Week 1 Project Report - Green Group

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4 / 1 / 2004

1 An Inverse Method for BI

Following the general method for producing a contraction-free weakening-free forward calculus from a backward sequent calculus, we adapt the sequent calculus for **BI** (Pym 1999, Ch. 6) into an inverse method.

$$\frac{}{\varphi \vdash^{\mathtt{I}} \varphi}\ INIT^{\mathtt{I}}$$

$$\frac{\Gamma \vdash^{\mathtt{I}} \varphi \quad \Delta[\psi] \vdash^{\mathtt{I}} \chi}{\Delta[\Gamma; \varphi \supset \psi] \vdash^{\mathtt{I}} \chi} \supset L^{\mathtt{I}} \qquad \frac{\Gamma; \varphi \vdash^{\mathtt{I}} \psi}{\Gamma \vdash^{\mathtt{I}} \varphi \supset \psi} \supset R_1^{\mathtt{I}} \qquad \frac{\Gamma \vdash^{\mathtt{I}} \psi}{\Gamma \vdash^{\mathtt{I}} \varphi \supset \psi} \supset R_2^{\mathtt{I}}$$

$$\frac{\Gamma(\varphi) \vdash^{\mathtt{I}} \chi}{\Gamma(\varphi \wedge \psi) \vdash^{\mathtt{I}} \chi} \wedge L_1^{\mathtt{I}} \qquad \frac{\Gamma(\psi) \vdash^{\mathtt{I}} \chi}{\Gamma(\varphi \wedge \psi) \vdash^{\mathtt{I}} \chi} \wedge L_2^{\mathtt{I}} \qquad \frac{\Gamma \vdash^{\mathtt{I}} \varphi \quad \Delta \vdash^{\mathtt{I}} \psi}{\Gamma \cup \Delta \vdash^{\mathtt{I}} \varphi \wedge \psi} \wedge R^{\mathtt{I}}$$

$$\frac{\Gamma(\varphi) \vdash^{\mathtt{I}} \chi \quad \Delta(\psi) \vdash^{\mathtt{I}} \chi}{\Gamma(\varphi \vee \psi) \cup \Delta(\varphi \vee \psi) \vdash^{\mathtt{I}} \chi} \vee L^{\mathtt{I}} \qquad \frac{\Gamma \vdash^{\mathtt{I}} \varphi_i}{\Gamma \vdash^{\mathtt{I}} \varphi_1 \vee \varphi_2} \vee R_i^{\mathtt{I}} \ \text{(for } i = 1, 2)$$

$$\frac{\Gamma(\varphi, \psi) \vdash^{\mathtt{I}} \chi}{\Gamma(\varphi * \psi) \vdash^{\mathtt{I}} \chi} *L^{\mathtt{I}} \qquad \frac{\Gamma \vdash^{\mathtt{I}} \varphi \quad \Delta \vdash^{\mathtt{I}} \psi}{\Gamma, \Delta \vdash^{\mathtt{I}} \varphi * \psi} *R^{\mathtt{I}}$$

$$\frac{\Gamma \vdash^{\mathtt{I}} \varphi \quad \Delta\{\psi\} \vdash^{\mathtt{I}} \chi}{\Delta\{\Gamma, \varphi \mathrel{-\!\!*} \psi\} \vdash^{\mathtt{I}} \chi} \mathrel{-\!\!*} L^{\mathtt{I}} \qquad \frac{\Gamma, \varphi \vdash^{\mathtt{I}} \psi}{\Gamma \vdash^{\mathtt{I}} \varphi \mathrel{-\!\!*} \psi} \mathrel{-\!\!*} R^{\mathtt{I}}$$

These rules require a bit of explanation. The main difference, between these judgements and those given by Pym is that in this case contexts are of a special set/multiset representation of bunches that capture equivalence classes of fully contracted bunches. The contexts are of type AddCtx, defined as follows:

$$\begin{array}{rcl} \Gamma : \text{AddCtx} & ::= & \boldsymbol{;}(\Delta : \text{MultCtx set}) \\ \text{and } \Delta : \text{MultCtx} & ::= & \varphi \\ & | & \boldsymbol{,}(\Gamma : \text{AddCtx multiset}) \end{array}$$

Where the outtermost $\boldsymbol{;}(-)$ is left implicit and we use φ to mean $\{\varphi\}$ on the left side of $\vdash^{\mathrm{I}}$. Also, we introduce the notation $\Gamma[\Delta]$ to mean that Δ is plugged into a hole in an additive context of Γ and $\Gamma\{\Delta\}$ to mean that Δ is plugged into a hole in a multiplicative context of Γ.

Following Pym, we define contracting coherent equivalence.

Definition 1 (Contracting Coherent Equivalence)
$\equiv_A$ is the least equivalence relation on bunches satisfying:

1. *Commutative monoid equations for $\emptyset_a$ and ;*
2. *Commutative monoid equations for $\emptyset_m$ and ,*
3. *Additive Idempotence (contraction):* $\Gamma; \Gamma \equiv_A \Gamma$
4. *Congruence: if* $\Delta \equiv_A \Delta'$ *then* $\Gamma(\Delta) \equiv_A \Gamma(\Delta')$

Pym defines an equivalence, $\equiv$ which is the same as $\equiv_A$ without condition 3 (Additive Idempotence). Uses of $\equiv$ refer to Pym's equivalence.

Theorem 2 ($\equiv_A$ Goodness)
If $\Gamma \equiv_A \Gamma'$ *then* $\Gamma \vdash \varphi \iff \Gamma' \vdash \psi$.

Proof: *This is clear, as $\equiv_A$ identifies things that are the same modulo Pym's rules E and C (non-contracting equivalence and contraction) and since we have additive contraction and weakening* $\Delta(\Sigma) \vdash \varphi \iff \Delta(\Sigma; \Sigma) \vdash \varphi$. ■

Our representation indeed captures the equivalence classes over $\equiv_A$ (or equivalently, equivalence classes of fully contracted bunches over $\equiv$). This can be seen by the following psuedo-isomorphisms.

Let π: MultCtx $\to$ Bunches/$\equiv_A$ and θ: AddCtx $\to$ Bunches/$\equiv_A$ be defined mutually recursively as follows:

$$\begin{aligned}
\pi(\varphi) &= [\varphi]_{\equiv_A} = \{\varphi\} \\
\pi(\textbf{,}(B)) &= \text{ Let } \beta_i \in \theta(b_i) \text{ for some fixed enumeration}(b_i)_{i\in I} \text{ of B in} \\
&\quad\; [(((\beta_0, \beta_1), \beta_2), \ldots)]_{\equiv_A} \\
\theta(\textbf{;}(A)) &= \text{ Let } \alpha_i \in \pi(a_i) \text{ for some fixed enumeration}(a_i)_{i\in I} \text{ of A} \\
&\quad\; \text{in } [(((\alpha_0, \alpha_1), \alpha_2), \ldots)]_{\equiv_A}
\end{aligned}$$

We can use θ and θ^{-1} to go back and forth between our representation and equivalence classes of bunches (or, if we replace $\equiv_A$ with $\equiv$, with equivalence classes of fully contracted bunches) in such a way that $(\theta^{-1} \circ \theta)(x) = x$ and $(\theta \circ \theta^{-1})(x) \equiv_A x$ (or, $(\theta \circ \theta^{-1})(x) \equiv x$ for fully contracted x). So our representation is good in the sense that roundtripping in either direction produces equivalent contexts.

To ease notational burden we will use $\overline{\Gamma}$ to denote an arbitrary fully contracted member of $\theta(\Gamma)$ for Γ:AddCtx. It is often convenient to think of Γ and $\overline{\Gamma}$ as (different ways of specifying) the same object. In particular the rules given can be viewed as operating on contexts that consist of fully contracted bunches if we replace "$\cup$" with ";" in the rules and implicitly contract fully after each rule.

Lemma 3 (Union is Contraction)
For all $\Gamma, \Gamma' : AddCtx$ *we have* $\overline{\Gamma \cup \Gamma'} \equiv_A \overline{\Gamma}; \overline{\Gamma'}$

Proof: *It is enough to see that, in our representation, two bunches are only ever contracted if they appear in the same additive context.*

To elaborate: By the equivalence we showed between our representation and equivalence classes of fully contracted bunches we can check for $\equiv_A$*-equivalence by converting to our representation and checking for equality. We can break apart the process of turning a bunch into our set/multiset representation as follows. First we flatten sections of the binary tree that use the same connective to get an n-ary tree, this is justified by the associativity of , and ;. Next we forget about ordering by going to multisets, this is justified by the commutativity of , and ;. Lastly we forget about the number of instances in additive multisets, essentially contracting everything within each additive context.* ■

Now we have the machinery we need to prove the soundness and completeness of our system.

Theorem 4 (Soundness)
If $\Gamma \vdash^{\mathcal{I}} \varphi$ *then* $\overline{\Gamma} \vdash \varphi$*.*

Proof: *[By enumerating a few cases, structural induction]*

case : *Last rule is* $\dfrac{}{\varphi \vdash^{\mathcal{I}} \varphi}\ INIT^{\mathcal{I}}$

We immediately have $\varphi \vdash \varphi$ *by rule INIT.*

case : *Last rule is* $\dfrac{\Gamma \vdash^{\mathcal{I}} \varphi \quad \Delta[\psi] \vdash^{\mathcal{I}} \chi}{\Delta[\Gamma; \varphi \supset \psi] \vdash^{\mathcal{I}} \chi} \supset L^{\mathcal{I}}$

By IH, have $\overline{\Gamma} \vdash \varphi$ *and* $\overline{\Delta[\psi]} \vdash \chi$*.*

Since ψ *is in an additive context,* $\Delta[\psi]$ *is of the form* $\Delta'(\Delta''; \psi)$ *where* Δ'' *is the rest of the context* ψ *is in (possibly* $\emptyset_a$*). So we can use rule* $\supset L$ *to get* $\Delta'(\Delta''; \overline{\Gamma}; \varphi \supset \psi) \vdash \chi$ *which is of the right form (bar of Delta without psi and with Gamma bar and phi implies psi added in the same context psi was in).*

case : *Last rule is* $\dfrac{\Gamma; \varphi \vdash^{\mathcal{I}} \psi}{\Gamma \vdash^{\mathcal{I}} \varphi \supset \psi} \supset R_1^{\mathcal{I}}$

By IH, have $\overline{\Gamma; \varphi} \vdash \psi$*.*

By the previous lemma we have $\overline{\Gamma; \varphi} \equiv_A \overline{\Gamma}; \varphi$*, so by rule* $\supset R$*, we have* $\overline{\Gamma} \vdash \varphi \supset \psi$*.*

note : *The rest should all work out fairly easily, I've thought about each of the rules and worked out a few more than this, and am convinced of soundness.*

■

Theorem 5 (Completeness)
If $\Gamma(\Delta) \vdash \varphi$ then $\Gamma^\circ(\Delta^\circ) \vdash^{\mathcal{I}} \varphi$ such that there exists Γ^w, a weakening instance of $\overline{\Gamma^\circ}$, such that $\Gamma \equiv \Gamma^w$ and there exists Δ^w, a weakening instance of $\overline{\Delta^\circ}$, such that $\Delta \equiv \Delta^w$.

Intuitively we think of Γ° as a bunch in our representation that we can convert to a normal representation and weaken to get Γ (these are the types of sequents our inverse method will prove).

Proof: *[By enumerating a few cases, structural induction]*

case : *Derivation is* $\dfrac{}{\varphi \vdash \varphi}\ INIT$

We have immediately $\dfrac{}{\varphi \vdash^{\mathcal{I}} \varphi}\ INIT^{\mathcal{I}}$ *which clearly works.*

case : *Last rule is* $\dfrac{\Gamma(\Delta) \vdash \varphi}{\Gamma(\Delta; \Delta') \vdash \varphi}\ W$

By IH, have $\Gamma^\circ(\Delta^\circ) \vdash^{\mathcal{I}} \varphi$ with $\Gamma(\Delta)$ equivalent to a weakening of $\overline{\Gamma^\circ(\Delta^\circ)}$, which we can just weaken one step further with Δ' to get a bunch equivalent to $\Gamma(\Delta; \Delta')$ so the IH sequent works directly.

case : *Last rule is* $\dfrac{\Gamma(\Delta; \Delta) \vdash \varphi}{\Gamma(\Delta) \vdash \varphi}\ C$

This is immediate from the IH also, since $\overline{\Gamma^\circ([\Delta; \Delta]^\circ)} \equiv \overline{\Gamma^\circ(\Delta^\circ)}$ (because all bunches in our representation are fully contracted).

case : *Last rule is* $\dfrac{\Gamma \vdash \varphi}{\Delta \vdash \varphi}\ (\Delta \equiv \Gamma)E$

Again, this is immediate from IH because $\Delta \equiv \Gamma$.

case : *Last rule is* $\dfrac{\Gamma \vdash \varphi \quad \Delta(\Delta'; \psi) \vdash \chi}{\Delta(\Gamma; \Delta'; \varphi \supset \psi) \vdash \chi} \supset L$

By IH have $\Gamma^\circ \vdash^{\mathcal{I}} \varphi$ and Γ^w, a weakening of $\overline{\Gamma^\circ}$, s.t. $\Gamma^w \equiv \Gamma$, and
By IH have $\Delta^\circ([\Delta'; \psi]^\circ) \vdash^{\mathcal{I}} \chi$ and Δ^w a weakening of $\overline{\Delta^\circ}$ s.t. $\Delta^w \equiv \Delta$

and $(\Delta'; \psi)^w$*, a weakening of* $\overline{[\Delta'; \psi]^\circ}$*, s.t.* $(\Delta'; \psi)^w \equiv \Delta'; \psi$

If $[\Delta'; \psi]^\circ$ *contains* ψ *then it must be in an additive context since its parent node was a ;. Therefore* $\Delta^\circ([\Delta'; \psi]^\circ)$ *is of the form* $\Delta''^\circ[\psi]$*, so we can apply rule* $\supset L^{\mathcal{I}}$ *to get* $\Delta''^\circ[\Gamma; \varphi \supset \psi] \vdash^{\mathcal{I}} \chi$ *which is of the right form.*

If $[\Delta'; \psi]^\circ$ *does not contain* ψ *then by a lemma to follow there is* $[\Delta'; \psi]^{w'}$*, a weakening of* $\overline{[\Delta'; \psi]^\circ}$*, s.t.* $[\Delta'; \psi]^{w'} \equiv \Delta'$ *so we can weaken* $\Delta^\circ([\Delta'; \psi]^\circ)$ *with* $\varphi \supset \psi$ *and* Γ° *to get* $\Delta^\circ(\Gamma^\circ; \Delta'^\circ; \varphi \supset \psi) \vdash^{\mathcal{I}} \chi$ *which works.*

case : *Last rule is* $\dfrac{\Gamma; \varphi \vdash \psi}{\Gamma \vdash \varphi \supset \psi} \supset R$

By IH have $[\Gamma; \varphi]^\circ \vdash \psi$*. We just need to use the following lemma and case on whether* φ *is in* $[\Gamma; \varphi]^\circ$ *and use* $\supset R_1^{\mathcal{I}}$ *if it is, or* $\supset R_2^{\mathcal{I}}$ *if it is not.*

note : *The rest should all work out,* $-\!\!*$ *is almost parallel to* $\supset$*,* $\wedge$ *and* $*$ *are straightforward and* $\vee$ *is easy given the correspondence between union in our representation and normal contraction.*

$\blacksquare$

Several cases in the previous proof use the following obvious lemma.

Lemma 6
Let $[\Delta; \psi]^\circ$ *be as above then if* $[\Delta; \psi]^\circ$ *does not contain* ψ *then there exists* $[\Delta; \psi]^{w'}$*, a weakening instance of* $\overline{[\Delta; \psi]^\circ}$*, such that* $\Delta \equiv [\Delta; \psi]^{w'}$*.*