On a Logical Foundation for Explicit Substitutions

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Some joint work with Aleks Nanevski and Brigitte Pientka

Work in progress!

Apologia

- No specific references. See:
  - Aleksandar Nanevski, Frank Pfenning, and Brigitte Pientka. *Contextual Modal Type Theory*. ToCL 2007, to appear.
  - Delia Kesner. *The Theory of Calculi with Explicit Substitutions Revisited*. Technical Report, October 2006.
- No theorems yet in dependent case
  - Substitution and identity theorems only up to $k = 2$
  - Cover here only non-dependent (simply typed) case

Motivation

- Logical Frameworks: *explicit substitutions*
  - Explicit substitutions used internally
  - Understand their meaning, properties
  - Make available for specifications?
- Logical Frameworks: *meta-variables*
  - Meta-variables used internally, for search
  - Understand their meaning, properties
  - Make available for specifications?
- Are explicit substitutions purely operational?

Preview of Answers

- Substitutions are *judgmental*
- Explicit substitutions are *categorical*
- Reductions are *propositional*
- Meta-variables and explicit substitutions are tightly linked

Outline

- Hypothetical judgments and substitutions
- Meta-variables and simultaneous substitutions
- A multi-level system with stratified substitutions

Judgments and Propositions

- *Judgments* are objects of knowledge, subject to inference
- *Propositions* are subjects of truth (and related judgments)
- Example judgments:
  - $A\ true$
  - $A\ valid$ (modal logic — truth in all worlds)
  - $A\ true\ at\ time\ t$ (temporal logic)
  - $A\ false$ (classical logic)
  - $M : A$ (type theory)
- Example propositions: $A \wedge B$, $A \supset B$, $\exists x.\ A$, $\ldots$

Meaning Explanations

- Meaning of logical connectives is determined by their *verifications* (= canonical proofs)
- Defined by *introduction* and *elimination* rules for truth
  - Introduction: how to *verify* truth

$$\frac{A\ \mathit{true} \quad B\ \mathit{true}}{A \wedge B\ \mathit{true}} \wedge I$$

  - Elimination: how to *use* truth

$$\frac{A \wedge B\ \mathit{true}}{A\ \mathit{true}} \wedge E_1 \qquad \frac{A \wedge B\ \mathit{true}}{B\ \mathit{true}} \wedge E_2$$

Computation and Reduction

- *Computation* reduces an arbitrary proof to a verification
- *Reduction* step where introduction is followed by elimination

$$\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots \\ A\ \mathit{true} & B\ \mathit{true}\end{array}}{A \wedge B\ \mathit{true}}\wedge I}{A\ \mathit{true}}\wedge E_1 \quad \longrightarrow \quad \begin{array}{c}\vdots \\ A\ \mathit{true}\end{array}$$

- Reduces complexity of propositions in proof
- Verifications have subformula property
  - Necessary for well-founded meaning explanation

Proof Terms

- *Proof terms* M record evidence for truth
- Analytic judgment $M : A$ (M *is a proof of* A *true*)

$$\frac{M : A \quad N : B}{\langle M, N \rangle : A \wedge B} \wedge I$$

$$\frac{M : A \wedge B}{\pi_1\, M : A} \wedge E_1 \qquad \frac{M : A \wedge B}{\pi_2\, M : B} \wedge E_2$$

- Computation via reduction on proof terms

$$\pi_1 \langle M, N \rangle \longrightarrow M$$

$$\pi_2 \langle M, N \rangle \longrightarrow N$$

Incomplete Deductions

- Incomplete deductions map proofs of open leaves to proofs of conclusion

$$\dfrac{\dfrac{A \wedge (B \wedge C)\ \mathit{true}}{B \wedge C\ \mathit{true}} \wedge E_2}{B\ \mathit{true}} \wedge E_1$$

- Complete deductions by *substituting* proofs for open leaves
- Write as *hypothetical judgment*

$$A \wedge (B \wedge C)\ \mathit{true} \vdash B\ \mathit{true}$$

Variables

- Label hypotheses with proof term *variables*

$$\dfrac{\dfrac{x : A \wedge (B \wedge C)}{\pi_2\, x : B \wedge C} \wedge E_2}{\pi_1\, \pi_2\, x : B} \wedge E_1$$

- Proof terms as evidence for hypothetical judgments

$$x{:}A \wedge (B \wedge C) \vdash \pi_1\, \pi_2\, x : B$$

- Filling in a proof substitutes for a variable

Structural Principles

- First form of hypothetical judgment

$$\underbrace{x_1{:}A_1, \ldots, x_n{:}A_n}_{\Gamma} \vdash M : C$$

- All x_i distinct; subject to tacit renaming (including M)
- Hypothesis rule (judgmental, not propositional)

$$\frac{x{:}A \in \Gamma}{\Gamma \vdash x : A}\ \text{hyp}$$

- Weakening principle (leaving M unchanged)

  If $\Gamma \vdash M : A$ then $\Gamma, x{:}B \vdash M : A$

Substitution Principle

- *Substitution principle* (judgmental, not propositional)

  If $\Gamma \vdash M : A$

  and $\Gamma, x{:}A \vdash N : C$

  then $\Gamma \vdash [M/x]N : C$
- Substitution operation $[M/x]N$ is *compositional* on N
  - Returns substitution-free term N'
  - $[M/x]x = M$
  - Corresponds to supplying missing proof
- Principle is *open-ended*
- Slightly more general weakening and substitution elided

Compositionality

- Extend definition of substitution compositionality

$$
\begin{aligned}
[M/x]\langle N_1, N_2\rangle &= \langle [M/x]N_1, [M/x]N_2\rangle \\
[M/x]\pi_1\, N &= \pi_1\, [M/x]N \\
[M/x]\pi_2\, N &= \pi_2\, [M/x]N
\end{aligned}
$$

- Equations can be oriented as rewrite rules
- Equality (judgmental) vs reduction (propositional)

$$
\begin{aligned}
\pi_1\, \langle N_1, N_2\rangle &\longrightarrow N_1 \\
\pi_2\, \langle N_1, N_2\rangle &\longrightarrow N_2
\end{aligned}
$$

Propositional Implication

- Define *implication* $A \supset B$ from hypothetical judgment

$$\frac{\Gamma, A\ \mathit{true} \vdash B\ \mathit{true}}{\Gamma \vdash A \supset B\ \mathit{true}} \supset I \qquad \frac{\Gamma \vdash A \supset B\ \mathit{true} \quad \Gamma \vdash A\ \mathit{true}}{\Gamma \vdash B\ \mathit{true}} \supset E$$

- Reflect hypothetical reasoning in propositions
- Implications can be nested arbitrarily

$$((A \supset B) \supset A) \supset A$$

Computation and Substitution

- Proof term assignment

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x.\, M : A \supset B} \supset I \qquad \frac{\Gamma \vdash M : A \supset B \quad \Gamma \vdash N : A}{\Gamma \vdash M\, N : B} \supset E$$

- Computation via proof reduction

$$(\lambda x.\, N)\, M \quad \longrightarrow \quad [M/x]N$$

- Proof reduction via (auxiliary) substitution operation
- Substitution is capture-avoiding (via tacit α-conversion)

$$[M/x](\lambda y.\, N) \quad = \quad \lambda y.\, [M/x]N \quad \text{for } x \neq y \text{ and } y \notin \mathrm{FV}(M)$$

Summary

- Hypothetical judgments from incomplete proofs
- Substitution operation $[M/x]N$ for hypothesis labeled x
- Reflects substitution principle for hypothetical judgments
- Compositional and open-ended
- Substitution (judgmental) vs. reduction (propositional)
- Implication $A \supset B$ internalizes hypothetical judgment
- Reduction via substitution $(\lambda x.\, N)\, M \longrightarrow [M/x]N$

Incomplete Proofs, Revisited

- Leaves of incomplete proofs are *hypothetical judgments*

$$\dfrac{A \wedge B \vdash B \qquad \dfrac{A \wedge B, A \supset C \vdash C}{A \wedge B \vdash (A \supset C) \supset C}\supset I}{\dfrac{A \wedge B \vdash B \wedge ((A \supset C) \supset C)}{\bullet \vdash (A \wedge B) \supset B \wedge ((A \supset C) \supset C)}\supset I}\wedge I$$

- Variables $x{:}A$ are insufficient to represent such obligations

Meta-Variables

- Introduce *meta-variables* U with $\Gamma \vdash U : A$

$$\dfrac{x{:}A \wedge B \vdash U : B \quad \dfrac{x{:}A \wedge B, y{:}A \supset C \vdash V : C}{x{:}A \wedge B \vdash \lambda y.\, V : (A \supset C) \supset C} \supset I}{\dfrac{x{:}A \wedge B \vdash \langle U, \lambda y.\, V\rangle : B \wedge ((A \supset C) \supset C)}{\bullet \vdash \lambda x.\, \langle U, \lambda y.\, V\rangle : (A \wedge B) \supset B \wedge ((A \supset C) \supset C)} \supset I} \wedge I$$

- Write $U : A[\Gamma]$ for $\Gamma \vdash U : A$ in hypothetical judgment

$$U : B[x{:}A \wedge B],$$
$$V : C[x{:}A \wedge B, y{:}A \supset C]$$
$$\vdash \lambda x.\, \langle U, \lambda y.\, V\rangle : (A \wedge B) \supset B \wedge ((A \supset C) \supset C)$$

Some Problems

- Substitution for meta-variables would *capture* variables

$$
\begin{array}{l}
U : B[x{:}A \wedge B], \\
V : C[x{:}A \wedge B, y{:}A \supset C] \\
\;\vdash \lambda x.\, \langle U, \lambda y.\, V \rangle : (A \wedge B) \supset B \wedge ((A \supset C) \supset C)
\end{array}
$$

  - $[\pi_2\, x/U](\lambda x.\, \langle U, \lambda y.\, V \rangle) = \lambda x.\, \langle \pi_2\, x, \lambda y.\, V \rangle$?
- Lack of α-conversion(!)
- Poor interaction with ordinary substitution, β-reduction
- Closedness restriction
  - Substitution for $U : A[\Gamma]$ can *only* use variables in Γ
  - Can it use other *meta-variables*?

Hypothetical Judgments, Revisited

- Distinguish meta-variables and variables

$$\underbrace{U_1{:}B_1[\Psi_1], \ldots, U_m{:}B_m[\Psi_m]}_{\Delta}; \underbrace{x_1{:}A_1, \ldots, x_n{:}A_n}_{\Gamma} \vdash M : C$$

  - Contexts Γ, Ψ_j
  - Meta-context Δ

- Hypothesis rule (as before)

$$\frac{x{:}A \in \Gamma}{\Delta; \Gamma \vdash x : A}\ \text{hyp}$$

Meta-Hypothesis Rule

- How to use meta-variables?

$$\frac{U : A[\Psi] \in \Delta}{\Delta; \Gamma \vdash \texttt{?} : A}\ \text{mhyp}$$

  - Meta-variable U can only use variables in Ψ
  - Term “?” can only use variables in Γ

- *Solution*: supply simultaneous substitution σ for variables in Ψ, using variables in Γ and meta-variables in Δ

$$\frac{U : A[\Psi] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Psi}{\Delta; \Gamma \vdash U[\sigma] : A}\ \text{mhyp}$$

Suspensions

- Meta-variable $U : A[\Psi]$ may mention variables in Ψ
- $\sigma : \Psi$ substitutes terms for these variables
- Suspension $U[\sigma] : A$ cannot be eliminated until U is known

Simultaneous Substitutions

- Substitutions match context structurally

$$\frac{}{\Delta;\Gamma \vdash (\bullet) : (\bullet)} \qquad \frac{\Delta;\Gamma \vdash \sigma : \Psi \quad \Delta;\Gamma \vdash M : A}{\Delta;\Gamma \vdash (\sigma, M) : (\Psi, x{:}A)}$$

- Write $(M_1, \ldots, M_m)$ for $(M_1/x_1, \ldots, M_m/x_m)$ for brevity
- Example with identity substitutions and renamed variables

$$\begin{array}{l} U : B[u{:}A \wedge B], \\ V : C[v{:}A \wedge B, w{:}A \supset C] \\ \vdash \lambda x.\, \langle U[x], \lambda y.\, V[x, y] \rangle : (A \wedge B) \supset B \wedge ((A \supset C) \supset C) \end{array}$$

- Remaining proof obligation in type of U and V

Explicit Substitutions

- Substitutions σ are now *inevitably* part of terms
- Substitutions must be *explicit*
- When we substitute term M for meta-variable U in *suspension* $U[\sigma]$, need to compute $M[\sigma]$
- Some questions:
  - How do we define $M[\sigma]$?
  - How do we substitute for meta-variables U?
  - How do we relate $[M/x]$ and $[\sigma]$?
  - How do we understand the logical meaning?

Definition of Substitution

- Typing guide

$$\frac{\Delta; \Psi \vdash M : A \quad \Delta; \Gamma \vdash \sigma : \Psi}{\Delta; \Gamma \vdash M[\sigma] : A}$$

- Propagation of substitution

$$\begin{array}{lcl}
\langle M, N \rangle[\sigma] & = & \langle M[\sigma], N[\sigma] \rangle \\
(\pi_i M)[\sigma] & = & \pi_i M[\sigma] \\
(\lambda x.\, M)[\sigma] & = & \lambda x.\, M[\sigma, x/x] \\
(M\; N)[\sigma] & = & (M[\sigma])\,(N[\sigma]) \\
x[\sigma] & = & M \qquad \text{for } M/x \in \sigma \\
(U[\tau])[\sigma] & = & U[\tau[\sigma]]
\end{array}$$

Composition of Substitution

- Typing guide

$$\frac{\Delta;\Psi \vdash \tau : \Theta \quad \Delta;\Gamma \vdash \sigma : \Psi}{\Delta;\Gamma \vdash \tau[\sigma] : \Theta}$$

- Composition of substitutions

$$\begin{array}{lcl} (\bullet)[\sigma] & = & (\bullet) \\ (\tau, M)[\sigma] & = & (\tau[\sigma], M[\sigma]) \end{array}$$

Substitution for Meta-Variables

- Substitution principle

  If $\Delta; \Psi \vdash M : A$

  and $\Delta, U{:}A[\Psi]; \Gamma \vdash N : C$

  then $\Delta; \Gamma \vdash [(\Psi. M)/U]N : C$

- Close $(\Psi. M)$ for variable naming hygiene
- Compositional, with two remarks:
  - $[(\Psi. M)/U](U[\sigma]) = M[\sigma'/\Psi]$ where $\sigma' = [(\Psi. M)/U]\sigma$ and σ'/Ψ renames domain
  - $[(\Psi. M)/U](\lambda x. N) = \lambda x. [(\Psi. M)/U]N$ since no capture possible ($\Psi. M$ closed)

Example

- Recall example

$$U : B[u{:}A \wedge B],$$
$$V : C[v{:}A \wedge B, w{:}A \supset C]$$
$$\vdash \lambda x.\, \langle U[x], \lambda y.\, V[x, y]\rangle : (A \wedge B) \supset B \wedge ((A \supset C) \supset C)$$

- Apply $[(v, w.\, w\,(\pi_1 v))/V]$
- Crucial step:

$$\lambda x.\, \langle U[x], \lambda y.\, [(v, w.\, w\,(\pi_1 v))/V]V[x, y]\rangle$$
$$= \lambda x.\, \langle U[x], \lambda y.\, (w\,(\pi_1 v))[x/v, y/w]\rangle$$
$$= \lambda x.\, \langle U[x], \lambda y.\, y\,(\pi_1 x)\rangle$$

Single Substitution, Revisited

- For $\Gamma = (x_1{:}A_1, \ldots, x_n{:}A_n)$ define $\mathrm{id}_\Gamma = (x_1/x_1, \ldots, x_n/x_n)$
- For $\Gamma, x{:}A \vdash N : C$

$$(\lambda x.\, N)\, M \quad \longrightarrow \quad N[\mathrm{id}_\Gamma, M/x]$$

- Problems:
  - Γ is unknown at redex
  - Terms no longer invariant under weakening
- Can unify at lower level of abstraction
  - Use polymorphic identity substitution
  - Use de Bruijn indexes and shifts

Categorical Judgments

- Logically, $U{:}A[\Psi]$ reads "A *valid relative* Ψ"
- Without proof terms, write judgment $A\ \mathit{valid}[\Psi]$
  - $A\ \mathit{true}$ in every world where Ψ is true
  - Defined by single judgmental rule

$$\frac{\Delta; \Psi \vdash A\ \mathit{true}}{\Delta; \Gamma \vdash A\ \mathit{valid}[\Psi]}$$

- Validity is *categorical* with respect to truth
  - Γ may not be used to prove $A\ \mathit{true}$

Logical Meaning

- Internalize judgment $A\ valid[\Psi]$ as *proposition* $[\Psi]A$

$$\frac{\Delta;\Psi \vdash A\ true}{\Delta;\Gamma \vdash [\Psi]A\ true}\,[\,]I \qquad \frac{\Delta;\Gamma \vdash [\Psi]A\ true \qquad \Delta, A[\Psi];\Gamma \vdash C\ true}{\Delta;\Gamma \vdash C\ true}\,[\,]E$$

- Multiple-world interpretation
  - $[\Psi]A$ is true if A is true in every world where Ψ is true
  - Interpret Ψ conjunctively
  - $[\bullet]A$ means A is *necessarily true* (intuitionistic S4)
- Substitutions $\Gamma \vdash \sigma : \Psi$ are *witnesses to accessibility* from worlds where Γ is true to worlds where Ψ is true

Summary, Two-Level System

- Incomplete proofs of hypothetical judgments necessitate meta-variables
- Uses of meta-variables require explicit substitutions in terms
- Substitutions witness accessibility under multiple world semantics
- Two-level system
  - Ordinary variables
  - Meta-variables, under context of ordinary variables

Abstracting Meta-Variables

- Propositional reflection of meta-variables

$$\frac{\Delta, u{:}A[\Psi]; \Gamma \vdash M : B}{\Delta; \Gamma \vdash \lambda U.\, M : [\Psi]A \to B} \to I$$

$$\frac{\Delta; \Gamma \vdash M : [\Psi]A \to B \quad \Delta; \Psi \vdash N : A}{\Delta; \Gamma \vdash M\,(\Psi.\, N) : B} \to E$$

- New reduction

$$(\lambda U.\, M)\,(\Psi.\, N) \quad \longrightarrow \quad [(\Psi.\, N)/U]M$$

Incomplete Proofs, Rerevisited

- Now open leaves have form $\Delta; \Gamma \vdash$? $: A$
- Need meta^2-variables U^2
- New meta^2-hypothesis rule

$$\frac{U^2 : A[\Sigma; \Psi] \in \Delta^2 \quad \Delta^2; \Delta; \Gamma \vdash (\sigma^2; \sigma) : (\Sigma; \Psi)}{\Delta^2; \Delta; \Gamma \vdash U^2[\sigma^2; \sigma] : A} \text{ m}^2\text{hyp}$$

- Not practical
- Not expressively complete unless we close system under formation of meta-variables at any level

A Multi-Level System

- Unify in a *multi-level* system
- Models open derivations at any level
- Variables x^k at level $k \geq 0$
  - Ordinary variables x^0 for $k = 0$
  - Meta-variables x^1 for $k = 1$ (so far: U)
- Unified contexts

$$\Delta \quad ::= \quad \bullet \mid \Delta, x^k : A[\Psi^k]$$

- Ψ^k means $n < k$ for all declarations $x^n : A[\Gamma^n]$ in Ψ
- For declarations $x^0 : A[\Psi^0]$, $\Psi^0 = (\bullet)$ is forced!

Variables and Substitutions

- Unified hypothesis rule

$$\frac{x^k{:}A[\Psi^k] \in \Delta \quad \Delta \vdash \sigma : \Psi^k}{\Delta \vdash x[\sigma] : A}\text{hyp}$$

- Substitution typing

$$\frac{}{\Delta \vdash (\bullet) : (\bullet)} \qquad \frac{\Delta \vdash \sigma : \Psi^k \quad \Delta|_n, \Gamma^n \vdash M : A \quad (n < k)}{\Delta \vdash (\sigma, (\Gamma^n.\, M)) : (\Psi^k, x^n{:}A[\Gamma^n])}$$

- $\Delta|_n$ keeps only y^m for $m \geq n$.
  - Enforces categorical restriction

Abstraction and Application

- Typing rules

$$\frac{\Delta, x^k{:}A[\Psi^k] \vdash M : B}{\Delta \vdash \lambda x^k.\, M : [\Psi^k]A \to B} \to I$$

$$\frac{\Delta \vdash M : [\Psi^k]A \to B \quad \Delta|_k, \Psi^k \vdash N : A}{\Delta \vdash M\,(\Psi^k.\,N) : B} \to E$$

- $[(\bullet)^0]A \to B$ as $A \supset B$
- $[(\bullet)^1]A \to B$ as $\Box A \supset B$ in IS_4
- $[(\bullet)^2]A \to B$ as $\Box^2 A \supset B$ where $\Box^2 A\ \mathit{true}$ if $A\ \mathit{true}$ without using assumptions about truth or validity

Substitution Principle

- Write σ^k if $\Delta \vdash \sigma : \Psi^k$
- $M[\sigma^k]$ substitutes
  - for *all variables* in M of level $n < k$
  - for *no variables* in M of level $n \geq k$
- Typing guide

$$\frac{\Delta|_k, \Psi^k \vdash M : A \quad \Delta \vdash \sigma : \Psi^k}{\Delta \vdash M[\sigma^k] : A}$$

Substitution Definition

- Critical cases, extended compositionally

$$
\begin{array}{lcll}
(x^n[\tau^n])[\sigma^k] & = & M[\tau^n[\sigma^k]] & \text{for } n < k, \\
 & & & M/x^n \in \sigma \\
 & = & x^n[\tau^n[\sigma^k]] & \text{for } n \geq k \\
(\lambda x^n.\, M)[\sigma^k] & = & \lambda x^n.\, M[\sigma^k, x/x] & \text{for } n < k \\
 & = & \lambda x^n.\, M[\sigma^k] & \text{for } n \geq k \\
(M\ (\Gamma^n.\, N))[\sigma^k] & = & (M[\sigma^k])\ (\Gamma^n.\, N[\sigma|_n, \mathrm{id}^n_\Gamma]) & \text{for } n < k \\
 & = & (M[\sigma^k])\ (\Gamma^n.\, N) & \text{for } n \geq k
\end{array}
$$

Substitution Composition

- Typing guide

$$\frac{\Delta|_k, \Psi^k \vdash \tau : \Theta \quad \Delta \vdash \sigma : \Psi^k}{\Delta \vdash \tau[\sigma^k] : \Theta}$$

- Definition

$$\begin{aligned}(\tau, (\Gamma^n. M)/x^n)[\sigma^k] &= (\tau[\sigma], (\Gamma^n. M[\sigma|_n, \mathrm{id}_\Gamma^n])/x^n) && \text{for } n < k \\ &= (\tau[\sigma], (\Gamma^n. M)/x^n) && \text{for } n \geq k\end{aligned}$$

Single Substitutions, Rerevisited

- Typing guide

$$\frac{\Delta|_k, \Psi^k \vdash N : B \quad \Delta, x{:}B[\Psi^k] \vdash M : A}{\Delta \vdash [(\Psi^k. N)/x^k]M : A}$$

- Compositional, similar to simultaneous substitution
- Show only one case

$$[(\Psi^k. N)/x^k](x^k[\sigma^k]) = N[\sigma_1^k/\Psi^k]$$

for $\sigma_1^k = [(\Psi^k. N)/x^k](\sigma^k)$

Example, Modified and Revisited

- Omit suspension $[(\bullet)^0]$ and closure $(\bullet)^0$.

  $s^1 : B[u^0{:}A \wedge B]$,

  $t^1 : C[v^0{:}A, w^0{:}A \supset C]$

  $\vdash \lambda x^0.\, \langle s^1[x^0], \lambda y^0.\, t^1[\pi_1 x^0, y^0]\rangle : (A \wedge B) \supset B \wedge ((A \supset C) \supset C)$

- Simultaneous substitution at level 2

$$\sigma^2 = ((u^0.\, \pi_2 u^0)/s^1, (v^0, w^0.\, w^0\, v^0)/t^1)$$

- Crucial part

$$\begin{aligned}
&(t^1[\pi_1 x^0, y_0])[\sigma^2, x^0/x^0, y^0/y^0] \\
&= (w^0\, v^0)[\pi_1 x^0/v^0, y^0/w^0] \\
&= y^0\, (\pi_1 x^0)
\end{aligned}$$

Summary, Multi-Level System

- Uniform system of meta^k-variables x^k
  - Contextual type $x^k : A[\Psi^k]$
  - Closed with respect to variables y^n for $n < k$
  - Suspensions $x^k[\sigma^k]$ where $\sigma^k : \Psi^k$
- Level 0: ordinary variables
- Level 1: meta-variables
- Variables at all levels can be abstracted and applied
- Satisfies α-conversion, subject reduction

Ongoing Work, Theory

- Identity principle, subject expansion
- Extension to dependent types
  - In $\Delta, x^k{:}A[\Psi^k]$, A can depend on variables in $\Delta|_k$ and Ψ^k
  - If $\Delta\ ctx$ then $\Delta|_k\ ctx$
  - Conjecture substitution and identity properties
  - Checked for $k = 2$ (contextual modal type theory)
- Polymorphism? Substitution variables?
- Structural vs nominal contexts

Ongoing Work, Pragmatics

- Integrating single-variable and simultaneous substitution
- De Bruijn representation
  - Uniform numbering of all levels(?)
  - $\Delta|_k$ marks variables x^n for $n < k$ as invisible
- Level annotations and reconstruction

Summary

- A logical explanation of
  - Meta-variables
  - Explicit substitutions
- Methodology
  - Separating judgments from propositions
  - Categorical judgments
- Uniform presentation of meta^k-variables and substitutions
- Dependent version conjectured
- *Do not think of explicit substitutions as something purely operational!*