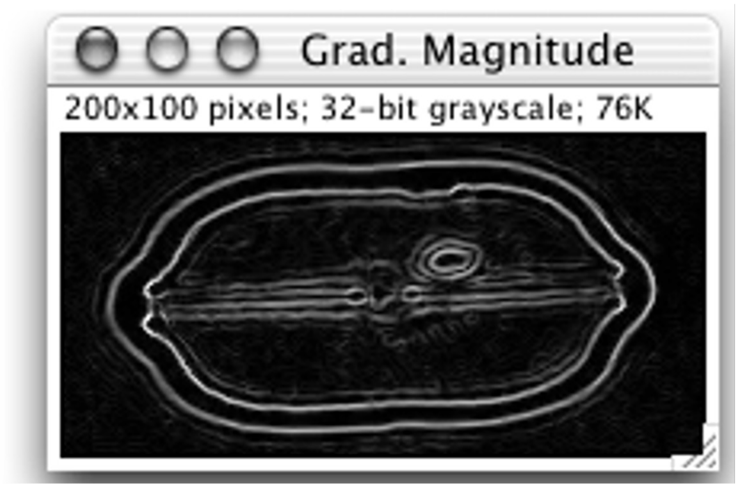
# Edge Detection (VERY IMPORTANT)

- Image areas where:
  - Brightness changes suddenly =
  - Some derivative has a large magnitude
- **<u>Often occur at object boundaries!</u>**
- Find by:
  - Estimating partial derivatives with kernels
  - Calculating magnitude and direction from partials



# Edge Detection




Diatom image (left) and its gradient magnitude (right).
(http://bigwww.epfl.ch/thevenaz/differentials/)

$$\nabla f = \left[ \frac{\partial f}{\partial x} \; \frac{\partial f}{\partial y} \right]^{\mathrm{T}} \equiv \left[ G_x \; G_y \right]^{\mathrm{T}}$$

$$\left|\nabla f\right| = \sqrt{G_x^2 + G_y^2} = \text{Edge Strength}$$

$$\angle \nabla f = \operatorname{atan}\left( \frac{G_x}{G_y} \right)$$

Then **threshold** the gradient magnitude image

<u>Detected edges are:</u>

- Too thick in places
- Missing in places
- Extraneous in places

# Convolving w/ Fourier

- Sometimes, the fastest way to convolve is to multiply in the frequency domain.
- Multiplication is fast. Fourier transforms are not.
- The Fast Fourier Transform (FFT) helps
- Pratt (Snyder ref. 5.33) figured out the details
  - Complex tradeoff depending on both the size of the kernel and the size of the image

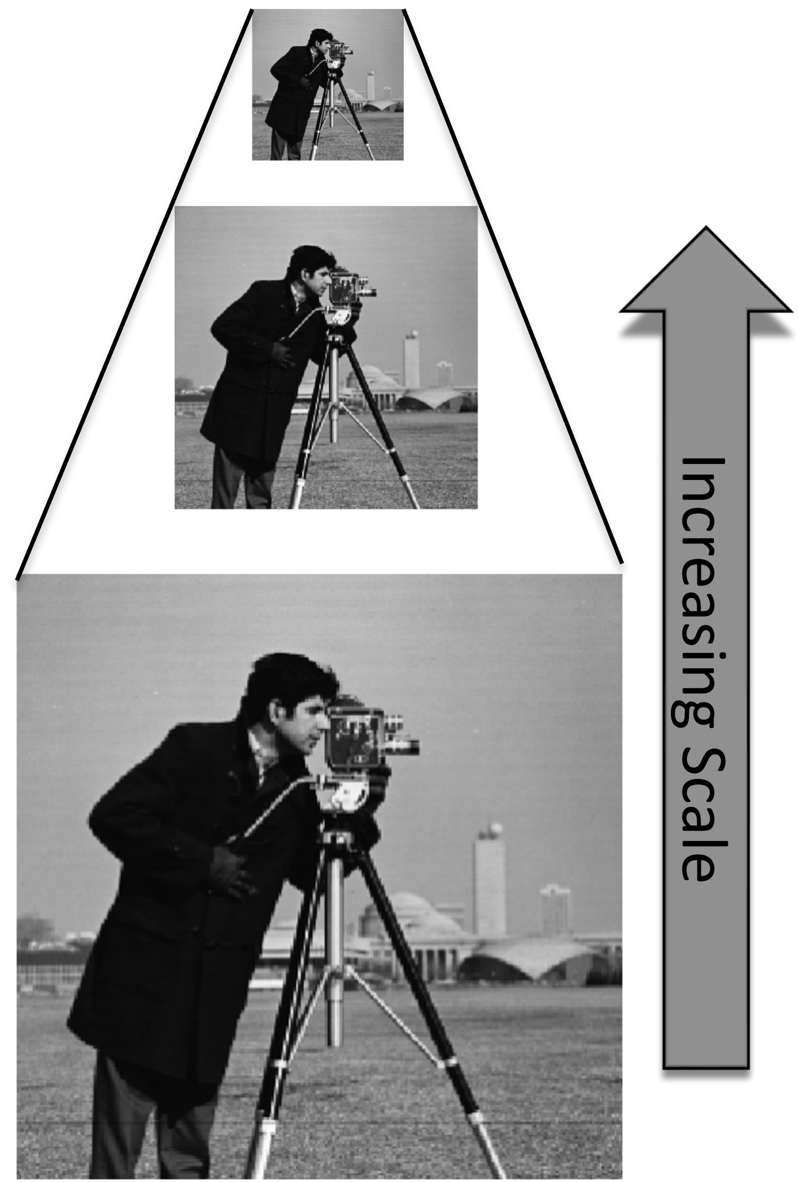
For kernels ≤ 7x7, normal (spatial domain) convolution is fastest*.

For kernels ≥ 13x13, the Fourier method is fastest*.

*For almost all image sizes

# Image Pyramids

- A series of representations of the same image
- Each is a 2:1 subsampling of the image at the next “lower level.
  - Subsampling = averaging = down sampling
  - The subsampling happens across all dimensions!
  - For a 2D image, 4 pixels in one layer correspond to 1 pixel in the next layer.
- To make a Gaussian pyramid:
  1. Blur with Gaussian
  2. Down sample by 2:1 in each dimension
  3. Go to step 1

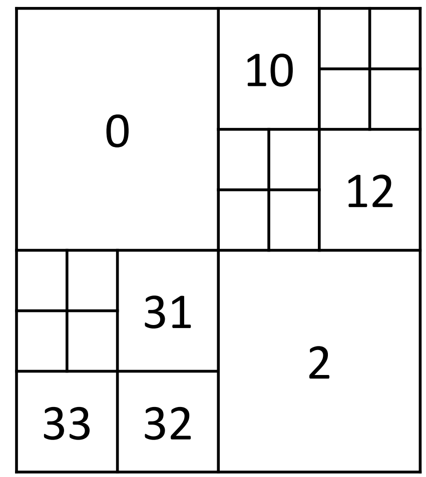


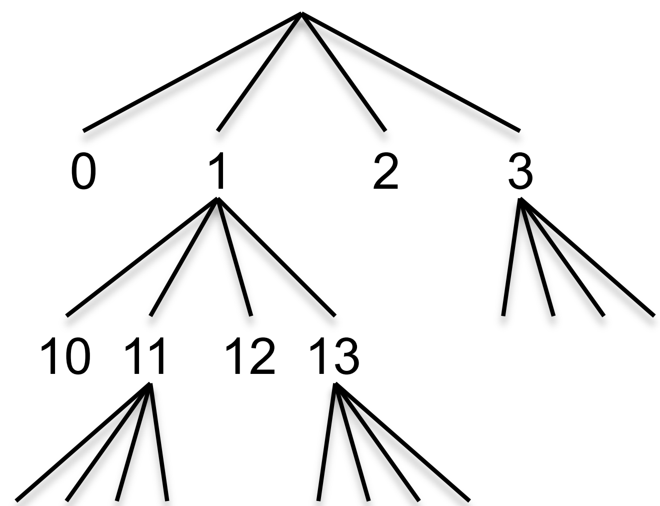
# Scale Space

- Multiple levels like a pyramid
- Blur like a pyramid
- <u>But don't subsample</u>
  - All layers have the same size
- Instead:
  - Convolve each layer with a Gaussian of variance $\sigma$.
  - $\sigma$ is the "scale parameter"
  - Only large features are visible at high scale (large $\sigma$).

# Quad/Oc Trees

- Represent an image
- Homogeneous blocks
- Inefficient for storage
  - Too much overhead
- Not stable across small changes
- But: Useful for representing scale space.



# Gaussian Scale Space

- Large scale = only large objects are visible
  - Increasing $\sigma \rightarrow$ coarser representations
- *Scale space causality*
  - Increasing $\sigma \rightarrow$ # extrema should not increase
  - Allows you to find “important” edges first at high scale.
- How features vary with scale tells us something about the image
- Non-integral steps in scale can be used
- Useful for representing:
  - Brightness
  - Texture
  - PDF (scale space implements clustering)

# How do People Do It?

- Receptive fields
- Representable by *Gabor functions*
  - 2D Gaussian +
  - A plane wave
- The plane wave tends to propagate along the short axis of the Gaussian
- But also representable by *Difference of offset Gaussians*
  - Only 3 extrema

# Canny Edge Detector

1. Use kernels to find at every point:
   - Gradient magnitude
   - Gradient direction
2. Perform Nonmaximum suppression (NMS) on the magnitude image
   - This thins edges that are too thick
   - Only preserve gradient magnitudes that are maximum compared to their 2 neighbors in the direction of the gradient

# Canny Edge Detector, contd.

- Edges are now properly located and 1 pixel wide
- But noise leads to false edges, and noise+blur lead to missing edges.
  - Help this with 2 thresholds
  - A high threshold does not get many false edges, and a low threshold does not miss many edges.
  - Do a “flood fill” on the low threshold result, seeded by the high-threshold result
    - Only flood fill along isophotes