1.2 Classical Logic

The inference rules so far only model *intuitionistic logic*, and some classically true propositions such as $A \vee \neg A$ (for an arbitrary A) are not derivable, as we will see in Section **??**. There are three commonly used ways one can construct a system of *classical natural deduction* by adding one additional rule of inference. $\bot_C$ is called *Proof by Contradiction* or *Rule of Indirect Proof*, $\neg\neg_C$ is the *Double Negation Rule*, and XM is referred to as *Excluded Middle*.

$$\dfrac{\begin{array}{c}\overline{\neg A}^{\,u} \\ \vdots \\ \bot\end{array}}{A}\,\bot_C^u \qquad \dfrac{\neg\neg A}{A}\,\neg\neg_C \qquad \dfrac{}{A \vee \neg A}\,\text{XM}$$

The rule for classical logic (whichever one chooses to adopt) breaks the pattern of introduction and elimination rules. One can still formulate some reductions for classical inferences, but natural deduction is at heart an intuitionistic calculus. The symmetries of classical logic are much better exhibited in sequent formulations of the logic. In Exercise 1.3 we explore the three ways of extending the intuitionistic proof system and show that they are equivalent.

Another way to obtain a natural deduction system for classical logic is to allow multiple conclusions (see, for example, Parigot [Par92]).

1.3 Localizing Hypotheses

In the formulation of natural from Section 1.1 correct use of hypotheses and parameters is a global property of a derivation. We can localize it by annotating each judgment in a derivation by the available parameters and hypotheses. Since hypotheses and their restrictions are critical for linear logic, we give here a formulation of natural deduction for intuitionistic logic with localized hypotheses, but not parameters. For this we need a notation for hypotheses which we call a *context*.

$$\textit{Contexts} \quad \Gamma \quad ::= \quad \cdot \mid \Gamma, u{:}A$$

Here, "$\cdot$" represents the empty context, and $\Gamma, u{:}A$ adds hypothesis $\vdash A$ labelled u to Γ. We assume that each label u occurs at most once in a context in order to avoid ambiguities. The main judgment can then be written as $\Gamma \vdash A$, where

$$\cdot, u_1{:}A_1, \ldots, u_n{:}A_n \vdash A$$

stands for

$$\begin{array}{c}\overline{\vdash A_1}^{\,u_1} \quad \ldots \quad \overline{\vdash A_n}^{\,u_n} \\ \vdots \\ \vdash A\end{array}$$

in the notation of Section 1.1.

We use a few important abbreviations in order to make this notation less cumbersome. First of all, we may omit the leading "$\cdot$" and write, for example, $u_1{:}A_1, u_2{:}A_2$ instead of $\cdot, u_1{:}A_1, u_2{:}A_2$. Secondly, we denote concatenation of contexts by overloading the comma operator as follows.

$$\begin{array}{rcl} \Gamma, \cdot & = & \Gamma \\ \Gamma, (\Gamma', u{:}A) & = & (\Gamma, \Gamma'), u{:}A \end{array}$$

With these additional definitions, the localized version of our rules are as follows.

Introduction Rules

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge\mathrm{I}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee\mathrm{I_L} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee\mathrm{I_R}$$

$$\frac{\Gamma, u{:}A \vdash B}{\Gamma \vdash A \supset B} \supset\mathrm{I}^u$$

$$\frac{\Gamma, u{:}A \vdash p}{\Gamma \vdash \neg A} \neg\mathrm{I}^{p,u}$$

$$\frac{}{\Gamma \vdash \top} \top\mathrm{I}$$

no $\bot$ *introduction*

$$\frac{\Gamma \vdash [a/x]A}{\Gamma \vdash \forall x.\ A} \forall\mathrm{I}^a$$

$$\frac{\Gamma \vdash [t/x]A}{\Gamma \vdash \exists x.\ A} \exists\mathrm{I}$$

Elimination Rules

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\mathrm{E_L} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge\mathrm{E_R}$$

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, u_1{:}A \vdash C \qquad \Gamma, u_2{:}B \vdash C}{\Gamma \vdash C} \vee\mathrm{E}^{u_1,u_2}$$

$$\frac{\Gamma \vdash A \supset B \qquad \Gamma \vdash A}{\Gamma \vdash B} \supset\mathrm{E}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash \neg A}{\Gamma \vdash C} \neg\mathrm{E}$$

no $\top$ *elimination*

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash C} \bot\mathrm{E}$$

$$\frac{\Gamma \vdash \forall x.\ A}{\Gamma \vdash [t/x]A} \forall\mathrm{E}$$

$$\frac{\Gamma \vdash \exists x.\ A \qquad \Gamma, u{:}[a/x]A \vdash C}{\Gamma \vdash C} \exists\mathrm{E}^{a,u}$$

We also have a new rule for hypotheses which was an implicit property of the hypothetical judgments before.

$$\frac{}{\Gamma_1, u{:}A, \Gamma_2 \vdash A} u$$

Other general assumptions about hypotheses, namely that they may be used arbitrarily often in a derivation and that their order does not matter, are indirectly reflected in these rules. Note that if we erase the context Γ from the judgments throughout a derivation, we obtain a derivation in the original notation.

When we discussed local reductions in order to establish local soundness, we used the notation

$$\begin{array}{c} \dfrac{\mathcal{D}}{\vdash A}\, u \\ \mathcal{E} \\ \vdash C \end{array}$$

for the result of substituting the derivation $\mathcal{D}$ of $\vdash A$ for all uses of the hypothesis $\vdash A$ labelled u in $\mathcal{E}$. We would now like to reformulate the property with localized hypotheses. In order to prove that the (now explicit) hypotheses behave as expected, we use the principle of *structural induction* over derivations. Simply put, we prove a property for all derivations by showing that, whenever it holds for the premisses of an inference, it holds for the conclusion. Note that we have to show the property outright when the rule under consideration has no premisses, which amounts to the base cases forof the induction.

Theorem 1.1 (Structural Properties of Hypotheses) *The following properties hold for intuitionistic natural deduction.*

1. *(Exchange) If* $\Gamma_1, u_1{:}A, \Gamma_2, u_2{:}B, \Gamma_2 \vdash C$ *then* $\Gamma_1, u_2{:}B, \Gamma_2, u_1{:}A, \Gamma_2 \vdash C$.
2. *(Weakening) If* $\Gamma_1, \Gamma_2 \vdash C$ *then* $\Gamma_1, u{:}A, \Gamma_2 \vdash C$.
3. *(Contraction) If* $\Gamma_1, u_1{:}A, \Gamma_2, u_2{:}A, \Gamma_2 \vdash C$ *then* $\Gamma_1, u{:}A, \Gamma_2, \Gamma_3 \vdash C$.
4. *(Substitution) If* $\Gamma_1, u{:}A, \Gamma_2 \vdash C$ *and* $\Gamma_1 \vdash A$ *then* $\Gamma_1, \Gamma_2 \vdash C$.

Proof: The proof is in each case by straightforward induction over the structure of the first given derivation.

In the case of exchange, we appeal to the inductive assumption on the derivations of the premisses and construct a new derivation with the same inference rule. Algorithmically, this means that we exchange the hypotheses labelled u_1 and u_2 in every judgment in the derivation.

In the case of weakening and contraction, we proceed similarly, either adding the new hypothesis $u{:}A$ to every judgment in the derivation (for weakening), or replacing uses of u_1 and u_2 by u (for contraction).

For substitution, we apply the inductive assumption to the premisses of the given derivation $\mathcal{D}$ until we reach hypotheses. If the hypothesis is different from u we can simply erase $u{:}A$ (which is unused) to obtain the desired derivation. If the hypothesis is $u{:}A$ the derivation looks like

$$\mathcal{D} = \dfrac{}{\Gamma_1, u{:}A, \Gamma_2 \vdash A}\, u$$

so $C = A$ in this case. We are also given a derivation $\mathcal{E}$ of $\Gamma_1 \vdash A$ and have to construct a derivation $\mathcal{F}$ of $\Gamma_1, \Gamma_2 \vdash A$. But we can just repeatedly apply weakening to $\mathcal{E}$ to obtain $\mathcal{F}$. Algorithmically, this means that, as expected, we substitute the derivation $\mathcal{E}$ (possibly weakened) for uses of the hypotheses $u{:}A$ in $\mathcal{D}$. Note that in our original notation, this weakening has no impact, since unused hypotheses are not apparent in a derivation. □

It is also possible to localize the derivations themselves, using *proof terms*. As we will see in Chapter **??**, these proof terms form a λ-calculus closely related to functional programming. When parameters, hypotheses, and proof terms are all localized our main judgment becomes decidable. In the terminology of Martin-Löf [ML94], the main judgment is then *analytic* rather than *synthetic*. We no longer need to go outside the judgment itself in order to collect evidence for it: An analytic judgment encapsulates its own evidence.

1.4 Exercises

Exercise 1.1 Prove the following by natural deduction using only intuitionistic rules when possible. We use the convention that $\supset$, $\wedge$, and $\vee$ associate to the right, that is, $A{\supset}B{\supset}C$ stands for $A{\supset}(B{\supset}C)$. $A \equiv B$ is a syntactic abbreviation for $(A \supset B) \wedge (B \supset A)$. Also, we assume that $\wedge$ and $\vee$ bind more tightly than $\supset$, that is, $A \wedge B \supset C$ stands for $(A \wedge B) \supset C$. The scope of a quantifier extends as far to the right as consistent with the present parentheses. For example, $(\forall x.\, P(x) \supset C) \wedge \neg C$ would be disambiguated to $(\forall x.\, (P(x) \supset C)) \wedge (\neg C)$.

1. $\vdash A \supset B \supset A$.
2. $\vdash A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.
3. (Peirce's Law). $\vdash ((A \supset B) \supset A) \supset A$.
4. $\vdash A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$.
5. $\vdash A \supset (A \wedge B) \vee (A \wedge \neg B)$.
6. $\vdash (A \supset \exists x.\, P(x)) \equiv \exists x.\, (A \supset P(x))$.
7. $\vdash ((\forall x.\, P(x)) \supset C) \equiv \exists x.\, (P(x) \supset C)$.
8. $\vdash \exists x.\, \forall y.\, (P(x) \supset P(y))$.

Exercise 1.2 We write $A \vdash B$ if B follows from hypothesis A and $A \dashv\vdash B$ for $A \vdash B$ and $B \vdash A$. Which of the following eight parametric judgments are derivable intuitionistically?

1. $(\exists x.\, A) \supset B \dashv\vdash \forall x.\, (A \supset B)$
2. $A \supset (\exists x.\, B) \dashv\vdash \exists x.\, (A \supset B)$
3. $(\forall x.\, A) \supset B \dashv\vdash \exists x.\, (A \supset B)$
4. $A \supset (\forall x.\, B) \dashv\vdash \forall x.\, (A \supset B)$

Provide natural deductions for the valid judgments. You may assume that the bound variable x does not occur in B (items 1 and 3) or A (items 2 and 4).

Exercise 1.3 Show that the three ways of extending the intuitionistic proof system are equivalent, that is, the same formulas are deducible in all three systems.

Exercise 1.4 Assume we had omitted disjunction and existential quantification and their introduction and elimination rules from the list of logical primitives. In the classical system, give a definition of disjunction and existential quantification (in terms of other logical constants) and show that the introduction and elimination rules now become *admissible rules of inference*. A rule of inference is *admissible* if any deduction using the rule can be transformed into one without using the rule.

Exercise 1.5 Assume we would like to design a system of natural deduction for a simple temporal logic. The main judgment is now "*A is true at time t*" written as

$$\vdash^t A.$$

1. Explain how to modify the given rules for natural deduction to this more general judgment and show the rules for implication and universal quantification.
2. Write out introduction and elimination rules for the temporal operator $\bigcirc A$ which should be true if A is true at the next point in time. Denote the "next time after t" by $t + 1$.
3. Show the local reductions and expansions which show the local soundness and completness of your rules.
4. Write out introduction and elimination rules for the temporal operator $\Box A$ which should be true if A is true at all times.
5. Show the local reductions and expansions.

Exercise 1.6 Design introduction and elimination rules for the connectives

1. $A \equiv B$, usually defined as $(A \supset B) \wedge (B \supset A)$,
2. $A \mid B$ (exclusive or), usually defined as $(A \wedge \neg B) \vee (\neg A \wedge B)$,

without recourse to other logical constants or operators. Also show the corresponding local reductions and expansions.

Chapter 2

Intuitionistic Linear Logic

Linear logic, in its original formulation by Girard [Gir87] and many subsequent investigations was presented as a refinement of classical logic. This calculus of *classical linear logic* can be cleanly related to classical logic and exhibits many pleasant symmetries. On the other hand, a number of applications in logic and functional programming can be treated most directly using the intuitionistic version. In this chapter we present the basic system of natural deduction defining intuitionistic linear logic. Further surveys and introductions to linear logic can be found in [Lin92, Sce93, Tro92]. A historical introduction [Došš93] and context for linear and other substructural logics outside computer science can be found in [SHD93].

We introduce linear logic by enriching our judgment forms by a *linear hypothetical judgment*. A linear hypothetical judgment has the form "J_2 provided resource J_1". We consider this judgment evident if we are prepared to make judgment J_2 when provided with the resource J_1. A *resource* or *linear hypothesis* behaves like an ordinary hypothesis except that it must be used exactly once.

As an example, consider the basic judgment "*I own X*" for objects X. We would be prepared to make the linear hypothetical judgment

> "*I own book* b provided *I own \$5*"

if we know that book b costs five dollars and that it is available. If we ever actually had \$5, we could then achieve a situation in which we owned the book. Obviously, we would no longer own the five dollars, since they would have been consumed in the process of obtaining the book. It is clear that we would not be prepared to make the judgment above if the book costs ten dollars due to insufficient resources. But we would reject the judgment even if the book cost only one dollar, since we would not use all the given resources.

We can already see that evidence for a judgment of this form is a derivation in which we have to keep track of resources. Implicit here is a notion of state and change of state which is not present in traditional mathematical logic. For this reason, linear logic is often referred to as a "logic of state".

In the following section we develop the logical connectives of linear logic, based on the notion of linear hypothetical judgment.

2.1 Purely Linear Natural Deduction

The main judgment of *purely linear natural deduction* is "A is true assuming linear hypotheses $A_1, \ldots, A_n$". Later we also admit unrestricted hypotheses, but we postpone this complication. We refer to $A_1, \ldots A_n$ as *resources* and A as the *goal* to be achieved. As for hypotheses in Section 1.3 we localize resources into a context Δ and write

$$\Delta \vdash A.$$

Just as for ordinary hypotheses, we can substitute concrete evidence for a linear hypothesis for its use in a derivation. The corresponding *substitution principle* is the following:

If $\Delta_1, w{:}A, \Delta_2 \vdash C$ and $\Delta \vdash A$ then $\Delta_1, \Delta, \Delta_2 \vdash C$.

Intuitively, it states that if we have a derivation of $\vdash C$ from resources Δ_1, Δ_2 and the additional resource $\vdash A$ labelled w, and if we have a derivation of $\vdash A$ requiring resources Δ, then we can obtain a derivation of $\vdash C$ from resources Δ_1, Δ, and Δ_2.

Resources also satisfy the principle of exchange, since their order is irrelevant.

If $\Delta_1, w_1{:}A, \Delta_2, w_2{:}B, \Delta_3 \vdash C$ then $\Delta_1, w_2{:}B, \Delta_2, w_1{:}A, \Delta_3 \vdash C$.

Note that unlike unrestricted hypotheses, resources cannot be weakened or contracted since they must be used exactly once. Weakening would allow resources to remain unused, while contraction would allow resources to be used more than once.

Finally, the use of a resource is restricted (when compared to hypotheses in the intuitionistic case) so that there are no other resources remaining.

$$\frac{}{\cdot, w{:}A \vdash A}\, w$$

We now examine, connective by connective, the introduction and elimination rules of linear logic and check their local soundness and completeness. In the local reductions and expansion we will have to carefully check the preservation of resources.

Simultaneous Conjunction. Assume we have some resources Δ and we want to achieve goals A and B. We then need to split our resources Δ into Δ_1 and Δ_2 and show that with resources Δ_1 we can achieve A and with Δ_2 we can achieve B. The introduction rule for the corresponding connective of *simultaneous conjunction*, written $A \otimes B$ and read "*A tensor B*", then requires a notation for splitting resources (when viewed bottom-up) or merging resources (when viewed top-down). We merge $\Delta_1 \times \Delta_2$ according to the following rules.

$$\begin{array}{rcl} \cdot \times \cdot & = & \cdot \\ \Delta_1 \times (\Delta_2, u{:}A) & = & (\Delta_1 \times \Delta_2), u{:}A \\ (\Delta_1, u{:}A) \times \Delta_2 & = & (\Delta_1 \times \Delta_2), u{:}A \end{array}$$

Note that this is a non-deterministic operation, since the last two rules might both be applicable. We use the convention that any way to merge two contexts in an inference rules yields a valid inference. With this preparation we can now state the introduction rule for simultaneous conjunction.

$$\frac{\Delta_1 \vdash A \qquad \Delta_2 \vdash B}{\Delta_1 \times \Delta_2 \vdash A \otimes B}\otimes\mathrm{I}$$

The elimination rule should capture what we can achieve if we know that we can achieve both A and B simultaneously from some hypothetical resources Δ. We reason as follows: If with A, B, and additional resources Δ' we could achieve goal C, then we could achieve C from resources Δ and Δ'.

$$\frac{\Delta \vdash A \otimes B \qquad \Delta', w_1{:}A, w_2{:}B \vdash C}{\Delta' \times \Delta \vdash C}\otimes\mathrm{E}^{w_1,w_2}$$

The way we achieve C is to commit resources Δ to achieving A and B by the derivation of the left premiss and then using the remaining resources Δ' together with A and B to achieve C.

As before, we should check that the rules above are locally sound and complete. First, the local reduction

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta_1 \vdash A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta_2 \vdash B\end{array}}{\Delta_1 \times \Delta_2 \vdash A \otimes B}\otimes\mathrm{I} \qquad \begin{array}{c}\mathcal{E}\\ \Delta', w_1{:}A, w_2{:}B \vdash C\end{array}}{\Delta' \times \Delta_1 \times \Delta_2 \vdash C}\otimes\mathrm{E}^{w_1,w_2} \quad \Longrightarrow_R \quad \begin{array}{c}[\mathcal{D}_1/w_1][\mathcal{D}_2/w_2]\mathcal{E}\\ \Delta' \times \Delta_1 \times \Delta_2 \vdash C\end{array}$$

which requires two substitutions for linear hypotheses and the application of the substitution principle. We have also used exchange implicitly on the right hand side: if $\Delta', \Delta_1, \Delta_2 \vdash C$ then $\Delta' \times \Delta_1 \times \Delta_2 \vdash C$ due to the principle of exchange. We will use exchange tacitly from now on together with the substitution principle. The derivation on the right shows that the elimination rules are not too strong: we cannot obtain more judgments than we used to introduce the simultaneous conjunction.

For local completeness we have the following expansion.

$$\begin{array}{c}\mathcal{D}\\ \Delta \vdash A \otimes B\end{array} \quad \Longrightarrow_E \quad \frac{\begin{array}{c}\mathcal{D}\\ \Delta \vdash A \otimes B\end{array} \qquad \dfrac{\dfrac{}{\cdot, w_1{:}A \vdash A}w_1 \qquad \dfrac{}{\cdot, w_2{:}B \vdash B}w_2}{\cdot, w_1{:}A, w_2{:}B \vdash A \otimes B}\otimes\mathrm{I}}{\Delta \vdash A \otimes B}\otimes\mathrm{E}^{w_1,w_2}$$

The derivation on the right verifies that the elimination rules are strong enough so that the simultaneous conjunction can be reconstituted from the parts we obtain from the elimination rule.

Alternative Conjunction. Assume there are two books b_1 and b_2, each of which costs five dollars. If we had five dollars, we could by each one, but not both at the same time. It is our choice which one we buy and it therefore is a form of conjunction. We call it *alternative conjunction* $A \& B$ and pronounce it "A with B". It is sometimes also called *internal choice*. In its introduction rule, the resources are made available in both premisses, since we have to make a choice which among A and B we want to achieve.

$$\frac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \& B}\,\&\mathrm{I}$$

Consquently, if we have a resource $A \& B$, we can recover either A or B, but not both simultaneously. Therefore we have two elimination rules.

$$\frac{\Delta \vdash A \& B}{\Delta \vdash A}\,\&\mathrm{E_L} \qquad \frac{\Delta \vdash A \& B}{\Delta \vdash B}\,\&\mathrm{E_R}$$

The local reductions formalize the reasoning above.

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}_1\\ \Delta \vdash A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \Delta \vdash B\end{matrix}}{\Delta \vdash A \& B}\,\&\mathrm{I}}{\Delta \vdash A}\,\&\mathrm{E_L} \quad \Longrightarrow_R \quad \begin{matrix}\mathcal{D}_1\\ \Delta \vdash A\end{matrix}$$

$$\cfrac{\cfrac{\begin{matrix}\mathcal{D}_1\\ \Delta \vdash A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \Delta \vdash B\end{matrix}}{\Delta \vdash A \& B}\,\&\mathrm{I}}{\Delta \vdash B}\,\&\mathrm{E_R} \quad \Longrightarrow_R \quad \begin{matrix}\mathcal{D}_2\\ \Delta \vdash B\end{matrix}$$

We recognize these rules from intuitionistic natural deduction, where the context Γ is also made available in both premisses. The embedding of intuitionistic in linear logic will therefore map intuitionistic conjunction $A \wedge B$ to alternative conjunction $A \& B$. The expansion is also already familiar.

$$\begin{matrix}\mathcal{D}\\ \Delta \vdash A \& B\end{matrix} \quad \Longrightarrow_E \quad \cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ \Delta \vdash A \& B\end{matrix}}{\Delta \vdash A}\,\&\mathrm{E_L} \qquad \cfrac{\begin{matrix}\mathcal{D}\\ \Delta \vdash A \& B\end{matrix}}{\Delta \vdash B}\,\&\mathrm{E_R}}{\Delta \vdash A \& B}\,\&\mathrm{I}$$

Linear Implication. The *linear implication* or *resource implication* internalizes the linear hypothetical judgment at the level of propositions. We say $A \multimap B$ for the goal of achieving B with resource A.

$$\frac{\Delta, w{:}A \vdash B}{\Delta \vdash A \multimap B} \multimap\mathrm{I}^w$$

If we know $A \multimap B$ we can obtain B from a derivation of A.

$$\frac{\Delta \vdash A \multimap B \qquad \Delta' \vdash A}{\Delta \times \Delta' \vdash B} \multimap\mathrm{E}$$

As in the case for simultaneous conjunction, we have to split the resources, devoting Δ to achieving $A \multimap B$ and Δ' to achieving A.

The local reduction carries out the expected substitution for the linear hypothesis.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Delta, w{:}A \vdash B\end{matrix}}{\Delta \vdash A \multimap B} \multimap\mathrm{I}^w \qquad \begin{matrix}\mathcal{E}\\ \Delta' \vdash A\end{matrix}}{\Delta \times \Delta' \vdash B} \multimap\mathrm{E} \quad \Longrightarrow_R \quad \begin{matrix}[\mathcal{E}/w]\mathcal{D}\\ \Delta \times \Delta' \vdash B\end{matrix}$$

The rules are also locally complete, as witnessed by the local expansion.

$$\begin{matrix}\mathcal{D}\\ \Delta \vdash A \multimap B\end{matrix} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Delta \vdash A \multimap B\end{matrix} \qquad \dfrac{}{\cdot, w{:}A \vdash A} w}{\Delta, w{:}A \vdash B} \multimap\mathrm{E}}{\Delta \vdash A \multimap B} \multimap\mathrm{I}^w$$

Unit. The trivial goal which requires no resources is written as $\mathbf{1}$.

$$\frac{}{\cdot \vdash \mathbf{1}} \mathbf{1}\mathrm{I}$$

If we can achieve $\mathbf{1}$ from some resources Δ we know that we can consume all those resources.

$$\frac{\Delta \vdash \mathbf{1} \qquad \Delta' \vdash C}{\Delta' \times \Delta \vdash C} \mathbf{1}\mathrm{E}$$

The rules above and the local reduction and expansion can be seen as a case of 0-ary simultaneous conjunction. In particular, we will see that $\mathbf{1} \otimes A$ is equivalent to A.

$$\dfrac{\dfrac{}{\cdot \vdash \mathbf{1}} \mathbf{1}\mathrm{I} \qquad \begin{matrix}\mathcal{E}\\ \Delta' \vdash C\end{matrix}}{\Delta' \vdash C} \mathbf{1}\mathrm{E} \quad \Longrightarrow_R \quad \begin{matrix}\mathcal{E}\\ \Delta' \vdash C\end{matrix}$$

$$\begin{matrix}\mathcal{D}\\ \Delta \vdash \mathbf{1}\end{matrix} \quad \Longrightarrow_E \quad \dfrac{\begin{matrix}\mathcal{D}\\ \Delta \vdash \mathbf{1}\end{matrix} \qquad \dfrac{}{\cdot \vdash \mathbf{1}} \mathbf{1}\mathrm{I}}{\Delta \vdash \mathbf{1}} \mathbf{1}\mathrm{E}$$

Top. There is also a goal which consumes all resources. It is the unit of alternative conjunction and follows the laws of intuitionistic truth.

$$\frac{}{\Delta \vdash \top}\ \top\mathrm{I}$$

There is no elimination rule for $\top$ and consequently no local reduction (it is trivially locally sound). The local expansion replaces an arbitrary derivation by the rule above.

$$\begin{array}{c}\mathcal{D}\\ \Delta \vdash \top\end{array} \quad \Longrightarrow_E \quad \frac{}{\Delta \vdash \top}\ \top\mathrm{I}$$

Disjunction. The *disjunction* $A \oplus B$ (also called *external choice*) is characterized by two introduction rules.

$$\frac{\Delta \vdash A}{\Delta \vdash A \oplus B}\ \oplus\mathrm{I_L} \qquad \frac{\Delta \vdash B}{\Delta \vdash A \oplus B}\ \oplus\mathrm{I_R}$$

As in the case for intuitionistic disjunction, we therefore have to distinguish two cases when we know that we can achieve $A \oplus B$.

$$\frac{\Delta \vdash A \oplus B \qquad \Delta', w_1{:}A \vdash C \qquad \Delta', w_2{:}B \vdash C}{\Delta' \times \Delta \vdash C}\ \oplus\mathrm{E}^{w_1,w_2}$$

Note that resources Δ' appear in both branches, since only one of those two derivations will actually be used to achieve C, depending on the derivation of $A \oplus B$. This can be seen from the local reductions.

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \vdash A\end{array}}{\Delta \vdash A \oplus B}\ \oplus\mathrm{I_L} \qquad \begin{array}{c}\mathcal{E}_1\\ \Delta', w_1{:}A \vdash C\end{array} \qquad \begin{array}{c}\mathcal{E}_2\\ \Delta', w_2{:}B \vdash C\end{array}}{\Delta' \times \Delta \vdash C}\ \oplus\mathrm{E}^{w_1,w_2} \quad \Longrightarrow_R \quad \begin{array}{c}[\mathcal{D}/w_1]\mathcal{E}_1\\ \Delta' \times \Delta \vdash C\end{array}$$

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \vdash B\end{array}}{\Delta \vdash A \oplus B}\ \oplus\mathrm{I_L} \qquad \begin{array}{c}\mathcal{E}_1\\ \Delta', w_1{:}A \vdash C\end{array} \qquad \begin{array}{c}\mathcal{E}_2\\ \Delta', w_2{:}B \vdash C\end{array}}{\Delta' \times \Delta \vdash C}\ \oplus\mathrm{E}^{w_1,w_2} \quad \Longrightarrow_R \quad \begin{array}{c}[\mathcal{D}/w_2]\mathcal{E}_2\\ \Delta' \times \Delta \vdash C\end{array}$$

The local expansion is also familiar from intuitionistic disjunction.

$$\begin{array}{c}\mathcal{D}\\ \Delta \vdash A \oplus B\end{array} \quad \Longrightarrow_E \quad \frac{\begin{array}{c}\mathcal{D}\\ \Delta \vdash A \oplus B\end{array} \qquad \dfrac{\dfrac{}{\cdot, w_1{:}A \vdash A}\ w_1}{\cdot, w_1{:}A \vdash A \oplus B}\ \vee\mathrm{I_L} \qquad \dfrac{\dfrac{}{\cdot, w_2{:}B \vdash B}\ w_2}{\cdot, w_2{:}B \vdash A \oplus B}\ \vee\mathrm{I_R}}{\Delta \vdash A \oplus B}\ \vee\mathrm{E}^{w_1,w_2}$$

Impossibility. The *impossibility* $\mathbf{0}$ is the case of a disjunction between zero alternatives and the unit of $\oplus$. There is no introduction rule. In the elimination rule we have to consider no branches.

$$\frac{\Delta \vdash \mathbf{0}}{\Delta' \times \Delta \vdash C}\,\mathbf{0}\mathrm{E}$$

There is no local reduction, since there is no introduction rule. However, as in the case of falsehood in intuitionistic logic, we have a local expansion.

$$\begin{array}{c}\mathcal{D}\\ \Delta \vdash \mathbf{0}\end{array} \quad \Longrightarrow_E \quad \frac{\begin{array}{c}\mathcal{D}\\ \Delta \vdash \mathbf{0}\end{array}}{\Delta \vdash \mathbf{0}}\,\mathbf{0}\mathrm{E}$$

Universal Quantification. Quantifiers do not interact much with linearity, since we make no restrictions on occurrences of the parameter. They are included here for reference, but we omit the local reductions and expansion which are given in Section 1.1.

$$\frac{\Delta \vdash [a/x]A}{\Delta \vdash \forall x.\ A}\,\forall\mathrm{I}^a \qquad\qquad \frac{\Delta \vdash \forall x.\ A}{\Delta \vdash [t/x]A}\,\forall\mathrm{E}$$

Existential Quantification. The idea remains the same as in the intuitionistic case, except that we have to split resources among the premisses of the elimination rule.

$$\frac{\Delta \vdash [t/x]A}{\Delta \vdash \exists x.\ A}\,\exists\mathrm{I} \qquad\qquad \frac{\Delta \vdash \exists x.\ A \qquad \Delta', w{:}[a/x]A \vdash C}{\Delta' \times \Delta \vdash C}\,\exists\mathrm{E}^{a,w}$$

We omit the local reduction and expansion, which are trivial modification of the rules in Section 1.1.

This concludes the purely linear operators. Negation and another version of falsehood are postponed to Section **??**, since they may be formally definable, but their interpretation is somewhat questionable in the context we have established so far.

The connectives we have introduced may be classified as to whether the resources are split among the premisses or distributed to the premisses. Connectives of the former kind are called *multiplicative*, the latter *additive*. For example, we might refer to simultaneous conjunction also as *multiplicative conjunction* and to *alternative conjunction* as *additive conjunction*. When we line up the operators against each other, we notice some gaps. For example, there seems to be only a multiplicative implication, but no additive implication. Dually, there seems to be only an additive disjunction, but no multiplicative disjunction. This is not an accident and is pursued further in Exercise **??**.

2.2 Intuitionistic Hypotheses in Linear Logic

So far, the main judgment permits only linear hypotheses. This means that the logic is too weak to embed intuitionistic logic, and we have failed so far to design a true extension. We now generalize the main judgment to

$$\Gamma; \Delta \vdash A$$

which we read as "*under unrestricted hypotheses* Γ *with resources* Δ *we can achieve goal* A". The hypotheses Γ are intended to satisfy all the structural properties of Section 1.3, that is, exchange, weakening, contraction, and substitution. Substitution is not completely straightforward, since we have to consider the interaction with linear hypotheses. It now reads as follows:

If $(\Gamma_1, u{:}A, \Gamma_2); \Delta \vdash C$ and $\Gamma_1; \cdot \vdash A$ then $(\Gamma_1, \Gamma_2); \Delta \vdash C$.

It is critical to understand why the derivation of $\vdash A$ may not use any linear hypotheses. This is because in the construction of the resulting derivation of $\vdash C$ we substitute the derivation of $\vdash A$ for any use of the hypothesis $u{:}A$ in the given derivation of C. But this hypothesis $u{:}A$ is unrestricted and my be used many times. The substitution could therefore replicate any resources used in the derivation of $\vdash A$, violating the basic principle that resources are used exactly once.

For a similar reason, we can only use an unrestricted hypothesis if there are no resources (which otherwise would not be used as required).

$$\frac{}{(\Gamma_1, u{:}A, \Gamma_2); \cdot \vdash A}\;u$$

All the other rules we presented for pure linear logic are extended by adding the unrestricted context to premisses and conclusion (see the rule summary on page 30). We now reflect the unrestricted hypotheses in the language of proposition by reintroducing the corresponding operator of *intuitionistic implication*.

Intuitionistic Implication. The intuistionistic implication is the familiar one, where we have to be careful in the elimination rule to capture the restriction on the substitution property.

$$\frac{(\Gamma, u{:}A); \Delta \vdash B}{\Gamma; \Delta \vdash A \supset B}\;{\supset}\mathrm{I}^u \qquad \frac{\Gamma; \Delta \vdash A \supset B \qquad \Gamma; \cdot \vdash A}{\Gamma; \Delta \vdash B}\;{\supset}\mathrm{E}$$

The local reduction uses the substitution principle for unrestricted hypotheses.

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}\\(\Gamma, u{:}A); \Delta \vdash B\end{array}}{\vdash A \supset B}\;{\supset}\mathrm{I}^u \qquad \begin{array}{c}\mathcal{E}\\ \Gamma; \cdot \vdash A\end{array}}{\vdash B}\;{\supset}\mathrm{E} \quad \Longrightarrow_R \quad \begin{array}{c}[\mathcal{D}/u]\mathcal{E}\\ \Gamma; \Delta \vdash B\end{array}$$

In Exercise **??** you are asked to show that the rules would be locally unsound (that is, local reduction is not possible), if the second premiss in the elimination rule would be allowed to depend on linear hypotheses. The local expansion is simpler.

$$\begin{array}{c}\mathcal{D}\\ \Gamma;\Delta\vdash A\supset B\end{array} \quad\Longrightarrow_E\quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\(\Gamma,u{:}A);\Delta\vdash A\supset B\end{array} \qquad \dfrac{}{(\Gamma,u{:}A);\cdot\vdash A}\,u}{(\Gamma,u{:}A);\Delta\vdash B}\,{\supset}\mathrm{E}}{\Gamma;\Delta\vdash A\supset B}\,{\supset}\mathrm{I}^u$$

Notice that we weaken $\mathcal{D}$ with the added (and unused) hypothesis u, which does not affect the structure of the derivation. This is not visible in the formulation of the local reduction in Section 1.1, since we did not make the hypotheses explicit.

"Of Course" Modality. Girard [Gir87] observed that there is an even simpler way to connect intuitionistic and linear logic by internalizing the notion of *intuitionistic truth* via a modal operator $!A$ he called "*of course* A" (often pronounced "*bang* A)". A formula is intuitionistically true if it can be derived without the use of any restricted resources.

$$\dfrac{\Gamma;\cdot\vdash A}{\Gamma;\cdot\vdash\, !A}\,!\mathrm{I}$$

The elimination rule states that if we can derive $\vdash\, !A$ than we are allowed to use A as an unrestricted hypothesis.

$$\dfrac{\Gamma;\Delta\vdash\, !A \qquad (\Gamma,u{:}A);\Delta'\vdash C}{\Gamma;(\Delta'\times\Delta)\vdash C}\,!\mathrm{E}^u$$

This pair of rules is locally sound and complete.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma;\cdot\vdash A\end{array}}{\Gamma;\cdot\vdash\, !A}\,!\mathrm{I} \qquad \begin{array}{c}\mathcal{E}\\(\Gamma,u{:}A);\Delta'\vdash C\end{array}}{\Gamma;\Delta'\vdash C}\,!\mathrm{E}^u \quad\Longrightarrow_R\quad \begin{array}{c}[\mathcal{D}/u]\mathcal{E}\\ \Gamma;\Delta'\vdash C\end{array}$$

$$\begin{array}{c}\mathcal{D}\\ \Gamma;\Delta\vdash\, !A\end{array} \quad\Longrightarrow_E\quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma;\Delta\vdash\, !A\end{array} \qquad \dfrac{\dfrac{}{(\Gamma,u{:}A);\cdot\vdash A}\,u}{(\Gamma,u{:}A);\cdot\vdash\, !A}\,!\mathrm{I}}{\Gamma;\Delta\vdash\, !A}\,!\mathrm{E}^u$$

Using the *of course* modality, one can define the intuitionstic implication $A\supset B$ as $(!A)\multimap B$. It was this observation which gave rise to Girard's development of linear logic. Under this interpretation, the introduction and elimination rules for intuitionistic implication are *derived rules of inference* (see Exercise **??**).

We now summarize the rules of intuitionistic linear logic. A very similar calculus was developed and analyzed in the categorical context by Barber [Bar96]. It differs from more traditional treatments by Abramsky [Abr93], Troelstra [Tro93], Bierman [Bie94] and Albrecht et al. [ABCJ94] in that structural rules remain completely implicit. The logic we consider here comprises the following logical operators.

Formulas	A	$::=$	P	Atoms
			$\mid A_1 \multimap A_2 \mid A_1 \otimes A_2 \mid \mathbf{1}$	Multiplicatives
			$\mid A_1 \& A_2 \mid \top \mid A_1 \oplus A_2 \mid \mathbf{0}$	Additives
			$\mid \forall x.\ A \mid \exists x.\ A$	Quantifiers
			$\mid A \supset B \mid\ !A$	Exponentials

It is instructive to compare the rules below with those of intuitionistic natural deduction on page 13, keeping in mind that hypotheses were left implicit in that formulation.

Hypotheses.

$$\frac{}{\Gamma; (\cdot, w{:}A) \vdash A}\, w \qquad \frac{}{(\Gamma_1, u{:}A, \Gamma_2); \cdot \vdash A}\, u$$

Multiplicative Connectives.

$$\frac{\Gamma; \Delta_1 \vdash A \qquad \Gamma; \Delta_2 \vdash B}{\Gamma; (\Delta_1 \times \Delta_2) \vdash A \otimes B}\, \otimes\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash A \otimes B \qquad \Gamma; (\Delta', w_1{:}A, w_2{:}B) \vdash C}{\Gamma; (\Delta' \times \Delta) \vdash C}\, \otimes\mathrm{E}^{w_1,w_2}$$

$$\frac{\Gamma; (\Delta, w{:}A) \vdash B}{\Gamma; \Delta \vdash A \multimap B}\, \multimap\mathrm{I}^w \qquad \frac{\Gamma; \Delta \vdash A \multimap B \qquad \Gamma; \Delta' \vdash A}{\Gamma; (\Delta \times \Delta') \vdash B}\, \multimap\mathrm{E}$$

$$\frac{}{\Gamma; \cdot \vdash \mathbf{1}}\, \mathbf{1}\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash \mathbf{1} \qquad \Gamma; \Delta' \vdash C}{\Gamma; (\Delta' \times \Delta) \vdash C}\, \mathbf{1}\mathrm{E}$$

Additive Connectives.

$$\frac{\Gamma; \Delta \vdash A \qquad \Gamma; \Delta \vdash B}{\Gamma; \Delta \vdash A \& B}\, \&\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash A \& B}{\Gamma; \Delta \vdash A}\, \&\mathrm{E_L} \qquad \frac{\Gamma; \Delta \vdash A \& B}{\Gamma; \Delta \vdash B}\, \&\mathrm{E_R}$$

$$\frac{}{\Gamma; \Delta \vdash \top}\, \top\mathrm{I} \qquad \textit{no } \top \textit{ elimination}$$

$$\frac{\Gamma; \Delta \vdash A}{\Gamma; \Delta \vdash A \oplus B} \oplus\mathrm{I_L} \qquad \frac{\Gamma; \Delta \vdash A \oplus B \quad \Gamma; (\Delta', w_1{:}A) \vdash C \quad \Gamma; (\Delta', w_2{:}B) \vdash C}{\Gamma; (\Delta' \times \Delta) \vdash C} \oplus\mathrm{E}^{w_1,w_2}$$

$$\frac{\Gamma; \Delta \vdash B}{\Gamma; \Delta \vdash A \oplus B} \oplus\mathrm{I_R}$$

$$\textit{no } \mathbf{0} \textit{ introduction} \qquad \frac{\Gamma; \Delta \vdash 0}{\Gamma; (\Delta' \times \Delta) \vdash C} \mathbf{0}\mathrm{E}$$

Quantifiers.

$$\frac{\Gamma; \Delta \vdash [a/x]A}{\Gamma; \Delta \vdash \forall x.\ A} \forall\mathrm{I}^a \qquad \frac{\Gamma; \Delta \vdash \forall x.\ A}{\Gamma; \Delta \vdash [t/x]A} \forall\mathrm{E}$$

$$\frac{\Gamma; \Delta \vdash [t/x]A}{\Gamma; \Delta \vdash \exists x.\ A} \exists\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash \exists x.\ A \qquad \Gamma; (\Delta', w{:}[a/x]A) \vdash C}{\Gamma; (\Delta' \times \Delta) \vdash C} \exists\mathrm{E}^{a,w}$$

Exponentials.

$$\frac{(\Gamma, u{:}A); \Delta \vdash B}{\Gamma; \Delta \vdash A \supset B} \supset\mathrm{I}^u \qquad \frac{\Gamma; \Delta \vdash A \supset B \qquad \Gamma; \cdot \vdash A}{\Gamma; \Delta \vdash B} \supset\mathrm{E}$$

$$\frac{\Gamma; \cdot \vdash A}{\Gamma; \cdot \vdash\ !A} !\mathrm{I} \qquad \frac{\Gamma; \Delta \vdash\ !A \qquad (\Gamma, u{:}A); \Delta' \vdash C}{\Gamma; (\Delta' \times \Delta) \vdash C} !\mathrm{E}^u$$

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