2.8 Deductions with Lemmas

One common way to find or formulate a proof is to introduce a lemma. In the sequent calculus, the introduction and use of a lemma during proof search is modelled by the rules of cut, *Cut* for lemmas used as linear hypotheses, and *Cut*! for lemmas used unrestrictedly. The corresponding rule for intuitionistic logic is due to Gentzen [Gen35]. We write $\Gamma;\Delta \overset{+}{\Longrightarrow} A$ for the judgment that A can be derived with the rules from before, plus one of the two cut rules below.

$$\frac{\Gamma;\Delta \overset{+}{\Longrightarrow} A \qquad \Gamma;(\Delta', A) \overset{+}{\Longrightarrow} C}{\Gamma;\Delta' \times \Delta \overset{+}{\Longrightarrow} C}\,Cut \qquad \frac{\Gamma;\cdot \overset{+}{\Longrightarrow} A \qquad (\Gamma, A);\Delta' \overset{+}{\Longrightarrow} C}{\Gamma;\Delta' \overset{+}{\Longrightarrow} C}\,Cut!$$

Note that the linear context in the left premiss of the *Cut*! rule must be empty, because the new hypothesis A in the right premiss is unrestricted in its use.

On the side of natural deduction, these rules correspond to substitution principles. They can be related to normal and atomic derivations only if we allow an additional coercion from normal to atomic derivations. This is because the left premiss corresponds to a derivation of $\Gamma;\Delta \vdash A \uparrow$ which can be substituted into a derivation of $\Gamma;\Delta', A \vdash C \uparrow$ only have the additional coercion has been applied. Of course, the resulting deductions are no longer normal in the sense we defined before, so we write $\Gamma;\Delta \vdash^{+} A \downarrow$ and $\Gamma;\Delta \vdash^{+} A \uparrow$. These judgments are defined with the same rules as $\Gamma;\Delta \vdash A \uparrow$ and $\Gamma;\Delta \vdash A \downarrow$, plus the following coercion.

$$\frac{\Gamma;\Delta \vdash^{+} A \uparrow}{\Gamma;\Delta \vdash^{+} A \downarrow}\uparrow\downarrow$$

It is now easy to prove that arbitrary natural deductions can be annotated with $\uparrow$ and $\downarrow$, since we can arbitrarily coerce back and forth between the two judgments.

Theorem 2.11 *If* $\Gamma;\Delta \vdash A$ *then* $\Gamma;\Delta \vdash^{+} A \uparrow$ *and* $\Gamma;\Delta \vdash^{+} A \downarrow$

Proof: By induction on the structure of $\mathcal{D} :: (\Gamma;\Delta \vdash A)$. □

Theorem 2.12

1. *If* $\Gamma;\Delta \vdash^{+} A \uparrow$ *then* $\Gamma;\Delta \vdash A$.
2. *If* $\Gamma;\Delta \vdash^{+} A \downarrow$ *then* $\Gamma;\Delta \vdash A$.

Proof: My mutual induction on $\mathcal{N} :: (\Gamma;\Delta \vdash^{+} A \uparrow)$ and $\mathcal{A} :: (\Gamma;\Delta \vdash^{+} A \downarrow)$. □

It is also easy to relate the Cut rules to the new coercions (and thereby to natural deductions), plus four substitution principles.

Property 2.13

1. *If* $\Gamma; (\Delta', w{:}A) \vdash^{+} C \uparrow$ *and* $\Gamma; \Delta \vdash^{+} A \downarrow$ *then* $\Gamma; (\Delta' \times \Delta) \vdash^{+} C \uparrow$.

2. *If* $\Gamma; (\Delta', w{:}A) \vdash^{+} C \downarrow$ *and* $\Gamma; \Delta \vdash^{+} A \downarrow$ *then* $\Gamma; (\Delta' \times \Delta) \vdash^{+} C \downarrow$.

3. *If* $(\Gamma, u{:}A); \Delta' \vdash^{+} C \uparrow$ *and* $\Gamma; \cdot \vdash^{+} A \downarrow$ *then* $\Gamma; \Delta' \vdash^{+} C \uparrow$.

4. *If* $(\Gamma, u{:}A); \Delta' \vdash^{+} C \downarrow$ *and* $\Gamma; \cdot \vdash^{+} A \downarrow$ *then* $\Gamma; \Delta' \vdash^{+} C \downarrow$.

Proof: By mutual induction on the structure of the given derivations. □

We can now extend Theorems 2.9 and 2.10 to relate sequent derivations with Cut to natural deductions with explicit lemmas.

Theorem 2.14 (Soundness of Sequent Derivations with Cut)
If $\Gamma; \Delta \overset{+}{\Longrightarrow} A$ *then* $\Gamma; \Delta \vdash^{+} A \uparrow$.

Proof: As in Theorem 2.9 by induction on the structure of the derivation of $\Gamma; \Delta \overset{+}{\Longrightarrow} A$. An inference with one of the new rules *Cut* or *Cut*! is translated into an application of the $\uparrow\downarrow$ coercion followed by an appeal to one of the substitution principles in Property 2.13. □

Theorem 2.15 (Completeness of Sequent Derivations with Cut)

1. *If* $\Gamma; \Delta \vdash^{+} A \uparrow$ *then there is a sequent derivation of* $\Gamma; \Delta \overset{+}{\Longrightarrow} A$, *and*

2. *if* $\Gamma; \Delta \vdash^{+} A \downarrow$ *then for any formula* C *and derivation of* $\Gamma; (\Delta', A) \overset{+}{\Longrightarrow} C$ *there is a derivation of* $\Gamma; (\Delta' \times \Delta) \overset{+}{\Longrightarrow} C$.

Proof: As in the proof of Theorem 2.10 by induction on the structure of the given derivations. In the new case of the $\uparrow\downarrow$ coercion, we use the rule of *Cut*. The other new rule, *Cut*!, is not needed for this proof, but is necessary for the proof of admissibility of cut in the next section. □

2.9 Cut Elimination

We viewed the sequent calculus as a calculus of proof search for natural deduction. The proofs of the soundness theorems 2.10 and 2.15 provide ways to translate cut-free sequent derivations into normal natural deductions, and sequent derivations with cut into arbitrary natural deductions.

This section is devoted to showing that the two rules of cut are redundant in the sense that any derivation in the sequent calculus which makes use of the rules of cut can be translated to one that does not. Taken together with the soundness and completeness theorems for the sequent calculi with and without cut, this has many important consequences.

First of all, a proof search procedure which looks only for cut-free sequent derivations will be complete: any derivable proposition can be proven this way. When the cut rule

$$\frac{\Gamma; \Delta \overset{+}{\Longrightarrow} A \qquad \Gamma; \Delta', A \overset{+}{\Longrightarrow} C}{\Gamma; \Delta' \times \Delta \overset{+}{\Longrightarrow} C}\ \mathit{Cut}$$

is viewed in the bottom-up direction the way it would be used during proof search, it introduces a new and arbitrary proposition A. Clearly, this introduces a great amount of non-determinism into the search. The cut elimination theorem now tells us that we never need to use this rule. All the remaining rules have the property that the premisses contain only instances of propositions in the conclusion, or parts thereof. This latter property is often called the *subformula property.*

Secondly, it is easy to see that the logic is *consistent*, that is, not every proposition is provable. In particular, the sequent $\cdot; \cdot \Longrightarrow \mathbf{0}$ does not have a cut-free derivation, because there is simply no rule which could be applied to infer it! This property clearly fails in the presence of cut: it is *prima facie* quite possible that the sequent $\cdot; \cdot \overset{+}{\Longrightarrow} \mathbf{0}$ is the conclusion of the cut rule.

Along the same lines, we can show that a number of propositions are *not derivable* in the sequent calculus and therefore not true as defined by the natural deduction rules. Examples of this kind are given at the end of this section.

We prove cut elimination by showing that the two cut rules are *admissible rules of inference* in the sequent calculus without cut. An inference rule is admissible if whenever we can find derivations for its premisses we can find a derivation of its conclusion. This should be distinguished from a *derived rule of inference* which requires a direct derivation of the conclusion from the premisses. We can also think of a derived rule as an evident hypothetical judgment where the premisses are (unrestricted) hypotheses.

Derived rules of inference have the important property that they remain evident under any extension of the logic. An admissible rule, on the other hand, represents a global property of the deductive system under consideration and may well fail when the system is extended. Of course, every derived rule is also admissible.

Theorem 2.16 (Admissibility of Cut)

1. *If* $\Gamma; \Delta \Longrightarrow A$ *and* $\Gamma; (\Delta', A) \Longrightarrow C$ *then* $\Gamma; \Delta' \times \Delta \Longrightarrow C$.
2. *If* $\Gamma; \cdot \Longrightarrow A$ *and* $(\Gamma, A); \Delta' \Longrightarrow C$ *then* $\Gamma; \Delta' \Longrightarrow C$.

Proof: By nested inductions on the structure of the cut formula A and the given derivations, where induction hypothesis (1) has priority over (2). To state this more precisely, we refer to the given derivations as $\mathcal{D} :: (\Gamma; \Delta \Longrightarrow A)$, $\mathcal{D}' :: (\Gamma; \cdot \Longrightarrow A)$, $\mathcal{E} :: (\Gamma; (\Delta, A) \Longrightarrow C)$, and $\mathcal{E}' :: ((\Gamma, A); \Delta' \vdash C)$. Then we may appeal to the induction hypothesis whenever

a. the cut formula A is strictly smaller, or

b. the cut formula A remains the same, but we appeal to induction hypothesis (1) in the proof of (2) (but when we appeal to (2) in the proof of (1) the cut formula must be strictly smaller), or

c. the cut formula A and the derivation $\mathcal{E}$ remain the same, but the derivation $\mathcal{D}$ becomes smaller, or

d. the cut formula A and the derivation $\mathcal{D}$ remain the same, but the derivation $\mathcal{E}$ or $\mathcal{E}'$ becomes smaller.

Here, we consider a formula smaller it is an immediate subformula, where $[t/x]A$ is considered a subformula of $\forall x.\ A$, since it contains fewer quantifiers and logical connectives. A derivation is smaller if it is an immediate subderivation, where we allow weakening by additional unrestricted hypothesis in one case (which does not affect the structure of the derivation).

The cases we have to consider fall into 5 classes:

Initial Cuts: One of the two premisses is an initial sequent. In these cases the cut can be eliminated directly.

Principal Cuts: The cut formula A was just inferred by a right rule in $\mathcal{D}$ and by a left rule in $\mathcal{E}$. In these cases we appeal to the induction hypothesis (possibly several times) on smaller cut formulas (item (a) above).

Dereliction Cut: The cases for the *Cut*! rule are treated as right commutative cuts (see below), except for the rule of dereliction which requires an appeal to induction hypothesis (1) with the same cut formula (item (b) above).

Left Commutative Cuts: The cut formula A is a side formula of the last inference in $\mathcal{D}$. In these cases we may appeal to the induction hypotheses with the same cut formula, but smaller derivation $\mathcal{D}$ (item (c) above).

Right Commutative Cuts: The cut formula A is a side formula of the last inference in $\mathcal{E}$. In these cases we may appeal to the induction hypotheses with the same cut formula, but smaller derivation $\mathcal{E}$ or $\mathcal{E}'$ (item (d) above).

[*Some cases to be filled in later.*]

□

Using the admissibility of cut, the cut elimination theorem follows by a simple structural induction.

Theorem 2.17 (Cut Elimination)
If $\Gamma; \Delta \overset{+}{\Longrightarrow} C$ *then* $\Gamma; \Delta \Longrightarrow C$.

Proof: By induction on the structure of $\mathcal{D} :: (\Gamma; \Delta \overset{+}{\Longrightarrow} C)$. In each case except *Cut* or *Cut*! we simply appeal to the induction hypothesis on the derivations of the premisses and use the corresponding rule in the cut-free sequent calculus. For the *Cut* and *Cut*! rules we appeal to the induction hypothesis and then admissibility of cut (Theorem 2.16) on the resulting derivations. □

2.10 Consequences of Cut Elimination

As a first consequence, we see that linear logic is *consistent*: not every proposition can be proved. A proof of consistency for both intuitionistic and classical logic was Gentzen's original motivation for the development of the sequent calculus and his proof of cut elimination.

Theorem 2.18 (Consistency of Intuitionistic Linear Logic)
$\cdot;\cdot \vdash \mathbf{0}$ *is not derivable.*

Proof: If the judgment were derivable, by Theorems 2.11, 2.15, and 2.17, there must be a cut-free sequent derivation of $\cdot;\cdot \Longrightarrow \mathbf{0}$. But there is no rule with which we could infer this sequent (there is no right rule for $\mathbf{0}$), and so it cannot be derivable. □

A second consequence is that every natural deduction can be translated to a normal natural deduction. The necessary construction is implicit in the proofs of the soundness and completeness theorems for sequent calculi and the proofs of admissibility of cut and cut elimination. In Chapter **??** we will see a much more direct, but in other respects more complicated proof.

Theorem 2.19 (Normalization for Natural Deductions)
If $\Gamma;\Delta \vdash A$ *then* $\Gamma;\Delta \vdash A \uparrow$.

Proof: Directly, using theorems from this chapter. Assume $\Gamma;\Delta \vdash A$. Then

$\Gamma;\Delta \vdash^{+} A$ by Theorem 2.11,
$\Gamma;\Delta \stackrel{+}{\Longrightarrow} A$ by completeness of sequent derivations with cut (Theorem 2.15),
$\Gamma;\Delta \Longrightarrow A$ by cut elimination (Theorem 2.17), and
$\Gamma;\Delta \vdash A \uparrow$ by soundness of cut-free sequent derivations (Theorem 2.9).

□

2.11 Exercises

Exercise 2.1 Give a counterexample which shows that the elimination $\supset$E would be locally unsound if its second premiss were allowed to depend on linear hypotheses.

Exercise 2.2 If we *define* intuitionistic implication $A \supset B$ in linear logic as an abbreviation for $(!A) \multimap B$, then the given introduction and elimination rules become *derived rules of inference*. Prove this by giving a derivation for the conclusion of the $\supset$E rule from its premisses under the interpretation, and similarly for the $\supset$I rule.

For the other direction, show how $!A$ could be defined from intuitionistic implication or speculate why this might not be possible.

Exercise 2.3 [*To be filled in: an exercise exploring the "missing connectives" of multiplicative disjunction and additive implication.*]

Exercise 2.4 In the blocks world example from Section 2.3, sketch the derivation for the same goal A_0 and initial situation Δ_0 in which block b is put on block c, rather than the table.

Exercise 2.5 Model the *Towers of Hanoi* in linear logic in analogy with our modelling of the blocks world.

1. Define the necessary atomic propositions and their meaning.
2. Describe the legal moves in *Towers of Hanoi* as unrestricted hypotheses Γ_0 independently from the number of towers or disks.
3. Represent the initial situation of three towers, where two are empty and one contains two disks in a legal configuration.
4. Represent the goal of legally stacking the two disks on some arbitrary other tower.
5. Sketch the proof for the obvious 3-move solution as in Section 2.3.

Exercise 2.6 Consider if $\otimes$ and $\&$ can be distributed over $\oplus$ or *vice versa*. There are four different possible equivalences based on eight possible entailments. Give natural deductions for the entailments which hold.

Exercise 2.7 In this exercise we explore distributive and related *interaction laws* for linear implication. In intuitionistic logic, for example, we have the following $(A \wedge B) \supset C \dashv\vdash A \supset (B \supset C)$ and $A \supset (B \wedge C) \dashv\vdash (A \supset B) \wedge (A \supset C)$, where $\dashv\vdash$ is mutual entailment as in Exercise 1.2.

In linear logic, we now write $A \dashv\vdash A'$ for linear mutual entailment, that is, A' follows from linear hypothesis A and *vice versa*. Write out appropriate interaction laws or indicate none exists, for each of the following propositions.

1. $A \multimap (B \otimes C)$
2. $(A \otimes B) \multimap C)$
3. $A \multimap \mathbf{1}$
4. $\mathbf{1} \multimap A$
5. $A \multimap (B \& C)$
6. $(A \& B) \multimap C$
7. $A \multimap \top$
8. $\top \multimap A$

9. $A \multimap (B \oplus C)$

10. $(A \oplus B) \multimap C$

11. $A \multimap \mathbf{0}$

12. $\mathbf{0} \multimap A$

13. $A \multimap (B \multimap C)$

14. $(A \multimap B) \multimap C$

Note that an interaction law exists only if there is a mutual linear entailment—we are not interested if one direction holds, but not the other.

Give the derivations in both directions for one of the interaction laws of a binary connective $\otimes$, $\&$, $\oplus$, or $\multimap$, and for one of the interaction laws of a logical constant $\mathbf{1}$, $\top$, or $\mathbf{0}$.

Exercise 2.8 Extend the interaction laws from Exercise 2.7 by laws showing how linear implication interacts with existential and universal quantification.

Exercise 2.9 Design an alternative translation $()^*$ from formulas and natural deductions in intuitionistic logic to intuitionistic linear logic in which conjunction ($\wedge$) and truth ($\top$) are mapped to simultaneous conjunction ($\otimes$) and its unit ($\mathbf{1}$) instead of the additive connectives as in $()^+$. Prove the correctness of the embedding and discuss the relative merits of the two translations.

Exercise 2.10 Extend the embedding from from Section 2.4 to encompass intuitionistic propositions $\neg A$ without adding any connectives to the linear logic. Modify the statements and proofs of embedding and conservativity (if necessary) and show the proof cases concerned with negation.

Exercise 2.11 Find a derivation $\mathcal{D} :: (\Gamma; \Delta \vdash A)$ which contains no local redex, but which is not normal in the sense that there is no derivation $\mathcal{N} :: (\Gamma; \Delta \vdash A \uparrow)$ such that $\mathcal{N}^- = \mathcal{D}$.

Exercise 2.12 Internalize the notion of mutual linear entailment from Exercise 2.7 as a new linear connective $A \circ\!\!-\!\!\circ A'$.

1. Give introduction and elimination rules. Your rules should be orthogonal to all other connectives and not mention, for example, linear implication.

2. Are your rules locally sound and complete? Give the local reduction and expansions, if they exist.

3. Annotate your rules, extending the definitions of normal and atomic derivations.

4. Give right and left sequent rules corresponding to the introduction and elimination rules, respectively.

5. Show the new cases in the proofs of soundness and completeness of the sequent calculus with respect to natural deduction (Theorems 2.9 and 2.10).

6. Show a new principal case in the proof of admissibility of cut (Theorem 2.16).

7. Would you classify the new connective as multiplicative, additive, or exponential? Can it be defined from the linear connectives introduced in Sections 2.1 and 2.2 in such a way that your introduction and elimination rules become derived rules of inference? If so, give the definition, if not explain informally why it is not possible.