15-259/559: Probability and Computing **Homework 1**
Due Friday 08/29/2025 at 12:50PM

This homework reviews the **prerequisites** for the class. Start by carefully reading **Chapter 1** in the PnC textbook. Everything in Chapter 1 will be used repeatedly throughout the course, so please ensure you are up to date with the prerequisites. Show all your work. Work through the problems carefully and **do not use online reference, mathematical software, or GenAI** as a shortcut for finding the solutions. You will regret it in the quizzes and examinations.

Part I

Exercises 1.2, 1.4, 1.5, 1.6, 1.7 in the textbook.

Part II

1.11 Limits and Derivatives

(a) Let a be a constant. Compute

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n. \tag{1}$$

Derive your answer using definition of e in Eq (1.8) of textbook.

(b) Let f be a continuous function. Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2+3} f(t)\mathrm{d}t. \tag{2}$$

1.12 Harmonic Number and Natural Logarithm

The nth harmonic number is denoted by H_n and defined as

$$H_n \coloneqq 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n} \qquad (n\geq 1). \tag{3}$$

Prove that

$$\ln(n+1) < H_n < 1+\ln(n). \tag{4}$$

1.13 Hilbert's Store

Hilbert's store has an *infinite* number of items. The original price of each item is \$1. However, every item gets a discount of 10% off of its current price each day. For example, the price of a 1-day old item is \$0.9, and the price of a 2-day old item is \$0.81, etc. The store has 1 item that is 1 day old, 2 items that are 2 days old, 3 items that are 3 days old, 4 items that are 4 days old, and so on. You come to the store and will buy everything in the store. How much should you pay in total?

1.14 The Fundamental Theorems of Calculus

For a given function $f : \mathbb{R} \to \mathbb{R}$, we say that another function $F : \mathbb{R} \to \mathbb{R}$ is an *anti-derivative* of f over an interval $I \subset \mathbb{R}$ if F is differentiable over I and $F'(x) = f(x)$ for all $x \in I$.

The First Fundamental Theorem of Calculus (FTC-I) states that if f is a continuous function on an interval I and $a \in I$, then the function

$$x \mapsto \int_a^x f(t)\mathrm{d}t \qquad (x \in I) \tag{5}$$

is an anti-derivative of f.

The Second Fundamental Theorem of Calculus (FTC-II) states that if f is an integrable function on an interval I and F is an anti-derivative of f, then

$$\int_a^b f(t)\mathrm{d}t = F(b) - F(a) \qquad (a, b \in I; a \leq b) \tag{6}$$

Prove FTC-II in the special case that f is continuous. Hint: Use FTC-I.