3.6 Weakly Uniform Derivations

In the preceding sections we have dealt with universal, existential, and resource non-determinism. The inversion principles in Sections 3.1 and 3.4 reduced *disjunctive non-determinism*, since invertible rules can always be applied eagerly in the bottom-up search for a derivation without losing completeness. In this section we extend this analysis in order to reduce disjunctive non-determinism even when rules are not necessarily invertible, resulting in a notion of *weakly uniform derivations.*

> [***Warning:*** *The material in this section is highly speculative. While the soundness of the procedure is clear, its completeness has not yet been proven and quite possibly fails.*]

As in the section for unification, we describe the search procedure as a deductive system. We do not explicitly deal with universal, existential, or resource non-determinism, which can easily be incorporated in the system following the ideas in the previous two sections. Unlike unification, however, there remains some residual disjunctive non-determinism which reflects the remaining difficult choices. This can be treated, for example, by backtracking in an iterative deepening implementation.

The basic structure of sequents in the procedure is

$$\Gamma; \Delta \longrightarrow A$$

where Γ contains only propositions whose corresponding unrestricted left rule is not invertible and Δ contains only propositions whose corresponding left rules are not invertible. If a right rule is invertible in this situation, we apply it, until we are in a situation where neither the left rules for the principal connectives on the hypotheses, nor the right rule for the conclusion is invertible. Then we have to make a choice and pick one of the hypotheses or the conclusion.

The surprising[4] observation is that we can now *focus* on this hypothesis or conclusion and break it down, applying a sequence of left or right rules as long as the corresponding rules for the principal connective are *not* invertible. We call the structure of the resulting derivation *weakly focussed.* In the strongly focussed case which is relevant to logic programming, we can always reduce a focussed hypothesis or conclusion to an atomic proposition; here we may have to stop when we encounter other propositions.

We now go through various classes of inference rules. We omit the cases for intuitionistic implication (which is left to Exercise **??**), since it is easily defined by $A_1 \supset A_2 = (!A_1) \multimap A_2$.

Invertible Right Rules. Whenever the right rule for the principal connective of the succedent is applicable, we must apply it.

$$\frac{\Gamma; \Delta \longrightarrow A_1 \qquad \Gamma; \Delta \longrightarrow A_2}{\Gamma; \Delta \longrightarrow A_1 \& A_2} \&R \qquad \frac{}{\Gamma; \Delta \longrightarrow \top} \top R$$

[4] [*and perhaps false*]

The right rule for linear implication is invertible, but it requires a new auxiliary judgment. When applying the right rule to $A_1 \multimap A_2$ we cannot add A_1 directly to Δ, since its principal connective may have an invertible left rule, violating our invariant. Instead, we employ an auxiliary judgment $\Gamma; (\Delta \mid \Delta') \longrightarrow A$. Interpreted algorithmically, it will apply all the invertible left rules to the propositions in Δ' and merge the remaining ones into Δ

$$\frac{\Gamma; (\Delta \mid A_1) \longrightarrow A_2}{\Gamma; \Delta \Longrightarrow A_1 \multimap A_2} \multimap\text{R} \qquad \frac{\Gamma; \Delta \longrightarrow [a/x]A}{\Gamma; \Delta \longrightarrow \forall x.\ A} \forall\text{R}^a$$

Invertible Left Rules. The invertible left rules require yet another additional judgment to merge unrestricted hypotheses into Γ in a way that eliminates all unrestricted invertible rules, written as $(\Gamma \mid \Gamma'); (\Delta \mid \Delta') \longrightarrow B$.

$$\frac{\Gamma; (\Delta \mid A_1, A_2, \Delta') \longrightarrow B}{\Gamma; (\Delta \mid A_1 \otimes A_2, \Delta') \longrightarrow B} \otimes\text{L} \qquad \frac{\Gamma; (\Delta \mid \Delta') \longrightarrow B}{\Gamma; (\Delta \mid \mathbf{1}, \Delta') \longrightarrow B} \mathbf{1}\text{L}$$

$$\frac{\Gamma; (\Delta \mid A_1, \Delta') \longrightarrow B \qquad \Gamma; (\Delta \mid A_2, \Delta') \longrightarrow B}{\Gamma; (\Delta \mid A_1 \oplus A_2, \Delta') \longrightarrow B} \oplus\text{L} \qquad \frac{}{\Gamma; (\Delta \mid \mathbf{0}, \Delta') \longrightarrow B} \mathbf{0}\text{L}$$

$$\frac{(\Gamma \mid A); (\Delta \mid \Delta') \longrightarrow B}{\Gamma; (\Delta \mid\, !A, \Delta') \longrightarrow B} !\text{L} \qquad \frac{\Gamma; (\Delta \mid [a/x]A, \Delta') \longrightarrow B}{\Gamma; (\Delta \mid \exists x.\ A, \Delta')} \exists\text{L}^a$$

There is also one rule which allows us to merge remaining propositions into Δ, and one rule which allows us to go back to work on the succedent when all hypotheses have been decomposed and merged into Δ.

$$\frac{\Gamma; (\Delta, D \mid \Delta') \longrightarrow B}{\Gamma; (\Delta \mid D, \Delta') \longrightarrow B} \text{md} \qquad \frac{\Gamma; \Delta \longrightarrow B}{\Gamma; (\Delta \mid \cdot) \longrightarrow B} \text{me}$$

In the md rule, we use D to stand for a proposition whose left rule is non-invertible. We we call them *left-critical propositions*.

Left-Critical Propositions $\quad D \quad ::= \quad P \mid A_1 \& A_2 \mid \top \mid A_1 \multimap A_2 \mid \forall x.\ A$

Invertible Unrestricted Left Rules. Next we show the rules which eagerly apply invertible left rules to unrestricted hypotheses.

$$\frac{(\Gamma \mid A_1, A_2, \Gamma'); (\Delta \mid \Delta') \longrightarrow B}{(\Gamma \mid A_1 \& A_2, \Gamma'); (\Delta \mid \Delta') \longrightarrow B}\,\&\mathrm{L}! \qquad \frac{(\Gamma \mid \Gamma'); (\Delta \mid \Delta') \longrightarrow B}{(\Gamma \mid \top, \Gamma'); (\Delta \mid \Delta') \longrightarrow B}\,\top\mathrm{L}!$$

$$\frac{(\Gamma \mid \Gamma'); (\Delta \mid \Delta') \longrightarrow B}{(\Gamma \mid \mathbf{1}, \Gamma'); (\Delta \mid \Delta') \longrightarrow B}\,\mathbf{1}\mathrm{L}! \qquad \frac{}{(\Gamma \mid \mathbf{0}, \Gamma'); (\Delta \mid \Delta') \longrightarrow B}\,\mathbf{0}\mathrm{L}!$$

$$\frac{(\Gamma \mid A, \Gamma'); (\Delta \mid \Delta') \longrightarrow B}{(\Gamma \mid\, !A, \Gamma'); (\Delta \mid \Delta') \longrightarrow B}\,!\mathrm{L}!$$

$$\frac{(\Gamma, E \mid \Gamma'); (\Delta \mid \Delta') \longrightarrow B}{(\Gamma \mid E, \Gamma'); (\Delta \mid \Delta') \longrightarrow B}\,\mathrm{md}! \qquad \frac{\Gamma; (\Delta \mid \Delta') \longrightarrow B}{(\Gamma \mid \cdot); (\Delta \mid \Delta') \longrightarrow B}\,\mathrm{me}!$$

In the md! rule we use E to stand for those propositions whose principal connective is not invertible when it appears among the unrestricted resources.

$$\begin{array}{lll} \textit{Left!-Critical Propositions} \quad E & ::= & P \mid A_1 \multimap A_2 \mid \forall x.\, A \\ & & \mid A_1 \otimes A_2 \mid A_1 \oplus A_2 \mid \exists x.\, A \end{array}$$

Focussing Rules. With the rules above we arrive at a situation where Γ consists entirely of left!-critical propositions, Δ of left-critical propositions, and the succedent is a *right-critical proposition* C, defined as a proposition whose principal connective does not have an invertible right rule.

$$\textit{Right-Critical Propositions} \quad C \quad ::= \quad P \mid A_1 \otimes A_2 \mid \mathbf{1} \mid A_1 \oplus A_2 \mid \mathbf{0} \mid\, !A \mid \exists x.\, A$$

We also use C^* to stand for a non-atomic right-critical proposition. The idea of focussing is to pick either the succedent of the sequent or one of the linear or unrestricted hypothesis and apply a sequence of either left or right rules to this one distinguished proposition. Thus we have two judgments,

$$\begin{array}{ll} \Gamma; \Delta \longrightarrow\!\!\gg A & A \text{ has a right-focussed derivation, and} \\ \Gamma; \Delta \longrightarrow A \gg C & C \text{ has a derivation left-focussed on } A. \end{array}$$

They arise in a derivation when a sequent consists entirely of critical propositions.

$$\frac{\Gamma; \Delta \longrightarrow\!\!\gg C^*}{\Gamma; \Delta \longrightarrow C^*} \gg\mathrm{R}$$

$$\frac{\Gamma; \Delta \longrightarrow D \gg C}{\Gamma; (\Delta, D) \longrightarrow C} \gg\mathrm{L} \qquad \frac{(\Gamma, E); \Delta \longrightarrow E \gg C}{(\Gamma, E); \Delta \longrightarrow C} \gg\mathrm{L}!$$

Here, we restrict the $\gg$ R rule to *non-atomic* critical propositions C^*. Note that these three rules represent a don't-know non-deterministic choice. In the presence of resource non-determinism, some of the hypotheses may only be potentially available—choosing them forces them to occur, possibly compromising other pending subgoals. This is also why, for example, the **1**R rule below is not trivial: in practice it constitutes a commitment that none of the potential hypotheses are used in this branch of the derivation.

The rules for right-focussed sequents and immediate entailment are once again just the right and left rules, until the focus proposition is no longer critical, at which point a new critical sequent may arise.

$$\frac{\Gamma; \Delta_1 \longrightarrow\gg A_1 \qquad \Gamma; \Delta_2 \longrightarrow\gg A_2}{\Gamma; \Delta_1 \times \Delta_2 \longrightarrow\gg A_1 \otimes A_2} \otimes\text{R} \qquad \frac{}{\Gamma; \cdot \longrightarrow\gg \mathbf{1}} \mathbf{1}\text{R}$$

$$\frac{\Gamma; \Delta \longrightarrow\gg A_1}{\Gamma; \Delta \longrightarrow\gg A_1 \oplus A_2} \oplus\text{R}_1 \qquad \frac{\Gamma; \Delta \longrightarrow\gg A_2}{\Gamma; \Delta \longrightarrow\gg A_1 \oplus A_2} \oplus\text{R}_2$$

No right rule for **0**

$$\frac{\Gamma; \cdot \longrightarrow\gg A}{\Gamma; \cdot \longrightarrow\gg\, !A} !\text{R}$$

$$\frac{\Gamma; \Delta \longrightarrow\gg [t/x]A}{\Gamma; \Delta \longrightarrow\gg \exists x.\, A} \exists\text{R} \qquad \frac{\Gamma; \Delta \longrightarrow \overline{C^*}}{\Gamma; \Delta \longrightarrow\gg \overline{C^*}} \text{ur}$$

In the last rule $\overline{C^*}$ is a proposition which is either atomic or not right critical.

$$\frac{\Gamma; \Delta \longrightarrow A_1 \gg C}{\Gamma; \Delta \longrightarrow A_1 \& A_2 \gg C} \&\text{L}_1 \qquad \frac{\Gamma; \Delta \longrightarrow A_2 \gg C}{\Gamma; \Delta \longrightarrow A_1 \& A_2 \gg C} \&\text{L}_2$$

No left rule for $\top$

$$\frac{\Gamma; \Delta_2 \longrightarrow A_2 \gg C \qquad \Gamma; \Delta_1 \longrightarrow A_1}{\Gamma; \Delta_1 \times \Delta_2 \longrightarrow A_1 \multimap A_2 \gg C} \multimap\text{L}$$

$$\frac{\Gamma; \Delta \longrightarrow [t/x]A \gg C}{\Gamma; \Delta \longrightarrow \forall x.\, A \gg C} \forall\text{L} \qquad \frac{}{\Gamma; \cdot \longrightarrow P \gg P} \text{I}$$

$$\frac{\Gamma; (\Delta \mid D) \longrightarrow C}{\Gamma; \Delta \longrightarrow D \gg C} \text{ul}$$

Note that the premisses of the $\multimap$L rule are another immediate entailment and a general, weakly uniform derivation of the antecedent of the linear implication.

The soundness of this inference system can be seen rather easily by induction, but we must generalize the statement to include all auxiliary judgments.

Theorem 3.14 (Soundness of Weakly Uniform Derivations)

1. *If* $\Gamma; \Delta \longrightarrow A$ *then* $\Gamma; \Delta \overset{-}{\Longrightarrow} A$.
2. *If* $\Gamma; (\Delta \mid \Delta') \longrightarrow A$ *then* $\Gamma; (\Delta, \Delta') \overset{-}{\Longrightarrow} A$.
3. *If* $(\Gamma \mid \Gamma'); (\Delta \mid \Delta') \longrightarrow A$ *then* $(\Gamma, \Gamma'); (\Delta, \Delta') \overset{-}{\Longrightarrow} A$.
4. *If* $\Gamma; \Delta \longrightarrow\gg A$ *then* $\Gamma; \Delta \overset{-}{\Longrightarrow} A$.
5. *If* $\Gamma; \Delta \longrightarrow A \gg C$ *then* $\Gamma; (\Delta, A) \overset{-}{\Longrightarrow} A$.

Proof: By a straightforward simultaneous induction on the given derivations. Each left and right rule in the weakly uniform derivation corresponds directly to a left and right rule on the sequent calculus. For the rules $\&$L!, $\top$L!, **1**L!, **0**L!, !L! we use their admissibility (Theorem 3.13). The remaining rules disappear in the translation, since the premiss and conclusion sequent are intpreted identically. □

The completeness is open at present. One appropriate proof technique would be to use the *permutability* of inference rules to show explicitly how to transform a derivation of $\Gamma; \Delta \overset{-}{\Longrightarrow} A$ into a weakly uniform one. The only difficult in this proof is the explosive number of cases which must be checked.